Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 11 4.02.2010

This set of homeworks should be handed in by Thursday, 11 February 2010 in the lecture.

1. Let $\omega \in \Omega^1(\mathbb{R}^2\backslash\{0\})$ be given by

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

and $\gamma : [0,8] \to \mathbb{R}^2$ be the piecewise smooth curve defined by

$$\gamma(t) = \begin{cases} (1,t) & \text{for } 0 \le t < 1, \\ (2-t,1) & \text{for } 1 \le t < 3, \\ (-1,4-t) & \text{for } 3 \le t < 5, \\ (t-6,-1) & \text{for } 5 \le t < 7, \\ (1,t-8) & \text{for } 7 \le t \le 8. \end{cases}$$

(i) Draw the curve γ.

(ii) Calculate directly $\int_\gamma \omega$. You can use (without proof) the fact that

$$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{\Delta}} \arctan \frac{2ax+b}{\sqrt{\Delta}},$$

if $\Delta = 4ac - b^2 > 0$.

(iii) Find a free homotopy between the curves γ and c (from Exercise 3 of Exercise Sheet 10) in $\mathbb{R}^2\backslash\{0\}$.

2. (i) Show that if $\omega \in \Omega^1(U)$ and $c : [a,b] \to U$ is a smooth curve with $\|F_\omega(c(t))\| \le M$ for all $t \in [a,b]$, then

$$|\int_c \omega| \le M \cdot L(c).$$

(ii) Let $\omega \in \Omega^1(\mathbb{R}^2\backslash\{0\})$ be a closed form. Assume that $\|F_\omega\|$ is bounded in a disk of centre 0. Use Corollary 6.19 to show that ω is exact in $\mathbb{R}^2\backslash\{0\}$.

(iii) Why is this not a contradiction to the non-exactness of the form

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

in Exercise 3 of Exercise Sheet 10?