Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 15 4.03.2010

This set of homeworks should be handed in by Thursday, 18 March 2010 in the lecture.

1. For $a, b, c > 0$ consider the ellipsoid

$$E := \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}.$$

   Let ω be the following differential form on $\mathbb{R}^3$;

$$\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

   (a) Calculate $d\omega$.

   (b) Find an almost global parametrisation of E such that the outward unit normal vector field is positively oriented. Calculate

$$\int_E \omega.$$

   Hint: Think of polar coordinates on the sphere.

2. Consider the 2-form ω, defined on $\mathbb{R}^3 \backslash \{0\}$ by

$$\omega = \frac{x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}.$$

   (a) Show that ω is closed.

   (b) Show that

$$\omega_x(v, w) = \frac{\langle v \times w, x \rangle}{\|x\|^3}.$$

   (c) For $r > 0$, let $S_r(0) := \{x \in \mathbb{R}^3 \mid \|x\| = r\}$ be the oriented sphere of radius r around the origin, such that the outward unit normal vector $n(x) = \frac{x}{r}$ is positively oriented. Show that

$$\int_{S_r(0)} \omega = 4\pi.$$