Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 12 11.02.2010

1. Let $\alpha \in \Lambda^n(\mathbb{R}^n)$ be the alternating n-form defined by

$$\alpha(e_1, \ldots, e_n) = 1,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Show that

$$\alpha(v_1, \ldots, v_n) = \det(v_1, \ldots, v_n).$$

2. Let φ, ψ, ϕ be the following forms in $\mathbb{R}^3$:

$$\begin{aligned}\varphi &= x\,dx - y\,dy,\\ \psi &= z\,dx \wedge dy + x\,dy \wedge dz,\\ \phi &= z\,dz.\end{aligned}$$

Calculate $\varphi \wedge \psi, \phi \wedge \varphi \wedge \psi$, $d\varphi$, $d\psi$ and $d\phi$.

3. A function $g : \mathbb{R}^3 \to \mathbb{R}$ is said to be homogeneous of degree k if $g(tx, ty, tz) = t^k g(x, y, z)$ for all $t > 0$ and $(x, y, z) \in \mathbb{R}^3$. Prove the following facts:

(a) If g is differentiable and homogeneous of degree k, then we have *Euler's relation*:

$$x\frac{\partial g}{\partial x}(x, y, z) + y\frac{\partial g}{\partial y}(x, y, z) + z\frac{\partial g}{\partial z}(x, y, z) = kg(x, y, z).$$

Hint: Differentiate $g(tx, ty, tz)$ in t.

(b) If the differential form

$$\omega = a\,dx + b\,dy + c\,dz$$

is such that a, b and c are homogeneous of degree k and ω is closed, then we have $\omega = df$ with

$$f = \frac{xa + yb + zc}{k + 1}.$$

(c) If the differential form

$$\omega = a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy$$

is such that a, b and c are homogeneous of degree k and ω is closed, then we have $\omega = d\eta$, where

$$\eta = \frac{(zb - yc)\,dx + (xc - za)\,dy + (ya - xb)\,dz}{k + 2}.$$

4. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Then we can associate to $F = (f_1, f_2, f_3)$ the following 1- and 2-form:

$$\omega_F^1 = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3, \quad \omega_F^2 = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2.$$

Show the following identities:

$$\begin{aligned} df &= \omega_{\nabla f}^1, \\ d\omega_F^1 &= \omega_{\text{curl}F}^2, \\ d\omega_F^2 &= \text{div}F\, dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

Derive from these identities and $d^2 = 0$ that $\text{curl} \circ \nabla f = 0$ and $\text{div} \circ \text{curl}F = 0$.