Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 1 15.10.2009

1. For $x \in \mathbb{R}^n$, write $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Define

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^{n} |x_i - y_i| \\ d_\infty(x,y) &= \max\{|x_i - y_i| \,:\, i = 1, \ldots, n\} \end{aligned}$$

   (a) Show that d_1 and d_∞ define metrics on $\mathbb{R}^n$.

   (b) Draw

$$\begin{aligned} D_1(0;1) &= \{x \in \mathbb{R}^2 \,|\, d_1(x,0) \le 1\} \\ D_\infty(0;1) &= \{x \in \mathbb{R}^2 \,|\, d_\infty(x,0) \le 1\} \end{aligned}$$

2. Recall that for $x, y \in \mathbb{R}^n$ we have $d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$

   (a) Show that $d_\infty(x,y) \le d_2(x,y) \le d_1(x,y)$.

   (b) Show that $d_1(x,y) \le n \cdot d_\infty(x,y)$.

   (c) Show that $d_2(x,y) \le \sqrt{n} \cdot d_\infty(x,y)$.

3. Let $V = C[0,2] = \{f : [0,2] \to \mathbb{R} \,|\, f \text{ continuous}\}$ and we have an inner product on V given by

$$\langle f, g \rangle = \int_0^2 f(x) g(x)\, dx$$

   for $f, g \in V$.

   Define a sequence $(f_n)_{n \in \mathbb{N}}$ by

$$f_n(x) = \begin{cases} 0 & \text{for } x \in [0, 1 - 1/n] \\ \frac{n}{2}x - \frac{n-1}{2} & \text{for } x \in (1 - 1/n, 1 + 1/n) \\ 1 & \text{for } x \in [1 + 1/n, 2] \end{cases}$$

   (a) Show that $(f_n)_{n \in \mathbb{N}}$ is a sequence in V, i.e., show that each f_n is continuous.

   (b) Show that $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, where V has the metric induced from the inner product $\langle \cdot, \cdot \rangle$.

Remark. The Cauchy sequence in Question 3(b) does not converge, so V is not complete. You should try to convince yourself that this is the case, but this is not part of the question.