Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 14 25.02.2010

1. Let U_1, U_2, U_3 be three open star-like subsets of $\mathbb{R}^n$. Suppose that the two intersections $U_1 \cap U_2$ and $U_2 \cap U_3$ are pathwise connected and $U_1 \cap U_3 = \emptyset$. Let $\omega \in \Omega^1(U_1 \cup U_2 \cup U_3)$ be closed. Show that ω is exact.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
$$f(x_1, x_2, x_3) = (y_1, y_2, y_3) = (x_1 \cos x_2, x_1 \sin x_2, x_3^2).$$
   (a) Calculate the pullback $\omega = f^*(y_3\, dy_1 \wedge dy_2 \wedge dy_3)$.

   (b) Calculate $\int_{(1,2)\times(0,2\pi)\times(-1,1)} \omega$.

3. Let $U \subset \mathbb{R}^n$ be open and starlike and $\omega \in \Omega^k(U)$, $k \geq 1$ with $d\omega = 0$. The aim of this exercise is to prove **Poincaré's Lemma**, i.e., that there is an $\alpha \in \Omega^{k-1}(U)$ with $d\alpha = \omega$.

   Henceforth we will denote the coordinate functions of $\mathbb{R}\times U$ by $t, x_1, \ldots, x_n$. Note that every differential form $\eta \in \Omega^k(\mathbb{R} \times U)$ is then of the form
$$\eta = \eta_1 + dt \wedge \eta_2, \tag{1}$$
   where
$$\eta_1 = \sum_{i_1 < \cdots < i_k} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
   and
$$\eta_2 = \sum_{j_1 < \cdots < j_{k-1}} g_{j_1,\ldots,j_{k-1}} dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}},$$
   with $f_{i_1,\ldots,i_k}, g_{j_1,\ldots,j_{k-1}} \in C^\infty(\mathbb{R} \times U)$.

   Since U is starlike, there is a point $p \in U$ and a map $H : \mathbb{R} \times U \to \mathbb{R}^n$, defined by $H(t, x) = p + t(x - p)$, such that $H(t, x) \in U$ for all $t \in [0, 1]$ and $x \in U$ (since $H([0, 1], x)$ is the straight line segment from p to x). Observe that $H(0, x) = p$ and $H(1, x) = x$. Let $i_t : U \to \mathbb{R} \times U$ be the inclusion of U into $\mathbb{R} \times U$ at "level" t, i.e., $i_t(x) = (t, x)$.

   Finally, let $I : \Omega^k(\mathbb{R} \times U) \to \Omega^{k-1}(U)$ be defined by
$$(I\eta)_x(v_1, \ldots, v_{k-1}) = \int_0^1 \eta_2(t, x)(Di_t(x)(v_1), \ldots, Di_t(x)(v_k))\, dt,$$
   if $\eta = \eta_1 + dt \wedge \eta_2$ as given in (1).

   (a) Prove that $i_1^*\eta - i_0^*\eta = d(I\eta) + I(d\eta)$.

(b) Using (a) and $H \circ i_1 = \mathrm{id}$ and $H \circ i_0 =$ constant, show that

$$\omega = d\alpha$$

with $\alpha = I(H^*\omega)$.

Hint: Note that if $F =$ constant, then $DF(x) = 0$ for all x, and therefore $F^*\omega = 0$.