11 Differential forms on oriented manifolds

Recall the definition of a *k-dimensional manifold* $M \subset \mathbb{R}^n$ from Chapter 4 in Dirk Schütz's notes: every $p \in M$ is contained in an open set $U \subset \mathbb{R}^n$ for which there is a smooth diffeomorphism $h : U \to V \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ with $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\})$. The map h is called a *chart* of the manifold.

More useful than chart are the so-called *coordinate systems* or *local parametrisations* of a manifold. A local parametrisation can be obtained from a chart $h : U \to V$ as follows: The restriction of h to $U \cap M$ maps this set essentially to an open set in $\mathbb{R}^k$, since the last $n-k$ coordinates in the image are all zero. Let $U_0 := U \cap M$ and $V_0 \subset \mathbb{R}^k$ be defined by $h(U_0) = V_0 \times \{0\}$. Then h restricts to a bijection $h_0 : U_0 \to V_0$ between an open neighbourhood U_0 of p within M and an open set $V_0 \subset \mathbb{R}^k$. The inverse of this map $\varphi := h_0^{-1} : V_0 \subset \mathbb{R}^k \to U_0 \subset M$ is called a *coordinate system* or a *local parametrisation* of M.

The image sets of all local parametrisations cover all of M and with their help one can consider M as a subset in $\mathbb{R}^n$ which is built up by local patches which are diffeomorphic to open sets in $\mathbb{R}^k$. A collection of local parametrisations covering all of M is called an *atlas of* M.

Example. *Stereographic projections from the north pole* $N = (0,0,1)$ *and south pole* $S = (0,0,-1)$ *of the unit sphere*

$$S^2 = \{p \in \mathbb{R}^3 \mid \|p\| = 1\} \subset \mathbb{R}^3$$

can be used to obtain two coordinate systems $\varphi_N : \mathbb{R}^2 \to S^2 \backslash \{N\}$ *and* $\varphi_S : \mathbb{R}^2 \to S^2 \backslash \{S\}$ *covering the sphere. They are:*

$$\begin{aligned} \varphi_N(x_1,x_2) &:= \left(\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1} \right), \\ \varphi_S(y_1,y_2) &:= \left(\frac{2y_1}{y_1^2+y_2^2+1}, \frac{2y_2}{y_1^2+y_2^2+1}, \frac{1-y_1^2-y_2^2}{y_1^2+y_2^2+1} \right). \end{aligned}$$

Their images overlap in all of $S^2 \backslash \{S, N\}$ *and the* coordinate change

$$\varphi_S^{-1} \circ \varphi_N : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2 \backslash \{0\}$$

is a diffeomorphism given by

$$\varphi_S^{-1} \circ \varphi_N(x_1,x_2) = \left(\frac{x_1}{x_1^2+x_2^2}, \frac{x_2}{x_1^2+x_2^2} \right).$$

This diffeomorphism is orientation reversing since we have

$$\begin{aligned} \det D(\varphi_S^{-1} \circ \varphi_N)(x_1,x_2) &= \det \begin{pmatrix} \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2} & \frac{-2x_1x_2}{(x_1^2+x_2^2)^2} \\ \frac{-2x_1x_2}{(x_1^2+x_2^2)^2} & \frac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2} \end{pmatrix} \\ &= \frac{(x_2^2-x_1^2)(x_1^2-x_2^2)-4x_1^2x_2^2}{(x_1^2+x_2^2)^4} = -\frac{1}{(x_1^2+x_2^2)^2} < 0. \end{aligned}$$

The task in the next exercise is to derive the above explicit formulas for φ_S, φ_N and $\varphi_S^{-1} \circ \varphi_N$.

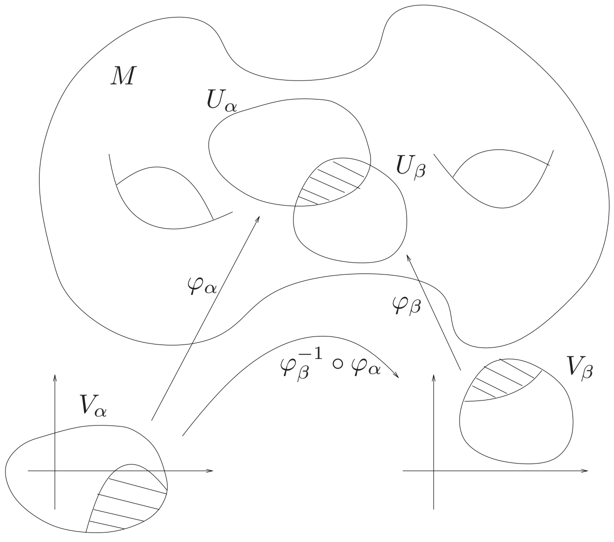

Figure 4: Local parametrisations and coordinate change of a manifold M

Exercise 15. *Let*

$$S^2 = \{p \in \mathbb{R}^3 \mid \|p\| = 1\} \subset \mathbb{R}^3.$$

Stereographic projection from the north pole $N = (0,0,1) \in S^2$ *is a map* $\varphi_N : \mathbb{R}^2 \to S^2 \backslash \{N\}$ *defined as follows: Let* $(x_1, x_2) \in \mathbb{R}^2$*. Consider the straight Euclidean line through the points* N *and* $(x_1, x_2, 0) \in \mathbb{R}^3$*. This line intersects* S^2 *in a unique point p besides the north pole. We define* $\varphi_N(x_1, x_2) := p \in S^2$*. Calculate* φ_N *explicitly. Similarly, let* $\varphi_S : \mathbb{R}^2 \to S^2 \backslash \{S\}$ *be stereographic projection from the south pole* $S = (0,0,-1)$*. Calculate* φ_S *explicitly as well as the coordinate change* $\varphi_S^{-1} \circ \varphi_N : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2 \backslash \{0\}$*. Why is this composition not defined at the point* 0*?*

Definition 11.1. *A k-dimensional manifold* $M \subset \mathbb{R}^n$ *is called* orientable*, if there exist a family* $\{\varphi_\alpha : V_\alpha \to U_\alpha \subset M\}_{\alpha \in A}$ *of local parametrisations covering* M *(i.e.* $\bigcup_{\alpha \in A} U_\alpha = M$*), such that the coordinate changes*

$$\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(U_\alpha \cap U_\beta) \to \varphi_\alpha^{-1}(U_\alpha \cap U_\beta)$$

are orientation preserving diffeomorphisms for all $\alpha, \beta \in A$*. Such a choice of local parametrisations* $\{\varphi_\alpha\}_{\alpha \in A}$ *is called an* orientation *of* M *and the local parametrisations* φ_α *are called* oriented local parametrisations*. A manifold together with an orientation is called an* oriented manifold.

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Example. *The local parametrisations in the previous example cover* S^2*, but they don't define an orientation of* S^2*. However, this doesn't mean that* S^2 *is not orientable. An orientation of* S^2 *is given, e.g. by* $\psi_N = \varphi_N$ *and* $\psi_S(y_1, y_2) = \varphi_S(y_2, y_1)$*. In this case we have* $D(\psi_S^{-1} \circ \psi_N) > 0$ *and also* $D(\psi_N^{-1} \circ \psi_S) > 0$*.*

Next, we explain how an orientation of a manifold M induces an orientation in all its tangent spaces T_pM, $p \in M$.

Definition 11.2. *Let W be a k-dimensional $\mathbb{R}$-vector space. Any choice of (ordered) basis $v_1, \dots, v_k$ of W defines an* orientation *on W in the following sense: For any other basis $w_1, \dots, w_k$ of W, we have a transformation matrix $A = (a_{ij})$ given by*

$$w_j = \sum_i a_{ij} v_i,$$

and we call $w_1, \dots, w_k$ an oriented basis *if* $\det A > 0$.

Remark 14. *Note that there are precisely two orientations which an $\mathbb{R}$-vector space can have. A canonical orientation on $\mathbb{R}^n$ is given by the standard basis $e_1, e_2, \dots, e_n$.*

Let $M \subset \mathbb{R}^n$ be an oriented k-dimensional manifold with an atlas of oriented local parametrisations. Let $p \in M$ be in the image of the oriented local parametrisation $\varphi : V \to U \subset M$ of M, i.e., $p = \varphi(x)$. Recall that the (k-dimensional) tangent space T_pM is given as the image $D\varphi(x)(\mathbb{R}^k)$ (see Proposition 5.2. in Dirk Schütz's notes) and that a basis of T_pM is given by the vectors

$$\frac{\partial \varphi}{\partial x_1}(x), \dots, \frac{\partial \varphi}{\partial x_k}(x). \tag{7}$$

This particular basis induces an orientation on T_pM. Note that the basis

$$\frac{\partial \psi}{\partial y_1}(y), \dots, \frac{\partial \psi}{\partial y_k}(y)$$

of any other oriented local parametrisation $\psi : V' \to U' \subset M$ with $p = \psi(y)$ induces the same orientation since the transformation matrix is given by $D(\varphi^{-1} \circ \psi)$, which has positive determinant.

In conclusion, we can speak of an *oriented basis* $v_1, \dots, v_k \in T_pM$ in an oriented manifold M. The bases generated by oriented local parametrisations (see (7) above) are oriented bases by definition.

Definition 11.3. *A manifold $M \subset \mathbb{R}^n$ of dimension $n-1$ is called a* hypersurface. *A hypersurface M has two unit normal vectors at every point $p \in M$, i.e., there is $v \in \mathbb{R}^n$ with $\|v\| = 1$ and*

$$-v, v \perp T_pM.$$

A unit normal vector $N(p)$ at the point p of an oriented hypersurface M is said to be positively oriented *if for any oriented basis $v_1, v_2, \dots, v_{k-1} \in T_pM$, the following basis of $\mathbb{R}^n$*

$$N(p), v_1, v_2, \dots, v_{n-1}$$

has the same orientation as the standard basis $e_1, e_2, \dots, e_n$.

Note that in a hypersurface $M \subset \mathbb{R}^n$ is orientable if and only if there is a well-defined smooth global unit normal vector field $N : M \to \mathbb{R}^n$. For a parametrised (hyper)surface $M \subset \mathbb{R}^3$ we can define

$$N(p) = \frac{D\varphi(x)(e_1) \times D\varphi(x)(e_2)}{\|D\varphi(x)(e_1) \times D\varphi(x)(e_2)\|},$$

with $\varphi : V \to U \subset M$ and $\varphi(x) = p$.

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Preimages of regular values of a map $f : \mathbb{R}^n \to \mathbb{R}^k$ are manifolds of dimension $n-k$ (see Theorem 5 in Dirk Schütz's notes). Note that the existence of charts and local parametrisations is guaranteed in this case by the Implicit Function Theorem. If $k=1$, the preimage $M = f^{-1}(c)$ of a regular value $c \in \mathbb{R}$ is a hypersurface in $\mathbb{R}^n$, and a global unit normal vector field $N : M \to \mathbb{R}^n$ is given by

$$N(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|},$$

since we have (see Theorem 6 in Dirk Schütz's notes)

$$T_pM = \ker Df(p)$$

and

$$Df(p)(v) = \langle \nabla f(p), v \rangle.$$

Are at all manifolds which cannot be oriented? The answer is YES, and the most prominent example is the Möbius strip.

Example. *Topologically, the Möbius can be obtained by taking a paper strip of the form of a long rectangle, twist it once, and glue its ends together. You obtain then a 2-dimensional manifold, which looks as illustrated in Figure 5. An explicit description of this set M reads as follows:*

$$\begin{aligned} x_1 &= \cos t(1 + s\cos t/2), \\ x_2 &= \sin t(1 + s\cos t/2), \\ x_3 &= s\sin t/2, \end{aligned}$$

where $0 \le t \le 2\pi$ and $-1/2 < s < 1/2$. A short explanation why this manifold is not orientable goes as follows: If M were orientable, then it would allow a smooth global unit normal vector field $N : M \to \mathbb{R}^3$. But if you choose a normal vector at some point of the manifold, let's say a a point of the circle determined by $s = 0$, and if you extend it continuously around this circle then, after one turn along the circle, you would end up with a normal vector directing into the opposite normal direction. This would contradict to continuity of the unit normal vector field.

Now, we introduce differential forms on manifolds:

Definition 11.4. *Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold. A* (differential) *l-*form ω on M *assigns to every point $p \in M$ an alternating l-form $\omega_p \in \Lambda^l(T_pM)$ and, for every choice of smooth vector fields $v_1, \dots, v_l : M \to \mathbb{R}^n$, the function $f : M \to \mathbb{R}$, $f(p) = \omega_p(v_1(p), \dots, v_l(p))$ is a smooth function.*

We denote the $\mathbb{R}$-vector space of all l-forms on M by $\Omega^l(M)$.

Remark 15. *(a) Let $\{\varphi_\alpha : V_\alpha \to U_\alpha\}_{\alpha \in A}$ be an atlas of local parametrisations of M, i.e., $M = \cup_{\alpha \in A} U_\alpha$. Note that the smoothness condition of an ω with $\omega_p \in \Lambda^l(T_pM)$ in the above definition is satisfied, if and only if the functions*

$$f^\alpha_{i_1,\dots,i_l} : V_\alpha \to \mathbb{R}, \quad f^\alpha_{i_1,\dots,i_l}(x) = \omega_{\varphi_\alpha(x)}(\frac{\partial \varphi_\alpha}{\partial x_{i_1}}(x), \dots, \frac{\partial \varphi_\alpha}{\partial x_{i_l}}(x))$$

are smooth for all $\alpha \in A$ and all $1 \le i_1 \le \dots \le i_l \le k$.

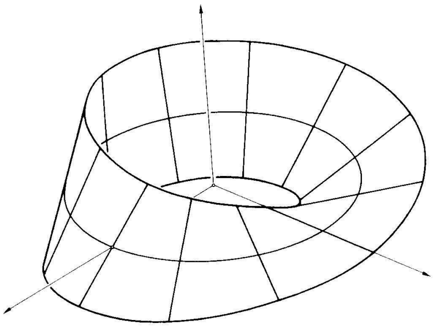

Figure 5: The Möbius strip

(b) Let $\omega \in \Omega^l(M)$*. The* pullback $\varphi_\alpha^*\omega$ *is defined as previously by*

$$(\varphi_\alpha^*\omega)_x(v_1, \dots, v_l) = \omega_{\varphi_\alpha(x)}(D\varphi_\alpha(v_1), \dots, D\varphi_\alpha(v_l)),$$

for all $x \in V_\alpha$ *and* $v_1, \dots, v_l \in \mathbb{R}^k$*. Note that every pullback* $\omega_\alpha = \varphi_\alpha^*\omega$ *lies in* $\Omega^l(V_\alpha)$*. Since all pullbacks are obtained from a global* l*-form on* M*, the (local) family of* l*-forms* $\{\omega_\alpha\}_{\alpha \in A}$ *satisfy the compatibility condition*

$$\omega_\beta = (\varphi_\alpha^{-1} \circ \varphi_\beta)^*\omega_\alpha, \tag{8}$$

for all $\alpha.\beta \in A$*. Conversely, every family* $\omega_\alpha \in \Omega(V_\alpha)_{\alpha \in A}$ *satisfying the above compatibility condition, uniquely determine a global* l*-form* ω *on the manifold* M*.*

Definition 11.5. *Let* $M \subset \mathbb{R}^n$ *be a* k*-dimensional manifold with an atlas* $\{\varphi_\alpha : V_\alpha \to U_\alpha, \alpha \in A\}$*. The exterior differential* $d : \Omega^l(M) \to \Omega^{l+1}(M)$ *is defined as follows: Let* $\omega \in \Omega^l(M)$ *and* $\{\omega_\alpha\}_{\alpha \in A}$ *be the corresponding family of pullbacks (under the local parametrisations* φ_α*). Then* $d\alpha \in \Omega^{l+1}(M)$ *is the (global)* $(l+1)$*-form on* M*, uniquely determined by the family* $\{d\omega_\alpha\}_{\alpha \in A}$*, where* $d\omega_\alpha \in \Omega^{l+1}(V_\alpha)$ *for all* $\alpha \in A$*.*

Remark 16. $d\omega$ *is well-defined, since the family* $\{d\omega_\alpha\}_{\alpha \in A}$ *satisfies the compatibility condition* (8)*:*

$$d\omega_\beta = d(\varphi_\alpha^{-1} \circ \varphi_\beta)^*\omega_\alpha = (\varphi_\alpha^{-1} \circ \varphi_\beta)^* d\omega_\alpha,$$

by Proposition 9.12.

Now we will discuss integration of a differential form $\omega \in \Omega^k(M)$ an a k-dimensional manifold $M \subset \mathbb{R}^n$. First we assume that $\omega_p = 0$ for all p outside a local parametrisation $\varphi_\alpha : V_\alpha \to U_\alpha$, i.e., $\omega_p = 0$ for all $p \notin U_\alpha$.

Definition 11.6. *Let* $\omega \in \Omega^k(M)$*, where* $M \subset \mathbb{R}^n$ *is an oriented* k*-dimensional manifold and* $\varphi : V \to U$ *is an oriented local parametisation such that* $\omega_p = 0$ *for all* $p \notin U$*. We define*

$$\int_M \omega := \int_V \varphi^*\omega,$$

where the right side is integration of a k-form over an open set V in $\mathbb{R}^k$ (see Definition 10.7 for this).

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Remark 17. *The above definition does not depend on the local parametrisation. Let $\varphi_\alpha : V_\alpha \to U_\alpha$, $\varphi_\beta : V_\beta \to U_\beta$ be oriented local parametrisations and $\omega_p \neq 0$ for all $p \notin U_\alpha \cap U_\beta$. Then $\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \varphi_\beta^{-1}(U_\alpha \cap U_\beta)$ is an orientation preserving diffeomorphism. We have*

$$\int_{V_\beta} \varphi_\beta^* \omega = \int_{\varphi_\beta^{-1}(U_\alpha \cap U_\beta)} \varphi_\beta^* \omega = \int_{\varphi_\alpha^{-1}(U_\alpha \cap U_\beta)} (\varphi_\beta^{-1} \circ \varphi_\alpha)^* \varphi_\beta^* \omega =$$

$$= \int_{\varphi_\alpha^{-1}(U_\alpha \cap U_\beta)} \varphi_\alpha^* (\varphi_\beta^{-1})^* \varphi_\beta^* \omega = \int_{\varphi_\alpha^{-1}(U_\alpha \cap U_\beta)} \varphi_\alpha^* \omega = \int_{V_\alpha} \varphi_\alpha^* \omega.$$

For the integration of general differential forms $\omega \in \Omega^k(M)$ on an oriented k-dimensional manifold M, we need the following tool, called partition of unity:

Theorem 11.7. *Let M be a k-dimensional manifold in $\mathbb{R}^n$ and $\{\varphi_\alpha : V_\alpha \to U_\alpha\}_{\alpha \in A}$ be a locally finite atlas on M. (Local finiteness means that for every $\alpha \in A$ there are only finitely many $\beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$.) Then there exists a family $\{f_\alpha\}_{\alpha \in A}$ of smooth functions $f_\alpha : M \to [0,1]$ with the following properties:*

- *(a) $\overline{\{p \in M \mid f_\alpha(p) \neq 0\}} \subset U_\alpha$ for all $\alpha \in A$. ($\overline{X} \subset M$ denotes the closure of X in M, i.e., all points $p \in M$ which are limits of sequences of points in M.)*
- *(b) $\sum_\alpha f_\alpha = 1$. (At each point p, there are only finite many non-zero terms in this sum because the atlas was chosen to be locally finite.)*

This family $\{f_\alpha\}$ is called a partition of unity *subordinated to the atlas $\{\varphi_\alpha\}$.*

Definition 11.8. *Let $M \subset \mathbb{R}^n$ be an oriented k-dimensional manifold, $\{\varphi_\alpha : V_\alpha \to U_\alpha\}_{\alpha \in A}$ an oriented atlas and $\{f_\alpha\}$ a subordinated partition of unity. Let $\omega \in \Omega^k(M)$. We define*

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega.$$

(Note that $\int_M f_\alpha \omega$ satisfies the requirements of the (local) Definition 11.6 for the integration of a form on a manifold and that $\omega = \sum_\alpha f_\alpha \omega$.)

Remark 18. *The definition is independent of the choice of the partition of unity. If $\{g_\alpha\}_{\alpha \in A}$ is another partition of unity subordinated to $\{\varphi_\alpha\}$, we have*

$$\int_M \omega := \sum_\alpha \int_M f_\alpha \omega = \sum_\alpha \int_M f_\alpha \left(\sum_{\beta \in A} g_\beta \right) \omega =$$

$$= \sum_{\alpha,\beta} \int_M f_\alpha g_\beta \omega = \sum_\beta \int_M g_\beta \left(\sum_\alpha f_\alpha \right) \omega = \sum_\beta \int_M g_\beta \omega.$$

Similarly, it can be shown that the integral is also independent of the choice of locally finite atlas of the manifold M, as long as both atlases have coinciding orientations.

Example. *Let*

$$M := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1, -1 < z < 1\}.$$

We choose the following oriented atlas:

$$\begin{aligned}
\varphi_1 &: V_1 := (-\pi,\pi) \times (-1,1) \to U_1 \subset M, \quad \varphi_1(\alpha,z) := (\cos\alpha, \sin\alpha, z),\\
\varphi_2 &: V_2 := (0,2\pi) \times (-1,1) \to U_2 \subset M, \quad \varphi_2(\alpha,z) := (\cos\alpha, \sin\alpha, z).
\end{aligned}$$

Note that φ_1, φ_2 *are* almost global parametrisations *of* M*, since they only miss out a vertical line of* M*, which is of measure zero in* M *and can be disregarded in the integration. (A subset* X *of a* k*-dimensional manifold* M *is* of measure zero*, if for a countable atlas* $\{\varphi_\alpha\}$ *of* M *all preimages* $\varphi^{-1}(X \cap U_\alpha) \subset \mathbb{R}^k$ *are sets of measure zero.)*

Let $i : M \to \mathbb{R}^3 \backslash \{z - axis\}$ *be the inclusion map, i.e.,* $i(p) = p$ *and*

$$\omega = -\frac{y}{x^2+y^2}\, dx \wedge dz + \frac{x}{x^2+y^2}\, dy \wedge dz \in \Omega^2(\mathbb{R}^3 \backslash \{z - axis\}).$$

Then $i^*\omega \in \Omega(M)$ *and we want to calculate* $\int_M i^*\omega$*. We have with Definition 11.6:*

$$\int_M i^*\omega = \int_{U_1} i^*\omega = \int_{V_1} \varphi_1^* i^*\omega = \int_{V_1} (i \circ \varphi_1)^*\omega.$$

Recall that $i \circ \varphi_1 : V_1 \to \mathbb{R}^3 \backslash \{z - axis\}$*,* $i \circ \varphi_1(\alpha, z) = (\cos\alpha, \sin\alpha, z)$*. Using the rules in Proposition 9.11, we obtain*

$$\begin{aligned}
(i\circ\varphi_1)^*\omega = (i\circ\varphi_1)^*\left(-\frac{y}{x^2+y^2}\, dx \wedge dz\right) + (i\circ\varphi_1)^*\left(\frac{x}{x^2+y^2}\, dy \wedge dz\right) =\\
= -\sin\alpha(((i\circ\varphi_1)^* dx) \wedge (i\circ\varphi_1)^* dz) + \cos\alpha((i\circ\varphi_1)^* dy) \wedge (i\circ\varphi_1)^* dz) =\\
= -\sin\alpha((-\sin\alpha\, d\alpha) \wedge dz) + \cos\alpha(\cos\alpha\, d\alpha) \wedge dz) = d\alpha \wedge dz.
\end{aligned}$$

This implies (with Fubini's Theorem) that

$$\int_M i^*\omega = \int_{V_1} d\alpha \wedge dz = \int_{-\pi}^{\pi}\int_{-1}^{1} d\alpha \wedge dz(e_1, e_2) = 4\pi.$$

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12 Manifolds with boundary and Stokes' Theorem

Next, we expand the definition of a manifold to allow the manifold to have a boundary. An example of a manifold with boundary is the upper hemisphere $M = S^2 \cap (\mathbb{R}^2 \times [0,\infty)) \subset \mathbb{R}^3$. The boundary of M is the equator in the horizontal coordinate plane. Another example is the finite cylinder

$$M := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1, z \in [a,b]\}.$$

In this case the boundary consists of two components, namely the two circles

$$\partial M := \{(x,y,a) \mid x^2+y^2=1\} \cup \{(x,y,b) \mid x^2+y^2=1\}.$$

We will also see that the boundary ∂M itself is again a manifold of dimension $\dim M - 1$ and without boundary. Moreover, if M carries an orientation, then ∂M carries a canonical induced orientation.

Exercise 16. *In this exercise, we like to construct a partition of unity for the closed interval* $M = [-10,10]$ *(one dimensional manifold with boundary* $\partial M = \{-10,10\}$*). Therefore, we introduce the functions*

$$g(x) = \begin{cases} 0, & x \le 0, \\ e^{-\frac{1}{x^2}}, & x > 0, \end{cases}, \qquad h(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x < 0, \\ 0, & x \ge 0. \end{cases}$$

(a) Let $a < b$*. Show that* $f_{a,b}(x) = g(x-a)h(x-b)$ *is a smooth function,* $f(x) > 0$ *for* $x \in (a,b)$ *and* $f(x) = 0$ *for* $x \le a$ *and* $x \ge b$*.*

(b) Let $F_1, F_2, F_3 : M \to [0,\infty)$ *be defined as*

$$F_1(x) = f_{-10,-3}(x), \quad F_2(x) = f_{-4,4}(x), \quad F_3(x) = f_{3,10}(x),$$

and $f_i : M \to [0,1]$, $f_i(x) = \frac{F_i(x)}{F_1(x)+F_2(x)+F_3(x)}$*. Show that* f_1, f_2, f_3 *is a partition of unity, subordinated to the covering* $U_1 = [-10,-2)$, $U_2 = (-5,5)$ *and* $U_3 = (2,10]$ *of* M*.*

To take care of the boundary, we introduce the half space

$$H^k := \{(x_1,\dots,x_k) \in \mathbb{R}^k \mid x_1 \le 0\}.$$

The set

$$\partial H^k := \{(0,x_2,\dots,x_k) \in \mathbb{R}^k\}$$

is called the *boundary of* H^k. A manifold with boundary is defined as follows (compare with Definition 4.5 in Dirk Schütz's notes):

Definition 12.1. *A subset* $M \subset \mathbb{R}^n$ *is called a* k-dimensional manifold with boundary*, if the following holds: every point* $p \in M$ *is contained in an open set* $U \subset \mathbb{R}^n$ *on which there is a diffeomorphism* $h : U \to U' \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ *with*

$$h(U \cap M) = U' \cap (H^k \times \{0\}).$$

Such an h is called a chart *of M. A family $\{h_\alpha : U_\alpha \to U'_\alpha\}_{\alpha \in A}$ of charts is called an* atlas, *if $M \subset \bigcup_\alpha U_\alpha$. The set of points $p \in M$ such that there is a chart $h : U \to U'$ with $h(p) \in \partial H^k \times \{0\}$ are called the* boundary points *of M. The boundary points of M don't depend on the choice of chart and the set of all boundary points is denoted by ∂M and called the* boundary of *M.*

Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold with boundary. As described at the beginning of Chapter 11 for manifolds without boundary, every chart $h : U(\subset \mathbb{R}^n) \to U'(\subset \mathbb{R}^k \times \mathbb{R}^{n-k})$, restricted to $U \cap M$, can be interpreted as a bijective map $h_0 : U_0 := U \cap M \to V_0 \subset H^k \subset \mathbb{R}^k$. The inverse map $\varphi : V_0 \to U_0 \subset M$ is called a local parametrisation or coordinate system of M. Some of the local parametrisations $\varphi : V_0(\subset H^k) \to U_0$ cover some of the boundary ∂M (this is the case if $V_0 \cap \partial H^k \neq \emptyset$), some other local parametrisations cover only interior patches of M. Recall also that a family $\{\varphi_\alpha : V_\alpha \to U_\alpha\}$ of local parametrisations is an atlas of M if $\bigcup_\alpha U_\alpha = M$.

Proposition 12.2. *Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold with boundary. Then $\partial M \subset \mathbb{R}^n$ is a $(k-1)$-dimensional manifold (without boundary).*

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Instead of a proof we describe how to obtain an atlas $\{\psi_\alpha\}$ of ∂M from an atlas $\{\varphi_\alpha : V_\alpha \to U_\alpha\}$ of M. Let

$$B := \{\alpha \in A \mid U_\alpha \cap \partial M \neq \emptyset\}.$$

For $\alpha \in B$, note that

$$V_\alpha \cap \partial H_k = \{(0, x_2, \ldots, x_k) \in V_\alpha\}.$$

We introduce the open subsets $W_\alpha \subset \mathbb{R}^{k-1}$ as

$$W_\alpha := \{(x_2, \ldots, x_k) \mid (0, x_2, \ldots, x_k) \in V_\alpha$$

and the maps

$$\psi_\alpha : W_\alpha \to \partial M, \quad \psi_\alpha(x_2, \ldots, x_k) = \varphi_\alpha(0, x_2, \ldots, x_k).$$

(See Figure 6 for illsutration.) Then $\{\psi_\alpha : W_\alpha \to Z_\alpha := U_\alpha \cap \partial M\}_{\alpha \in B}$ is an atlas of ∂M.

Next, we show that an orientation on M induces a canonical orientation on ∂M.

Proposition 12.3. *Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold with boundary and $\{\varphi_\alpha : V_\alpha \to U_\alpha\}_{\alpha \in A}$ an oriented atlas. Let*

$$B := \{\alpha \in A \mid U_\alpha \cap \partial M \neq \emptyset\}.$$

Then the induced atlas $\{\psi_\alpha : W_\alpha \to Z_\alpha\}_{\alpha \in B}$ satisfies

$$\det D(\psi_\beta^{-1} \circ \psi_\alpha)(x) > 0,$$

for all $\alpha, \beta \in B$ and $x \in \psi_\alpha^{-1}(Z_\alpha \cap Z_\beta)$. Hence, $\{\psi_\alpha\}$ defines an orientation on ∂M, the induced orientation *on ∂M.*

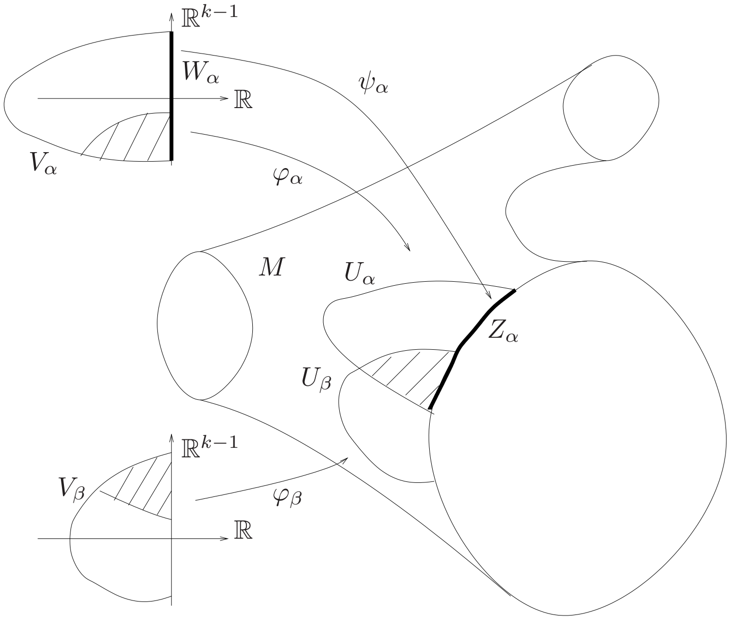

Figure 6: Local parametrisations ψ_α of ∂M derived from local parametrisations φ_α of M

Proof. Let $\alpha, \beta \in B$. Assume that $\varphi_\alpha(x) = p \in \partial M$. Then

$$D(\varphi_\beta^{-1} \circ \varphi_\alpha)(x) : \mathbb{R}^k \to \mathbb{R}^k$$

is a vector space isomorphism with $\partial H^k = \{0\} \times \mathbb{R}^{k-1}$ as invariant subspace, i.e.,

$$D(\varphi_\beta^{-1} \circ \varphi_\alpha)(x)(\partial H^k) = \partial H^k.$$

Let

$$D(\varphi_\beta^{-1} \circ \varphi_\alpha)(x)(e_1) = \lambda_1 e_1 + \sum_{j=2}^{k} \lambda_j e_j.$$

Note that $\lambda_1 \neq 0$ (since $D\varphi_\beta^{-1} \circ \varphi_\alpha(x)$ is an isomorphism). We like to show that $\lambda_1 > 0$. Consider the curve

$$c(t) = (c_1(t), \ldots, c_k(t)) = \varphi_\beta^{-1} \circ \varphi_\alpha(x + te_1).$$

Since, for $t < 0$ we have $c_1(t) < 0$, we have

$$\lambda_1 = c_1'(0) \geq 0.$$

Hence, we have with $x = (0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{k-1}$:

$$D(\varphi_\beta^{-1} \circ \varphi_\alpha)(x) = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ \lambda_2 & & & \\ \vdots & & D(\psi_\beta^{-1} \circ \psi_\alpha)(\bar{x}) & \\ \lambda_k & & & \end{pmatrix},$$

and thus

$$\det D(\psi_\beta^{-1} \circ \psi_\alpha)(\bar{x}) = \frac{1}{\lambda_1} \det D(\varphi_\beta^{-1} \circ \varphi_\alpha)(x) > 0.$$

□

Remark 19. *Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold with boundary and $\partial M \subset \mathbb{R}^n$ its $(k-1)$-dimensional boundary. Let $\{\varphi_\alpha\}_{\alpha \in A}$ be an oriented atlas of M and $\{\psi_\alpha\}_{\alpha \in B}$ be the induced atlas. As described earlier (see (7)), the orientations given by these atlases induce orientations on all tangent spaces T_pM for $T_p\partial M$. Now, if $p \in \partial M \subset M$, $T_p\partial M$ is a $(k-1)$-dimensional subspace of the k-dimensional vector space T_pM. Let $\varphi_\alpha : V_\alpha \to U_\alpha$ be a local parametrisation with $\varphi_\alpha(x) = p$ and $v_1 := D\varphi_\alpha(x)(e_1)$. Then $v_2, \ldots, v_k \in T_p\partial M$ is an oriented basis of $T_p\partial M$ if and only if $v_1, \ldots, v_k \in T_pM$ is an oriented basis of T_pM. The pictures in Figure 7 illustrate the situation.*

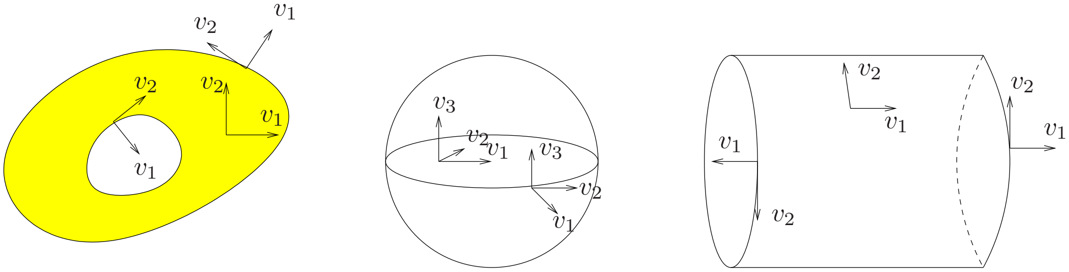

Figure 7: Orientations of M and ∂M

Now, we present the main result of these notes:

Theorem 12.4 (Stokes' Theorem for differential forms)**.** *Let $M \subset \mathbb{R}^n$ be an oriented k-dimensional manifold with boundary and $i : \partial M \to M$ be the inclusion map, i.e., $i(p) = p$. Let $\omega \in \Omega^{k-1}(M)$ be a differential form. Then we have*

$$\int_M d\omega = \int_{\partial M} i^*\omega,$$

where ∂M carries the induced orientation.

Remark 20. *To simplify notation, we often omit the inclusion map and simply write*

$$\int_M d\omega = \int_{\partial M} \omega.$$

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Exercise 17. *For $a, b, c > 0$ consider the ellipsoid*

Homework

$$E := \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}.$$

Let ω be the following differential form on $\mathbb{R}^3$;

$$\omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy.$$

(a) Calculate $d\omega$.

(b) *Find an almost global parametrisation of E such that the outward unit normal vector field is positively oriented. Calculate*

$$\int_E \omega.$$

Hint: *Think of polar coordinates on the sphere.*

(c) *Conclude from (a) and (b) that the volume of*

$$K := \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$$

is

$$\mathrm{vol}\,(K) = \int_K 1\,dx\,dy\,dz = \frac{4}{3}\pi abc.$$

Proof of Stokes' Theorem. The proof is based on the Fundamental Theorem of Calculus and consists of three steps:

Step 1: We first consider the case $M = H^k \subset \mathbb{R}^k$ with standard orientation (i.e., $e_1, \ldots, e_k$ is an oriented basis) and

$$\omega = f_i\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k.$$

(Here $\widehat{dx_i}$ means that this term is omitted!) To have a finite integral, assume that the function vanishes for large coordinates, i.e., there is a $K > 0$ such that $f_i(x_1, \ldots, x_k) = 0$ if one of the x_i satisfies $|x_i| > K$. Then

$$\begin{aligned} d\omega &= \frac{\partial f_i}{\partial x_i}\,dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\ &= (-1)^{i-1}\frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k. \end{aligned}$$

Note that ∂M is equal to ∂H^k with the induced orientation (i.e., $e_2, \ldots, e_k$ is an oriented basis). Consider the oriented local parametrisation $\varphi : \mathbb{R}^{k-1} \to \partial H^k$,

$$\varphi(y_1, \ldots, y_{k-1}) = (0, y_1, \ldots, y_{k-1}).$$

The components of φ are denoted by $\varphi_1, \ldots, \varphi_k$. Note that $\varphi_1(y) = 0$, hence $d\varphi_1 = 0$ and $\varphi_j(y_1, \ldots, y_{k-1}) = y_{j-1}$ and

$$\varphi^*\omega = (f_i \circ \varphi) d\varphi_1 \wedge \cdots \wedge \widehat{d\varphi_i} \wedge \cdots \wedge d\varphi_k. \tag{9}$$

Now, we consider two cases:

(a) Assume $i = 1$. Then

$$d\omega = \frac{\partial f_1}{\partial x_1}\,dx_1 \wedge \cdots \wedge dx_k,$$

and

$$\begin{aligned} \varphi^*\omega &= (f_1 \circ \varphi)\,d\varphi_2 \wedge \cdots \wedge d\varphi_k \\ &= (f_1 \circ \varphi)\,dy_1 \wedge \cdots \wedge dy_{k-1}. \end{aligned}$$

Now, on one side

$$\begin{aligned}\int_{H^k} d\omega &= \int_{H^k} \frac{\partial f_1}{\partial x_1} dx_1 \wedge \cdots \wedge dx_k \\ &= \int_{\mathbb{R}^{k-1}} \int_{(-\infty,0]} \frac{\partial f_1}{\partial x_1}(x_1, \ldots, x_k)\, dx_1 \ldots dx_k \\ &= \int_{\mathbb{R}^{k-1}} f_1(0, x_2, \ldots, x_k)\, dx_2 \ldots dx_k,\end{aligned}$$

using the Fundamental Theorem of Calculus and the fact that $f_1(x_1, \ldots, x_k) = 0$ for $x_1 < -K$. On the other side, we have

$$\begin{aligned}\int_{\partial H^k} \omega &= \int_{\mathbb{R}^{k-1}} \varphi^* \omega \\ &= \int_{\mathbb{R}^{k-1}} (f_1 \circ \varphi)\, dy_1 \wedge \cdots \wedge dy_{k-1} \\ &= \int_{\mathbb{R}^{k-1}} f_1(0, y_1, \ldots, y_{k-1})\, dy_1 \ldots dy_{k-1}.\end{aligned}$$

This shows for this case that

$$\int_{H^k} d\omega = \int_{\partial H^k} \omega.$$

(b) Assume $i \geq 2$. Then

$$d\omega = (-1)^{i-1} \frac{\partial f_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_k,$$

and $\varphi^* \omega = 0$, since $d\varphi_1 = 0$ shows up in the pullback (9). Moreover, using the Fundamental Theorem of Calculus again,

$$\begin{aligned}\int_{H^k} d\omega &= (-1)^{i-1} \int_{H^k} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k \\ &= (-1)^{i-1} \int_{\mathbb{R}} \cdots \underbrace{\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i}(x_1, \ldots, x_k)\, dx_i}_{=0}\, dx_1 \ldots \widehat{dx_i} \ldots dx_k = 0,\end{aligned}$$

since $f_i(x_1, \ldots, x_k) = 0$ for $|x_i| > K$. So we have also in this case

$$\int_{H^k} d\omega = 0 = \int_{\mathbb{R}^{k-1}} \varphi^* \omega = \int_{\partial H^k} \omega.$$

Combining (a) and (b), and using linearity, we have for all $\omega \in \Omega^k(H^k)$ with compact support:

$$\int_{H^k} d\omega = \int_{\partial H^k} \omega.$$

Step 2: Now, let $M \subset \mathbb{R}^n$ be an oriented k-dimensional manifold with boundary and $\omega \in \Omega^{k-1}(M)$ supported on an oriented local parametrisation $\varphi : V \to U$, with $V \subset H^k$ and $U \subset M$. This case is reduced to the case in Step 1 by pullback with respect to this parametrisation. We also use that pullbacks and exterior differentials commute:

$$\int_M d\omega = \int_U d\omega = \int_V \varphi^* d\omega = \int_V d\varphi^* \omega =$$
$$= \int_{H^k} d\varphi^* \omega = \int_{\partial H^k} \varphi^* \omega = \int_{V \cap \partial H^k} \varphi^* \omega = \int_{\partial M} \omega.$$

Step 3: Finally, consider a general $\omega \in \Omega^{k-1}(M)$ (with compact support in order to guarantee a finite integral). Let $\{\varphi_\alpha : V_\alpha \to U_\alpha\}_{\alpha \in A}$ be a locally finite atlas and $\{f_\alpha\}_{\alpha \in A}$ be a subordinated partition of unity. Since ω has compact support, it is non-zero only on finitely many local parametrisations φ_α with associated finite index set $B \subset A$. Then, using linearity of the exterior differential and Step 2, we obtain

$$\int_M d\omega = \sum_{\alpha \in B} \int_M d(f_\alpha \omega) = \sum_{\alpha \in B} \int_{\partial M} f_\alpha \omega = \int_{\partial M} \omega,$$

finishing the proof of Stokes' Theorem. □

Corollary 12.5. *Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented manifold with $\partial M = \emptyset$. Then we have for every $\omega \in \Omega^k(M)$ with compact support*

$$\int_M \omega = 0.$$

Proof. Clear from Stokes' Theorem, since integration over an empty set is zero. □

14 March 2008

13 Applications

Differential Forms and Stokes' Theorem, even though they might look a bit scary at first sight, can be considered as the "grand unifying theme" behind several other courses in mathematics. In Algebraic Geometry, meromorphic differential forms are very important in the Theorem of Riemann-Roch (see Chapter 6 in the book *Complex Algebraic Curves* by F. Kirwan or Chapters 2 and 3 in the book *Riemann Surfaces* by H. M. Farkas and I. Kra); in Differential Geometry, the most elegant proof of the Theorem of Gauss-Bonnet is a reduction to Stokes' Theorem (see Chapter 9 in the book *Riemannian Manifolds* by J. M. Lee and Chapter 6 in the book *Differential Forms and Applications* by M. do Carmo); in Algebraic Topology, the DeRham Cohomology plays a prominent role and is defined as the quotient of the vector space of closed k-forms by the subspace of exact k-forms (see the book *Differential forms in algebraic topology* by R. Bott and L.W. Tu). Connections to Vector Analysis and the Classical Theorems of Integration were already mentioned at the beginning of these notes. We like to finish these notes by discussing two more applications of the Theory of Differential Forms and Stokes' Theorem in Complex Analysis and Topology.

13.1 Cauchy's Integral Theorem

Theorem 13.1. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Then we have*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

for two freely homotopic closed curves γ_1, γ_2 in U. Moreover, if U is star-like (or more generally, simply connected), then we have for every closed curve γ

$$\int_\gamma f(z)dz = 0.$$

Proof. Let $f(x+iy) = f_1(x,y) + if_2(x,y)$. Recall from Definition 8.13 that the 1-forms $\omega = f_2dx + f_1dy$ and $\eta = f_1dx - f_2dy$ are closed if we have

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \quad \text{and} \quad \frac{\partial f_2}{\partial x} = -\frac{\partial f_1}{\partial y}.$$

These are the Cauchy-Riemann Differential Equations, which are satisfied since f is holomorphic. Let $\gamma : [a,b] \to \mathbb{C}$ be a differentiable curve with $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$. Then

$$\omega_{\gamma(t)}(\gamma'(t)) = f_2(\gamma(t))\gamma_1'(t) + f_1(\gamma(t))\gamma_2'(t) = \mathrm{Im}(f(\gamma(t))\gamma'(t)),$$

and

$$\eta_{\gamma(t)}(\gamma'(t)) = f_1(\gamma(t))\gamma_1'(t) - f_2(\gamma(t))\gamma_2'(t) = \mathrm{Re}(f(\gamma(t))\gamma'(t)),$$

and, consequently,

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \eta_{\gamma(t)}(\gamma'(t))dt + i\int_a^b \omega_{\gamma(t)}(\gamma'(t))dt = \int_\gamma \eta + i\int_\gamma \omega.$$

Using Corollary 8.19, we have for freely homotopic curves γ_1, γ_2:

$$\int_{\gamma_1} f(z)dz \quad = \quad \int_{\gamma_1} \omega \; + \; i \int_{\gamma_1} \eta \quad = \quad \int_{\gamma_2} \omega \; + \; i \int_{\gamma_2} \eta \quad = \quad \int_{\gamma_2} f(z)dz.$$

If U is star-like, both ω and η are exact, by Poincaré's Lemma. Proposition 8.11 implies then for every closed curve γ in U:

$$\int_\gamma f(z)dz = \int_\gamma \omega + i \int_\gamma \eta = 0 + i \cdot 0 = 0.$$

□

13.2 Brouwer's Fixed Point Theorem

Theorem 13.2. *Let $B := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ be the unit ball. Then every continuous map $f : B \to B$ has a fixed point, i.e., there is an $x \in B$ with $g(x) = x$.*

A proof of this theorem (following Hirsch, without differential forms but based on the classification of compact one-manifolds) can be found in J.W. Milnor's book *Topology from the differentiable viewpoint.* In this book is also an argument how to reduce the continuous case to the smooth case.

We present here a proof (by E. Lima) in the case that f is smooth. Moreover, we use the fact that every compact k-dimensional orientable manifold $M \subset \mathbb{R}^n$ without boundary admits a k-form $\omega \in \Omega(M)$ which is everywhere non-zero. Such a form can be constructed with the help of an oriented atlas φ_α and a subordinated partition of unity f_α as

$$\omega = \sum_\alpha f_\alpha d(\varphi_{\alpha 1}) \wedge \dots d(\varphi_{\alpha k}).$$

(Note that, since we use an oriented atlas, $\varphi_\alpha^* d(\varphi_{\beta_1}) \wedge \dots d(\varphi_{\beta_k}) = g dx_1 \wedge \dots \wedge dx_k \in \Omega(\varphi_\alpha^{-1}(U_\alpha \cap U_\beta))$ with a non-negative function g.)

Proof. The ball B is a canonically oriented n-dimensional manifold with boundary S^{n-1} (the $(n-1)$-dimensional unit sphere). Let $i : S^{n-1} \to B$ be the inclusion map, i.e., $i(x) = x$ and let $\omega \in \Omega^{n-1}(S^{n-1})$ be a nowhere vanishing $(n-1)$-form. Then we obviously have

$$\int_{S^{n-1}} \omega \neq 0. \tag{10}$$

Assume that there is a smooth map $f : B \to B$ without fixed points. Then we obtain a smooth map $g : B \to S^{n-1}$, defined as follows: Let $g(x)$ be the intersection point of S^{n-1} with the Euclidean ray starting at $f(x)$ and passing through x. (Note that for this construction it is necessary that $f(x) \neq x$.) Note that the restriction of g to S^{n-1} is the identity, i.e., $g \circ i = \mathrm{id}_{S^{n-1}}$. Note also that $d\omega = 0$, since there is no non-zero n-form on the $(n-1)$-dimensional manifold S^{n-1}. Using Stokes' Theorem, we obtain

$$0 = \int_B g^* d\omega = \int_B d(g^*\omega) = \int_{S^{n-1}} i^* g^* \omega = \int_{S^{n-1}} (g \circ i)^* \omega = \int_{S^{n-1}} \omega,$$

contradicting (10). □