Analysis 1 Problems and Solutions in Revision Lectures (Easter Term 2015)

Logic, sets

Problem 1 Calculate $\bigcap_{x\in[0,1]}(x-1,x+1)$ and $\bigcup_{x\in[0,1]}(x-1,x+1)$.

Solution: Thinking of the sets $(x-1,x+1)$ as moving over the real line axis with x running through $[0,1]$ leads to the guess that the intersection should be $(0,1)$ and the union should be $(-1,2)$.

Now we proof our guesses in full detail: For simplicity, let $I_x = (x-1,x+1)$.

$\bigcap_{x\in[0,1]} I_x = (0,1)$: Let $z\in(0,1)$. We show that z lies in each of the sets I_x. If $x\in[0,1]$, we have

$$x-1 \le 1-1 = 0 < z < 1 = 0+1 \le x+1,$$

i.e., $z \in I_x = (x-1,x+1)$. This shows that

$$(0,1) \subset \bigcup_{x\in[0,1]} I_x.$$

Finally, for any $z\le 0$ we have $z\notin I_1 = (0,2)$ and for any $z\ge 1$ we have $z\notin I_0 = (-1,1)$. This finishes the first proof.

$\bigcup_{x\in[0,1]} I_x = (-1,1)$: We first have

$$(-1,2) = I_0 \cup I_{1/2} \cup I_1 = (-1,1)\cup(-1/2,1/2)\cup(0,2) \subset \bigcup_{x\in[0,1]} I_x,$$

so it only remains to show that no other real number is in the union. Let $z\le -1$. Then we have for all $x\in[0,1]$: $z\le 0-1\le x-1$, i.e., $z\notin I_x$. Let $z\ge 2$. Then we have for all $x\in[0,1]$: $z\ge 1+1\ge x+1$, i.e. $z\notin I_x$. This finishes the second proof.

Problem 2 Show: If $Y\cap X_i=\emptyset$ for all $i\in I$, then $Y\cap\left(\bigcup_{i\in I}X_i\right)=\emptyset$.

Solution: We give an indirect proof: We assume that $Y\cap\left(\bigcup_{i\in I}X_i\right)\ne\emptyset$. Then there exists $x\in Y\cap\left(\bigcup_{i\in I}X_i\right)$. This means that $x\in Y$ and $x\in\bigcup_{i\in I}X_i$. Since $x\in\bigcup_{i\in I}X_i$, there exists $i_0\in I$ such that $x\in X_{i_0}$. But then also $x\in Y\cap X_{i_0}$ (since x lies in both sets), contradicting to the assumption that $Y\cap X_i=\emptyset$ for all $i\in I$.

Problem 3 Formulate De Morgan's laws for infinite sets.

Solution: Let X_i for $i \in I$ be sets, all contained in a bigger set X. Then we have

$$X \backslash \left(\bigcup_{i \in I} X_i \right) = \bigcap_{i \in I} (X \backslash X_i)$$

and

$$X \backslash \left(\bigcap_{i \in I} X_i \right) = \bigcup_{i \in I} (X \backslash X_i).$$

Numbers, inequalities

Problem 4* We write $x > y$ iff $x - y > 0$. Deduce from

(i) $\forall x \in \mathbb{R}$: either $x > 0$ or $x = 0$ or $x < 0$,

(ii) $\forall x, y > 0$: $x + y > 0$ and $xy > 0$

the law of transitivity: If $y > x$ and $z > y$ then $z > x$.

Solution: $y > x$ means $a = y - x > 0$. $z > y$ means $b = z - y > 0$. Since $a, b > 0$, we conclude with (ii) that $a + b > 0$. Note that

$$a + b = (y - x) + (z - y) = z - x.$$

So we have $z - x > 0$, which means that $z > x$.

Problem 5 Calculate explicitly $\{z \in \mathbb{C} \mid |z + 1| = |z - 1|\}$.

Solution: Geometrically, $|z - w|$ is the distance between the two points z and w. So the condition $|z + 1| = |z - 1|$, which agrees with $|z - (-1)| = |z - 1|$, describes the set of all points in the Argand plane which have the same distance from the two points $-1 \in \mathbb{C}$ and $1 \in \mathbb{C}$, i.e., the set under consideration is the perpendicular bisector of these two points, i.e., the vertical imaginary axis $\{z = iy \mid y \in \mathbb{R}\}$.

Now we prove this fact algebraically: Let $z = x+iy \in \{z \in \mathbb{C} \mid |z+1| = |z-1|\}$: This means that

$$(x + 1)^2 + y^2 = (x - 1)^2 + y^2.$$

This simplifies to $2x = -2x$, i.e., $x = 0$. So we see that

$$\{z \in \mathbb{C} \mid |z + 1| = |z - 1|\} = \{z = x + iy \in \mathbb{C} \mid x = 0\} = \{iy \mid y \in \mathbb{R}\}.$$

Problem 6 Find all real solutions of $\left|\frac{x}{x-1}\right| \leq 2$.

Solution: If we exclude $x = 1$, the inequality is equivalent to $|x| \leq 2|x - 1|$. Since all expressions involved are non-negative (we are dealing with absolute values), we can square both sides, leading to the equivalent formulation (Note: For the equivalence we need the fact that all expressions are positive!!)

$$x^2 \leq 4(x^2 - 2x + 1).$$

So we end up with the quadratic inequality

$$0 \le 3x^2 - 8x + 4 = 3(x-2)(x-2/3).$$

Consequently, all real solutions x satisfy ($x - 2 \ge 0$ and $x - 2/3 \ge 0$) (which simplifies to $x \ge 2$) OR the satisfy ($x - 2 \le 0$ and $x - 2/3 \le 0$) (which simplifies to $x \le 2/3$). The excluded value $x = 1$ does not fall into the union $S = (-\infty, 2/3] \cup [2, \infty)$, so S is the set of all real solutions.

Problem 7 Formulate with quantifiers:

(a) (x_n) does not converge to x.

(b) (x_n) is divergent.

Solution: See Lecture.

Problem 8 (a) Let $a, b > 0$. Show: $\frac{a}{b} + \frac{b}{a} \ge 2$.

(b) Pove by Induction, using (a): Let $a_1, \ldots, a_n > 0$. Then

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \ge n^2.$$

Solution: Ad (a): Since $a, b > 0$, the inequality if equivalent to

$$a^2 + b^2 \ge 2ab,$$

which, in turn, is equivalent to $(a-b)^2 = a^2 - 2ab + b^2 \ge 0$, which is obviously true.

Ad (b):

Start of Induction ($n = 1$): We have $a_1 \cdot \frac{1}{a_1} = 1 \ge 1$, so the statement is true for $n = 1$.

Induction Step ($n \to n+1$): Assume the statement is true for some $n \in \mathbb{N}$. Let $a_1, \ldots, a_{n+1} > 0$. Then we have, using the induction hypothesis and (a),

$$\begin{aligned}
\left(\sum_{k=1}^{n+1} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) &= \left(a_{n+1} + \sum_{k=1}^{n} a_k\right)\left(\frac{1}{a_{n+1}} + \sum_{k=1}^{n} \frac{1}{a_k}\right) \\
\ge 1 + n^2 + a_{n+1}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) + \frac{1}{a_{n+1}}\left(\sum_{k=1}^{n} a_k\right) &= 1 + n^2 + \sum_{k=1}^{n}\left(\frac{a_{n+1}}{a_k} + \frac{a_k}{a_{n+1}}\right) \\
&\ge 1 + n^2 + \sum_{k=1}^{n} 2 = 1 + 2n + n^2 = (n+1)^2,
\end{aligned}$$

showing that then the statement is also true for $n + 1$.

Sequences/Completeness Axiom for $\mathbb{R}$

Problem 9 Show that if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$.

Solution: Assume that $x_n \to x$ and $y_n \to y$. We have to estimate $|x_n y_n - xy|$, and first carry out the following calculation:

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |x_n - x| \cdot |y|.$$

We see that we can control the terms $|y_n - y|$ and $|x_n - x|$ because of convergence, and we only need to control the factors $|x_n|$. But since (x_n) is convergent, (x_n) is bounded and we can find $C > 0$ such that $|x_n| \leq C$ for all $n \in \mathbb{N}$. So, introducing $\epsilon > 0$, the estimates $|x_n - x|, |y_n - y| < \epsilon$ would lead to

$$|x_n y_n - xy| \leq C\epsilon + \epsilon|y| = \epsilon(C + |y|).$$

If we want to have a clean expression ϵ at the end, we can assume, e.g., that we have the estimates $|x_n - x| < \epsilon/(2(|y|+1))$ and $|y_n - y| < \epsilon/(2C)$ and then conclude

$$|x_n y_n - xy| \leq \frac{\epsilon}{2}\left(\frac{|y|}{|y|+1} + \frac{C}{C}\right) \leq \epsilon.$$

Now we have all ingredients. The arguments go now as follows: Since $x_n \to x$, we can find $N \in \mathbb{N}$ with $|x_n - x| \leq \epsilon/(2(|y|+1))$ for all $n \geq N$. Since $y_n \to y$, we can find $M \in \mathbb{N}$ with $|y_n - y| \leq \epsilon/(2C)$ for all $n \geq M$. Then both estimates hold for all $n \geq \max(N, M)$, and therefore we obtain for all such indices n that

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |x_n - x| \cdot |y| \leq \frac{\epsilon}{2}\left(\frac{|y|}{|y|+1} + \frac{C}{C}\right) \leq \epsilon.$$

But this means that $x_n y_n \to xy$, finishing the proof.

Problem 10

(a) Check whether $a_n = \sqrt{n+1} - \sqrt{n}$ is convergent.

(b) Prove of disprove: If $a_{n+1} - a_n \to 0$ then (a_n) is convergent.

(c) Show that if $|a_{n+1} - a_n| \leq 1/2^n$ then (a_n) is convergent.

Solution: Ad (a): We have

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

This implies that

$$0 \leq a_n \leq \frac{1}{2\sqrt{n}},$$

and since $1/\sqrt{n} \to 0$, we also have $a_n \to 0$, by the Squeezing Theorem.

Ad (b): The statement is false. A counterexample is $a_n = \sum_{k=1}^{n} \frac{1}{k}$, the harmonic series. We know that a_n is divergent but we have $a_{n+1} - a_n = 1/(n+1) \to 0$.

Ad (c): See Lecture.

Problem 11 Prove that every Cauchy sequence in $\mathbb{R}$ is convergent, using Bolzano-Weierstrass.

Solution: Let (x_n) be a Cauchy sequence. We know that this implies that (x_n) is bounded. Now, using Bolzano-Weierstrass, we can conclude that there is a subsequence (x_{n_j}), which is convergent, i.e., there exists $x \in \mathbb{R}$ such that $x_{n_j} \to x$ as $j \to \infty$. The task now is to prove that x is not only the limit of the subsequence $(x_{n_j}$, but of the whole sequence (x_n), using the "closeness" of elements with large indices in a Cauchy sequence. Let $\epsilon > 0$ be given. Since $x_{n_j} \to x$, we can find $J \in \mathbb{N}$ such that $|x_{n_j} - x| < \epsilon$ for all $j \geq J$. On the other hand, since (x_n) is a Cauchy sequence, we can find $N \in \mathbb{N}$ such that for all $n, m \geq N$: $|x_n - x_m| < \epsilon$. Now choose $j_0 \geq J$ large enough such that $n_{j_0} > N$ (this is possible since $n_j \to \infty$ as $j \to \infty$). Then, comparing elements x_n with x_{j_0}, we obtain, using the triangle inequality, for all $n \geq N$

$$|x_n - x| = |x_n - x_{j_0} + x_{j_0} - x| \leq |x_n - x_{j_0}| + |x_{j_0} - x|.$$

Since $j_0, n \geq N$, we have $|x_n - x_{j_0}| < \epsilon$. Since $j_0 \geq J$, we have $x_{j_0} - x| < \epsilon$. Combining both facts, we conclude that

$$|x_n - x| \leq |x_n - x_{j_0}| + |x_{j_0} - x| < \epsilon + \epsilon = 2\epsilon,$$

for all $n \geq N$. This is sufficient to conclude that $x_n \to x$, what we wanted to prove. (If you prefer to end up with a clean ϵ, then your earlier estimates should be made with $\epsilon/2$.)

Problem 12* Let (x_n) be a real sequence and $\sigma_n = \frac{1}{n}\sum_{j=1}^n x_j$ (the so called *Cesaro mean* of the sequence (x_n)).

(a) If $x_n \to x$ then also $\sigma_n \to x$.

(b) Find an example such that (x_n) is divergent but (σ_n) is convergent.

Solution: Ad (a): We assume that $x_n \to x$. So we know that $|x_n - x|$ is arbitrarily small for large enough indices n. Let us consider $|\sigma_n - x|$. We write

$$|\sigma_n - x| = \left| \left(\frac{1}{n} \sum_{j=1}^n x_j \right) - x \right| \leq \frac{1}{n} \sum_{j=1}^n |x_j - x|.$$

Let $\epsilon > 0$ be given. Since $x_n \to x$, we can find $N \in \mathbb{N}$ such that $|x_n - x| \leq \epsilon$ for all $n \geq N$. Now we split the above sum $\sum_{j=1}^n$ into $\sum_{j=1}^N$ and $\sum_{j=N+1}^n$ and obtain for all $n \geq N$:

$$|\sigma_n - x| \leq \frac{1}{n} \left(\sum_{j=1}^N |x_j - x| \right) + \frac{1}{n} \left(\sum_{j=N+1}^n \underbrace{|x_j - x|}_{\leq \epsilon} \right) \leq \frac{1}{n} \left(\sum_{j=1}^N |x_j - x| \right) + \epsilon.$$

Note that N is now a fixed natural number and $\sum_{j=1}^N |x_j - x|$ a fixed value. Therefore,

$$\frac{1}{n} \left(\sum_{j=1}^N |x_j - x| \right) \to 0 \quad \text{as } n \to \infty.$$

So we can find $N^* \in \mathbb{N}$ such that

$$\frac{1}{n}\left(\sum_{j=1}^{N} |x_j - x|\right) < \epsilon \quad \text{for all } n \geq N^*.$$

This implies that we have for all $n \geq \max(N, N^*)$:

$$|\sigma_n - x| \leq \frac{1}{n}\left(\sum_{j=1}^{N} |x_j - x|\right) + \epsilon < 2\epsilon.$$

This shows that $\sigma_n \to x$, as well.

Ad (b): Choose $x_n = (-1)^n$. Then (x_n) is not convergent and

$$\sigma_n = \frac{1}{n}\sum_{j=1}^{n}(-1)^n.$$

Since $-1 \leq$ ‘$sum_{j=1}^n (-1)^n \leq 0$, we see that $|\sigma_n| \leq 1/n \to 0$. So (σ_n) is a convergent sequence with limit 0.

Problem 13 Let (x_n) be a bounded real sequence. $xin\mathbb{R}$ is an *accumulation point* of (x_n) if there exists a subsequence (x_{n_j}) with $x_{n_j} \to x$.

(a) Explain why there is always at least one accumulation point.

(b) Give an example with 2 accumulation points (3 accumulation points).

(c)* Give an example where the set of all accumulation points is $[0,1]$.

Solution: Ad (a): This follows from Bolzano-Weierstrass: Every bounded real sequence (x_n) has a convergent subsequence. The limit of this subsequence is then necessarily an accumulation point.

Ad (b): The sequence $x_n = (-1)^n$ has precisely two accumulation points, namely -1 and 1. (The subsequence x_{2n} converges to 1 and the subsequence x_{2n+1} converges to -1. Moreover, no other real value x is a limit of a subsequence, since there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon)$ does not contain -1 or 1, and therefore none of the elements of the sequence fall within this interval.) The sequence $x_n = \cos(n\pi/2)$ has precisely three accumulation points (-1, 0 and 1), since the sequence assumes the values $0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, \dots$.

Ad (c): We choose the sequence x_n with subsequent values

$$0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \dots,$$

i.e., we concatenate the finite sequences $S_n = (1, 1/n, 2/n, \dots, (n-1)/n, 1)$ for $n = 1$, $n = 2$, $n = 3$, etc. Since every real number $x \in [0,1]$ lies within $1/(2n)$ distance of a suitable element of S_n, we can construct a subsequence of (x_n) converging to x. For every real number $x > 1$ or $x < 0$, we find $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap [0,1] = \emptyset$, so x cannot be an accumulation point.

Problem 14* Show: If a bounded sequence (x_n) has precisely one accumulation point x, then $x_n \to x$.

Solution: See Lecture.

Definition. *Let (x_n) be a bounded sequence and $A \subset \mathbb{R}$ be its set of accumulation points (A is then also bounded). The* limit inferior *of (x_n) is defined as*

$$\liminf_{n\to\infty} x_n = \inf A$$

and the limit superior *of (x_n) is defined as*

$$\limsup_{n\to\infty} x_n = \sup A.$$

Problem 15** Show that

$$\limsup_{n\to\infty} x_n = \underbrace{\lim_{n\to\infty} \left(\sup\{x_k \mid k \geq n\}\right)}_{=\bar{x}}$$

via the following steps:

(a) Show that the limit on the RHS (=right hand side) exists.

(b) Let $x > \bar{x}$. Show that there are only finitely many x_n with $x_n > x$.

(c) Let $x < \bar{x}$. Show that there are infinitely many x_n with $x_n > x$.

(d) Prove the identity.

Solution: Ad (a): Since (x_n) is a bounded sequence, the set $X_n = \{x_k \mid k \geq n\}$ is non-empty and bounded and has, by the Completeness Axiom for $\mathbb{R}$, a supremum $s_n = \sup X_n$. Since $X_{n+1} \subset X_n$, we have $s_{n+1} \leq s_n$, i.e., the sequence (s_n) is monotone decreasing and bounded below, since (x_n) is bounded. Therefore, (s_n) is convergent, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\sup\{x_k \mid k \geq n\}\right)$$

exists. So the RHS of the identity exists and will henceforth be denoted by $\bar{x}$.

Ad (b): Let $x > \bar{x}$. If there were infinitely many elements of (x_n) larger than x, we could combine them to a subsequence and conclude that there is a subsequence (x_{n_j}) of x_n with $x_{n_j} \geq x$ for all $j \in \mathbb{N}$. This would imply that

$$s_{n_j} = \sup\{x_k \mid k \geq n_j\} \geq x_{n_j} > x,$$

and therefore $\bar{x} = \lim_{n\to\infty} s_n = \lim_{j\to\infty} s_{n_j} \geq x$. But this contradicts to $x > \bar{x}$.

Ad (c): Let $x < \bar{x}$. If there were only finitely many elements of (x_n) larger than x, then we could find an index $N \in \mathbb{N}$ such that $x_n \leq x$ for all $n \geq N$. But then we had for all $n \geq N$

$$s_n = \sup\{x_k \mid k \geq n\} \leq x,$$

and therefore also $\bar{x} = \lim s_n \leq x$. But this contradicts to $x < \bar{x}$.

Ad (d): Recall that $\limsup x_n$ is the supremum of the set of all accumulation points of (x_n). To prove the identity, we first show that every accumulation point of (x_n) is smaller than or equal to $\bar{x}$. Let $x \in \mathbb{R}$ be an accumulation point of (x_n), i.e., $x_{n_j} \to x$ for some subsequence. If we had $x > \bar{x}$, we could choose $\epsilon > 0$ such that $x - \epsilon > \bar{x}$, and there were only finitely many x_n satisfying $x_n > x - \epsilon$, by (b). On the other hand, since $x_{n_j} \to x$, there exists $J \in \mathbb{N}$ such that $x_{n_j} \in (x - \epsilon, x + \epsilon)$ for all $j \geq J$, which contradicts to the previous statement. So we cannot have $x > \bar{x}$, i.e., all accumulation points of (x_n) are $\leq \bar{x}$. It only remains to show that $\bar{x}$ is an accumulation point to conclude that the supremum of all accumulation points of (x_n) agrees with $\bar{x}$, which we wanted to prove. Recall that we have $s_n \to \bar{x}$ for $n \to \infty$. Let us now construct a subsequence (x_{n_j}) with $s_{n_j} \to \bar{x}$ inductively: Assume we have already constructed the sequence $x_{n_1}, \ldots, x_{n_j}$ and we need to construct $x_{n_{j+1}}$. To do so, we choose an index $n > n_j$ such that $|s_n - \bar{x}| < 1/(2n)$ (we can find such an index n since $s_n \to \bar{x}$). Since $s_n = \sup\{x_k \mid k \geq n\}$, we can find $k \geq n$ such that $|s_n - x_k| < 1/(2n)$. This index $k \geq n > n_j$ becomes now n_{j+1}. We then have $n_{j+1} > n_j$ and

$$|\bar{x} - x_{n_{j+1}}| \leq |\bar{x} - s_n| + |s_n - x_{n_{j+1}}| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

This estimate shows that $x_{n_j} \to \bar{x}$, and we found a subsequence of (x_n) converging to $\bar{x}$. Therefore, $\bar{x}$ is an accumulation point of (x_n), finishing the proof.

Problem 16 Find lim inf and lim sup of $x_n = \begin{cases} \frac{n}{n+1} & n \text{ odd} \\ \frac{1}{n+1} & n \text{ even} \end{cases}$.

Solution: The subsequence $\frac{n}{n+1}$ converges to 1 and the subsequence $\frac{1}{n+1}$ converges to 0, so 0 and 1 are accumulation points. Note also that $0 \leq x_n \leq 1$, so no convergent subsequence of (x_n) can have limits x with $x < 0$ or $x > 1$. Therefore, 0 must be smallest accumulation point and 1 must be the largest accumulation point and we have

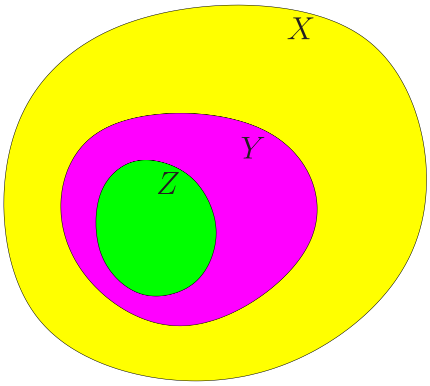
$$\liminf_{n\to\infty} x_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} x_n = 1.$$

Problem 17 Let

$$\begin{aligned} X &= \text{set of all bounded real sequences,} \\ Y &= \text{set of all convergent real sequences,} \\ Z &= \text{set of all bounded and monotone real sequences.} \end{aligned}$$

Draw a Venn-Diagram illustrating the relations between the three sets and give examples of sequences in the larger but not in the smaller of the sets (to show that inclusions of sets are "proper").

Solution: Since every convergent real sequence is bounded, we have $Y \subset X$. Since every bounded and monotone real sequence is convergent, we have $Z \subset Y$. So the Venn-Diagram looks as follows:



An example of a sequence in X and not in Y is $x_n = (-1)^n$, since (x_n) is bounded ($-1 \leq x_n \leq 1$) but not convergent. An example of a sequence in Y and not in Z is $x_n = (-1)^n/n$, since $x_n \to 0$ but x_n is not monotone.

Injectivity/Surjectivity/Bijectivity/Preimages

Problem 18 Let $f : X \to Y$ be injective (surjective) and $g : Y \to Z$ be injective (surjective). Show that $g \circ f : X \to Z$ is injective (surjective).

Solution: Let f and g be both injective. For $x_1, x_2 \in X$ let $g \circ f(x_1) = g \circ f(x_2)$. This means that $g(f(x_1)) = g(f(x_2))$. Since g is injective, this implies that $f(x_1) = f(x_2)$. Since f is injective, this implies that $x_1 = x_2$. So we showed that $g \circ f(x_1) = g \circ f(x_2)$ implies $x_1 = x_2$, which means that $g \circ f$ is injective.

Let f and g be both surjective. Let $z \in Z$. Since g is surjective, there exists $y \in Y$ such that $g(y) = z$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Combining both facts leads to

$$g \circ f(x) = g(f(x)) = g(y) = z,$$

showing that $g \circ f$ is surjective.

Problem 19* Let $f : X \to Y$. Show equivalence of the following two statements:

(a) f is surjective.

(b) $f(f^{-1}(Y_0)) = Y_0$ for all subsets $Y_0 \subset Y$.

Solution: Let us first show that (b) implies (a). So we need to show that f, satisfying property (b), is surjective. Let $y \in Y$. Then we have $f(f^{-1}(\{y\})) = \{y\}$, i.e., there is an element in $f^{-1}(\{y\}) \subset X$ whose image under f is y. Let us call this element $x \in X$. Then we have $f(x) = y$ and we showed surjectivity.

Now let us show that (a) implies (b). Assume that f is surjective. Let $Y_0 \subset Y$. We first show that $f(f^{-1}(Y_0)) \supset Y_0$. Let $y \in Y_0$. Then there exists $x \in X$ such that $f(x) = y$ and, therefore, $x \in f^{-1}(Y_0)$. But then also $f(x) \in f(f^{-1}(Y_0))$ and, since $f(x) = y$, $y \in f(f^{-1}(Y_0))$. This shows that $f(f^{-1}(Y_0)) \supset Y_0$. Now assume we had $y \in f(f^{-1}(Y_0)) \backslash Y_0$. Then we had $x \in f^{-1}(Y_0)$ such that $y = f(x)$. But, by the definition of the preimage, this would mean that $f(x) \in Y_0$. Since $y = f(x)$ we would end up with $y \in Y_0$, which is a contradiction to our assumption $y \in f(f^{-1}(Y_0)) \backslash Y_0$. So we must have $f(f^{-1}(Y_0)) \subset Y_0$. Both results together show that $f(f^{-1}(Y_0)) = Y_0$.

Problem 20 Preimages behave well under set operations.

(a) Show that $f^{-1}(\bigcap_{\alpha\in I} A_\alpha) = \bigcap_{\alpha\in I} f^{-1}(A_\alpha)$.

(b) Find an example that $f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2)$.

Solution: Ad (a): Assume that $f : X \to Y$ and that $A_\alpha \subset Y$ for all $\alpha \in I$. Then we have

$$\begin{aligned} x \in f^{-1}(\bigcap_{\alpha\in I} A_\alpha) &\Leftrightarrow f(x) \in \bigcap_{\alpha\in I} A_\alpha \\ &\Leftrightarrow \forall\, \alpha \in I : f(x) \in A_\alpha \\ &\Leftrightarrow \forall\, \alpha \in I : x \in f^{-1}(A_\alpha) \\ &\Leftrightarrow x \in \bigcap_{\alpha\in I} f^{-1}(A_\alpha), \end{aligned}$$

showing that we have, indeed, the identity $f^{-1}(\bigcap_{\alpha\in I} A_\alpha) = \bigcap_{\alpha\in I} f^{-1}(A_\alpha)$.

Ad (b): The strategy here is to choose a particularly simple, non-injective, map, say $f : \{a, b\} \to \{c\}$ with $f(a) = f(b) = c$. Choosing then $X_1 = \{a\}$ and $X_2 = \{b\}$, we have $X_1 \cap X_2 = \emptyset$ and, therefore, $f(X_1 \cap X_2) = f(\emptyset) = \emptyset$. On the other hand, we have $f(X_1) = f(X_2) = \{c\}$ and, therefore,

$$\emptyset = f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2) = \{c\}.$$

Limits of functions, Continuity

Problem 21 Using the ϵ/δ-formalism, formulate with quantifiers that $f : \mathbb{R} \to \mathbb{R}$ is not continuous at $x_0 \in \mathbb{R}$.

Solution: f is continuous at x_0 iff

$$\forall\, \epsilon > 0\, \exists\, \delta > 0 \text{ such that } |f(x) - f(x_0)| < \epsilon \text{ for all } x \in \mathbb{R} \text{ with } |x - x_0| < \delta.$$

The negation then reads as

$$\exists\, \epsilon > 0 \text{ such that } \forall\, \delta > 0 \text{ there exists } x \in \mathbb{R} \text{ with } |x - x_0| < \delta \text{ such that } |f(x) - f(x_0)| \geq \epsilon.$$

In plain words, there exists $\epsilon > 0$, such that we can find an infinite sequence of points x_n converging to x_0 such that $|f(x_n) - f(x_0)| \geq \epsilon$.

Problem 22 Using the ϵ/δ-definition of continuity, show: If $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 and $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0.

Solution: Let f and g be as described. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$:

$$|g \circ f(x) - g \circ f(x_0)| < \epsilon.$$

Note that $|g \circ f(x) - g \circ f(x_0)| = |g(f(x)) - g(f(x_0))|$. Since g is continuous at $f(x_0)$, there exists $\alpha > 0$ such that for all $y \in \mathbb{R}$ with $|y - f(x_0)| < \alpha$:

$$|g(y) - g(f(x_0))| < \epsilon. \tag{1}$$

Since $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$:

$$|f(x) = f(x_0)| < \alpha. \tag{2}$$

Combining (1) and (2), we have for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$: $|f(x) - f(x_0)| < \alpha$, and therefore, by (**??**):

$$|g(f(x)) - g(f(x_0))| < \epsilon.$$

Here it is important to start your arguments with g and not with f to end up with the desired result.

Problem 23 Using the method of proof of the Intermediate Value Theorem, find the solution $x > 0$ of $\cos x = e^x - 0.1$ within accuracy of $1/1000$ (you are allowed to use a pocket calculator to evaluate cos and e^x).

Solution: Let $f(x) = \cos x + 0.1 - e^x$. Then we have $f(0) = 0.1$ and $f(1) = -2.0779795\ldots$ Since f is continuous, there is a zero of f in the interval $[0, 1]$, by the Intermediate Value Theorem. We carry out consecutive bisections to get better and better approximations of a zero. We have

$$f(0.5) = -0.6711387\ldots$$

Therefore, we must have a zero of f in $[0, 0.5]$. We have

$$f(0.25) = -0.21511\ldots$$

Therefore, we must have a zero of f in $[0, 0.25]$. We have

$$f(0.125) = -0.04095\ldots$$

Therefore, we must have a zero of f in $[0, 0, 125]$. We have

$$f(0.0625) = 0.033553\ldots$$

Therefore, we must have a zero of f in $[0.0625, 0.125]$. We have

$$f(0.09375) = -0.002676\ldots$$

Therefore, we must have a zero of f in $[0.0625, 0.09375]$. We have

$$f(0.078125) = 0.01569\ldots$$

Therefore, we must have a zero of f in $[0.078125, 0.09375]$. We have

$$f(0.0859375) = 0.00657\ldots$$

Therefore, we must have a zero of f in $[0.0859375, 0.09375]$. We have

$$f(0.08984375) = 0.00196\ldots$$

Therefore, we must have a zero of f in $[0.08984375, 0.09375]$. We have

$$f(0.091796875) = -0.00035\ldots$$

Therefore, we must have a zero of f in $[0.08984375, 0.091796875]$. We have

$$f(0.0908203125) = 0.0008\dots$$

Therefore, we must have a zero of f in $[0.0908203125, 0.091796875]$. Since

$$0.091796875 - 0.0908203125 = 0.0009765625 < \frac{1}{1000},$$

we calculated a zero $x.0$ of f within an error range of $1/1000$.

Problem 24 Let $X \subset \mathbb{X}$ and $f : X \to \mathbb{R}$ be a function. Explain the difference between continuity of f on X and uniform continuity with the help of quantifiers. Which property is stronger?

Solution: See Lecture.

Problem 25* Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the compact interval $[a, b]$. Show that f is uniformly continuous via the following steps:

(a) Assume there exists $x_n, x'_n \in [a, b]$ with

$$|x_n - x'_n| < \frac{1}{n} \quad \text{and } |f(x_n) - f(x'_n)| > \epsilon.$$

Apply Bolzano-Weierstrass to the sequence (x_n) and conclude that there exists a subsequence (x_{n_j}) with

$$|f(x_{n_j}) - f(x'_{n_j})| \to 0 \quad \text{as } j \to \infty.$$

(b) Use your result from (a) to construct an Indirect Proof for uniform continuity.

Solution: Ad (a): We assume that there exists $x_n, x'_n \in [a, b]$ with

$$|x_n - x'_n| < \frac{1}{n} \quad \text{and } |f(x_n) - f(x'_n)| > \epsilon.$$

Since (x_n) is bounded (its elements lie in the bounded interval $[a, b]$), there exists a convergent subsequence (x_{n_j}) of (x_n), by Bolzano-Weierstrass. Let x^* be the limit of (x_{n_j}), i.e., $x_{n_j} \to x^* \in [a, b]$. By COLT, we conclude that

$$\lim_{j\to\infty} x_{n_j} \pm \frac{1}{n_j} = \lim_{j\to\infty} x_{n_j} \pm \lim_{j\to\infty} \frac{1}{n_j} = x^*.$$

Since $x_{n_j} - 1/n_j \le x'_{n_j} \le x_{n_j} + 1/n_j$, we conclude that

$$lim_{j\to\infty} x'_{n_j} = x^*,$$

by the Squeezing Theorem. By continuity of f, we conclude that

$$\lim_{j\to\infty} f(x_{n_j}) = f(x^*) = \lim_{j\to\infty} f(x'_{n_j}).$$

Using continuity of the absolute value and COLT, we obtain

$$\lim_{j\to\infty} |f(x_{n_j}) - f(x'_{n_j})| = |\lim_{j\to\infty} f(x_{n_j}) - f(x'_{n_j})| = |f(x^*) - f(x^*)| = 0.$$

This finishes the proof of (a).

Ad (b): We start our Indirect Proof by assuming that $f : [a,b] \to \mathbb{R}$ is continuous but not uniformly continuous. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, in particular for $\delta = 1/n > 0$, we can find two points, denoted by $x_n, x'_n \in [a,b]$ with $|x_n - x'_n| < \delta = 1/n$ and

$$|f(x_n) - f(x'_n)| > \epsilon. \tag{3}$$

Following the arguments in (a), we can then find subsequences (x_{n_j}) and (x'_{n_j})

$$|f(x_{n_j}) - f(x'_{n_j})| \to 0.$$

This means that for some $j \in \mathbb{N}$, we must have

$$|f(x_{n_j}) - f(x'_{n_j})| < \epsilon.$$

On the other hand we know from (3) that

$$|f(x_{n_j}) - f(x_{n_j})| > \epsilon$$

for all $j \in \mathbb{N}$. Both inequalities contradict each other, which finishes our Indirect Proof.

Differentiable functions

Problem 26 Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at $x = 1$ but not differentiable.

Solution: The function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable since

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

and

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

To construct a function continuous but not differentiable at $x = 1$, we shift this function and consider $f(x) = f(x-1) = |x-1|$.

Problem 27 Use the equivalent definition of differentiability to prove the following fact: If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at $c \in \mathbb{R}$, then so is fg. Give the formula for $(fg)'(c)$.

Solution: Differentiabity of f at c means that we have

$$f(x) = f(c) + (x-c)f_1(x)$$

with f_1 continuous at c and $f_1(c) = f'(c)$. Differentiabity of g at c means that we have

$$g(x) = g(c) + (x-c)g_1(x)$$

with g_1 continuous at c and $g_1(c) = g'(c)$. We have to show: There exists a function $h : \mathbb{R} \to \mathbb{R}$, continuous at c, such that

$$f(x)g(x) = f(c)g(c) + (x - c)h(x).$$

We have (following the goal to bring $f_1.g_1$ into the game)

$$f(x)g(x) - f(c)g(c) = f(x)g(x) \underbrace{-f(c)g(x) + f(c)g(x)}_{=0} -f(c)g(c) =$$

$$= (f(x)-f(c))g(x)+f(c)(g(x)-g(c)) = (x-c)f_1(x)g(x)+f(c)(x-c)g_1(x) =$$
$$= (x - c)(f_1(x)g(x) + f(c)g_1(x)).$$

Now, $h(x) = f_1(x)g(x)+f(c)g_1(x)$ is continuous at c, since f_1, g_1 are and since g is differentiable at c and, therefore, also continuous at c. So we know that fg is differentiable at c and we find an expression of the derivative via

$$(fg)'(c) = h(c) = f_1(c)g(c) + f(c)g_1(c) = f'(c)g(c) + f(c)g'(c).$$

Problem 28 Calculate $\lim_{x\to 0} \frac{e^x}{e^x-1} - \frac{1}{x}$ and $\lim_{x\to 0} \left(\int_{-x}^{2x} \cos(t^2)dt\right) / \left(\int_0^x \sin(t^2)dt\right)$.

Solution: We first consider $\lim_{x\to 0} \frac{e^x}{e^x-1} - \frac{1}{x}$. We have

$$\frac{e^x}{e^x - 1} - \frac{1}{x} = \frac{xe^x - (e^x - 1)}{x(e^x - 1)} = \frac{xe^x - e^x + 1}{xe^x - x}.$$

Let $f(x) = xe^x - e^x + 1$ and $g(x) = xe^x - x$. Then $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$ and we can consider $\lim_{x\to 0} f'(x)/g'(x)$ instead. We have $f'(x) = xe^x$ and $g'(x) = xe^x + e^x - 1$. We have again $\lim_{x\to 0} f'(x) = 0 = \lim_{x\to 0} g'(x)$ and we can consider $\lim_{x\to 0} f''(x)/g''(x)$ instead. We have $f''(x) = xe^x + e^x$ and $g''(x) = xe^x + 2e^x$ which leads to, using L'Hôpital twice,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{xe^x + e^x}{xe^x + 2e^x} = \frac{1}{2}.$$

Next, let $f(x) = \int_{-x}^{2x} \cos(t^2)dt$ and $g(x) = \int_0^x \sin(t^2)dt$. We have $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, so we can consider the derivatives of f and g which are, by the Fundamental Theorem of Calculus,

$$f'(x) = \cos(4x^2) - \cos(x^2) \quad \text{and } g'(x) = \sin(x^2).$$

We have again $\lim_{x\to 0} f'(x) = 1 - 1 = 0$ and $\lim_{x\to 0} g'(x) = 0$, so we can consider

$$f''(x) = -8x \sin(4x^2) + 2x \sin(x^2) = 2x \left(\sin(x^2) - 4 \sin(4x^2)\right)$$

and

$$g''(x) = 2x \cos(x^2).$$

We have again $\lim_{x\to 0} f''(x) = \lim_{x\to 0} g''(x) = 0$, so we can consider

$$f^{(3)}(x) = 2 \left(\sin(x^2) - 4 \sin(4x^2)\right) + 4x^2 \left(\cos(x^2) - 16 \cos(4x^2)\right)$$

and

$$g^{(3)}(x) = 2\cos(x^2) - 4x^2 \sin(x^2).$$

Now we have $\lim_{x\to 0} f^{(3)}(x) = 0$ and $\lim_{x\to 0} g^{(3)}(x) = 2 - 0 = 2$, so we obtain by applying L'Hopital three times,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f^{(3)}(x)}{g^{(3)}(x)} = \frac{0}{2} = 0.$$

Problem 29 Recall Newton iteration: Let $f \in C^2(\mathbb{R})$, i.e., f is twice differentiable on $\mathbb{R}$ with continuous second derivative f''. Let $f(c) = 0$. To find the zero c of f, Newton iteration is given via

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

To avoid any problems in Newton iteration of the kind "division by zero", we assume $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Use from Calculus the identity

$$f(c) = f(x) + f'(x)(c - x) + \frac{f''(\eta)}{2}(c - x)^2$$

for some η between x and c, to show

$$|c - x_{n+1}| = \left|\frac{f''(\eta)}{2f'(x_n)}\right| \cdot |c - x_n|^2 ,$$

for some η between x_n and c. If $|f''|$ and $|1/f'|$ can be estimated from above near c, this provides "quadratic" convergence $x_n \to c$ for the start value x_1 close enough to c.

Solution: Plugging in $x = x_n$, we obtain

$$0 = f(c) = f(x_n) + f'(x_n)(c - x_n) + \frac{f''(\eta)}{2}(c - x_n)^2$$

for some η between x_n and c. On the other hand, the recursion formula yields

$$-f'(x_n)x_{n+1} = -x_n f'(x_n) + f(x_n).$$

Combining both formulas leads to

$$0 = -f'(x_n)x_{n+1} + f'(x_n)c + \frac{f''(\eta)}{2}(c - x_n)^2 = f'(x_n)(c - x_{n+1}) + \frac{f''(\eta)}{2}(c - x_n)^2.$$

Dividing both sides by $f'(x_n) \neq 0$, we end up with

$$c - x_{n+1} = -\frac{f''(\eta)}{2f'(x_n)}(c - x_n)^2,$$

and we obtain the required formula by taking absolute values.

Infinite series

Problem 30 Give an elementary proof that $\sum 1/k$ is divergent.

Solution: We bracket the sum as follows:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \sum_{j=1}^{\infty} \left(\frac{1}{2^j+1} + \frac{1}{2^j+2} + \cdots + \frac{1}{2^{j+1}} \right).$$

We can estimate each bracket as follows:

$$\frac{1}{2^j+1} + \frac{1}{2^j+2} + \cdots + \frac{1}{2^{j+1}} \geq 2^j \cdot \frac{1}{2^{j+1}} = \frac{1}{2}.$$

This shows that

$$\sum_{k=1}^{\infty} \frac{1}{k} \geq 1 + \frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{2} = \infty.$$

Problem 31
(a) Formulate the Alternating Sign Test,
(b) Give an example of a series which is conditionally convergent.

Solution: Ad (a): The Alternating Sign Test states the following: Let (a_n) be a monotone decreasing sequence of non-negative numbers with $a_n \to 0$ as $n \to \infty$. Then the series $\sum(-1)^n a_n$ is convergent.

Ad (b): The series $\sum(-1)^k 1/k$ is convergent by the Alternating Sign Test. This series is not absolutely convergent, since $\sum \left|(-1)^k 1/k\right|$ is the harmonic series, which is divergent (see Problem 30). So $\sum(-1)^k 1/k$ is conditionally convergent.

Problem 32 Investigate convergence and divergence of $\sum_{k=1}^{\infty} 1/k^\alpha$.

Solution: Obviously, $\sum_{k=1}^{\infty} 1/k^\alpha$ is not convergent for $\alpha \leq 0$, since then $1/k^\alpha \geq 1$ and we do not have $1/k^\alpha \to 0$, which is necessary for $\sum_{k=1}^{\infty} 1/k^\alpha$ to converge.

For $\alpha > 0$, we use the Integral Test for series. Let $f(x) = 1/x^\alpha$. Then $f : [1, \infty) \to [0, \infty)$ is monotone decreasing and $a_k = f(k) = 1/k^\alpha$. We know that $\int_1^\infty 1/x^\alpha dx$ is convergent iff $\alpha > 1$. Then the Integral Test implies that $\sum_{k=1}^{\infty} 1/k^\alpha$ is convergent iff $\alpha > 1$.

Problem 33
(a) Formulate the n-th root test.
(b) Use the n-th root test and check convergence/divergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{3n-2n^2} \left(\frac{1}{4}\right)^n$.

Solution: Ad (a): Assume that $|a_n|^{1/n} \to L$ as $n \to \infty$. Then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $L < 1$ and divergent if $L > 1$.

Ad (b): Let $a_n = \left(\frac{2n}{2n+1}\right)^{3n-2n^2} \left(\frac{1}{4}\right)^n$. Then we have

$$|a_n|^{1/n} = \left(\frac{2n}{2n+1}\right)^{3-2n} \cdot \frac{1}{4} = \left(\frac{1}{1+1/(2n)}\right)^3 \cdot \left(1 + \frac{1}{2n}\right)^{2n} \cdot \frac{1}{4} \to 1^3 \cdot e \cdot \frac{1}{4} = \frac{e}{4} < 1.$$

This shows that the series converges absolutely.

Problem 34 (a) Formulate the result about the Cauchy Product.

(b) Use (a) to prove: $e^{x+y} = e^x e^y$.

Solution: Ad (a): Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be two absolutely convergent series. Then the following series

$$\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} a_l b_{k-l}\right)$$

is called the Cauchy Product of these two series and it is also absolutely convergent and we have

$$\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} a_l b_{k-l}\right) = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right).$$

Ad (b): We know that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is absolutely convergent for all $x \in \mathbb{C}$. Using the formula for the Cauchy Product, we then obtain

$$\begin{aligned} e^x e^y = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{y^k}{k!}\right) &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} \frac{x^l}{l!}\frac{y^{k-l}}{(k-l)!}\right) = \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{l=0}^{k} \frac{k!}{l!(k-l)!} x^l y^{k-l}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{l=0}^{k} \binom{k}{l} x^l y^{k-l}\right) \end{aligned}$$

Finally, we apply the Binomial Formula and end up with

$$e^x e^y = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{l=0}^{k} \binom{k}{l} x^l y^{k-l}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}(x+y)^k = e^{x+y}.$$

Problem 35 Decide about convergence or divergence of $\sum_{n=1}^{\infty} (2n-i)^{-2} e^{i(n^2+1)}$.

Solution: It is really important that you know that, for real $x \in \mathbb{R}$, the complex number $e^{ix} \in \mathbb{C}$ lies on the unit circle, where x describes the angle (modulo 2π) which the vector representing this complex number makes with the positive real axis. (This follows from Euler's identity $e^{ix} = \cos(x) + i\sin(x)$ and from $\cos^2(x) + \sin^2(x) = 1$.) Therefore, the complex numbers $e^{i(n^2+1)}$ lie on the unit circle and have modulus 1. We use the fact, which holds also for complex series, that absolute convergence implies ordinary convergence. Let $a_n = (2n-i)^{-2} e^{i(n^2+1)}$. Then

$$|a_n| = \frac{1}{|2n-i|^2}\left|e^{i(n^2+1)}\right| = \frac{1}{4n^2+1} \le \frac{1}{4n^2}.$$

The series is then absolute convergent, by comparison with the convergent series $\sum 1/(4n^2)$ and, therefore, also convergent.

Integrals

Problem 36 Let $f : [a, b] \to \mathbb{R}$ be monotone increasing. Show that f is Riemann integrable.

Solution: Let $\mathcal{P} = \{a = x_0, x_1, \ldots, x_n = b\}$ with $x_0 < x_1 < \cdots < x_n$ be a partition of $[a, b]$. Since f is monotone increasing, we conclude that

$$\begin{aligned} m_j(f) &= \inf_{x\in[x_{j-1},x_j]} f(x) = f(x_{j-1}), \\ M_j(f) &= \sup_{x\in[x_{j-1},x_j]} f(x) = f(x_j). \end{aligned}$$

Therefore, we have

$$U(f,\mathcal{P})-L(f,\mathcal{P}) = \sum_{j=1}^{n}(M_j(f)-m_j(f))(x_j-x_{j-1}) = \sum_{j=1}^{n} \underbrace{(f(x_j) - f(x_{j-1}))}_{\geq 0} \cdot \underbrace{(x_j - x_{j-1}}_{\geq 0} \leq$$
$$\leq \left(\max_{j=1,2,\ldots,n}(x_j - x_{j-1})\right) \cdot \sum_{j=1}^{n}(f(x_j) - f(x_{j-1})).$$

Note that $\sum_{j=1}^{n}(f(x_j)-f(x_{j-1}))$ is a telescope sum, i.e., the intermediate terms cancel each other out, and we have $\sum_{j=1}^{n}(f(x_j) - f(x_{j-1})) = f(b) - f(a)$. So we end up with the inequality

$$0 \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) \leq \left(\max_{j=1,2,\ldots,n}(x_j - x_{j-1})\right)(f(b) - f(a)).$$

If we choose an equidistant partition, i.e., $x_j - x_{j-1} = \frac{b-a}{n}$, we have

$$0 \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) \leq \frac{b-a}{n}(f(b) - f(a)).$$

As $n \to \infty$, we see that the difference $U(f,\mathcal{P}) - L(f,\mathcal{P})$ goes to zero which implies that $\mathcal{L}(f) = \mathcal{U}(f)$ and $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Problem 37 (a) State the Mean Value Theorem for Integrals.

(b) Show that all conditions in the theorem are necessary by providing counterexamples when individual conditions in the Theorem are removed.

Solution: Ad (a): Let $f, g : [a, b] \to \mathbb{R}$ be two functions with f being continuous and $g \geq 0$ being Riemann integrable. Then fg is also Riemann integrable and there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

ad (b): Assume first that f is continuous and g is Riemann integrable, but we no longer have $g \geq 0$. Then we could choose $f = g$ and would have the identity

$$\int_a^b f^2(x)dx = f(c)\int_a^b f(x)dx$$

for some $c \in [a, b]$. Note that the left hand side is always non-negative and strictly positive if $f \neq 0$. So we could choose a function $f : [a, b] \to \mathbb{R}$ with integral $\int_a^b f(x)dx$ equal to zero. Then the right hand side would always be zero, no matter how we choose $c \in [a, b]$. So a counterexample is, for example, $[a, b] = [-\pi, \pi]$ and $f(x) = g(x) = \sin(x)$. Then we have

$$0 < \int_{-\pi}^{\pi} \sin^2(x)dx \neq f(c) \int_{-\pi}^{\pi} \sin(x)dx = 0$$

for any choice of $c \in [-\pi, \pi]$.

Assume next that $g \geq 0$ is Riemann integrable but f is no longer continuous. We could choose, for simplicity $g \equiv 1$ and would have the identity

$$\int_a^b f(x)dx = f(c) \int_a^b dx = (b - a)f(c).$$

If we assume that f takes only the values 0 and 1 on $[a, b]$, then the right hand side could only assume two values, namely 0 or $b - a$. So we can construct a counterexample by choosing, for example, $[a, b] = [0, 1]$ and $f(x) = 0$ on $[0, 3/4]$ and $f(x) = 1$ on $(3/4, 1]$. Then we end up with

$$\frac{3}{4} = \int_0^1 f(x)dx \neq f(c) \in \{0, 1\}$$

for any choice of $c \in [0, 1]$.

This shows that the assumptions of the theorem are all essential anc cannot be removed.

Problem 38 Calculate $\int_0^{\sqrt{\pi}} xe^{ix^2} dx$ by decomposing it into real and imaginary part.

Solution: We have $e^{ix^2} = \cos(x^2) + i \sin(x^2)$, so we obtain

$$\int_0^{\sqrt{\pi}} xe^{ix^2} dx = \int_0^{\sqrt{\pi}} x \cos(x^2)dx + i \int_0^{\sqrt{\pi}} x \sin(x^2)dx.$$

Now, a primitive of $x \cos(x^2)$ is $\sin(x^2)/2$, and a primitive of $x \sin(x^2)$ is $-\cos(x^2)/2$. So we obtain

$$\int_0^{\sqrt{\pi}} xe^{ix^2} dx = \left[\frac{\sin(x^2)}{2}\right]_0^{\sqrt{\pi}} + i \left[\frac{-\cos(x^2)}{2}\right]_0^{\sqrt{\pi}} =$$
$$= \frac{\sin(\pi)}{2} - \frac{\sin(0)}{2} + i\left(-\frac{\cos(\pi)}{2} + \frac{\cos(0)}{2}\right) = -i\left(\frac{-1}{2} + \frac{1}{2}\right) = i.$$

Improper Integrals

Problem 39 Show: If $f : [x_0, \infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = 0$, then f is bounded, i.e., there exists $C > 0$ such that $|f(x)| \leq C$ for all $x \in [x_0, \infty)$.

Solution: See Lecture.

Problem 40 Prove the following "Limit Comparison Test": Let $a > 0$ and $f : (0, a] \to \mathbb{R}$ be continuous and $\frac{f(x)}{x^\alpha} \to c \neq 0$ as $x \to 0$. Then the improper integral

$$\int_0^a f(x)dx$$

is convergent iff the improper integral $\int_0^a x^\alpha dx$ is convergent (which we know to be equivalent with $\alpha > -1$).

Solution: Without loss of generality, we can assume that $c > 0$. (If $c < 0$, simply replace f by $-f$.) We first show that there exists $0 < b < a$ such that $\frac{c}{2}x^\alpha \leq f(x) \leq 2cx^\alpha$ for all $x \in (0, b]$. $\frac{f(x)}{x^\alpha} \to c > 0$ as $x \to 0^+$ implies that $|\frac{f(x)}{x^\alpha} - c| < \frac{c}{2}$ for x close to 0, i.e., there exists b with $0 < b < a$ such that

$$|\frac{f(x)}{x^\alpha} - c| < \frac{c}{2} \quad \text{for all } x \in (0, b].$$

But this is equivalent to

$$\frac{c}{2} = c - \frac{c}{2} < \frac{f(x)}{x^\alpha} < c + \frac{c}{2} \leq 2c$$

for all $x \in [0, b)$. Multiplying with x^α leads to

$$\frac{c}{2}x^\alpha \leq f(x) \leq 2cx^\alpha \quad \text{for all } x \in (0, b].$$

Now we split the integral into

$$\int_0^a f(x)dx = \int_0^b f(x)dx + \int_b^a f(x)dx$$

and observe that the integral $\int_b^a f(x)dx$ causes no problem since $f : [b, a] \to \mathbb{R}$ is continuous and, therefore, it is a perfectly well defined expression. As for the improper integral $\int_0^b f(x)dx$, we now have the ordinary integral comparison test, since $0 \leq cx^\alpha/2 \leq f(x) \leq 2cx^\alpha$ for all $x \in (0, b]$.

Problem 41 Find all $\beta \in \mathbb{R}$ such that

$$\int_0^\infty \frac{x^\beta}{\sqrt{x^4 + x\cos(x) + 3}}dx$$

is convergent.

Solution: We split the integral into two parts

$$\int_0^\infty \frac{x^\beta}{\sqrt{x^4 + x\cos(x) + 3}}dx = \int_0^C \frac{x^\beta}{\sqrt{x^4 + x\cos(x) + 3}}dx + \int_C^\infty \frac{x^\beta}{\sqrt{x^4 + x\cos(x) + 3}}.$$

A good separating point here is $C = 2$, since then we have for $x \in [C, \infty)$

$$x^4 + x\cos x + 3 \geq x^4 - x \geq x^4 - \frac{x^4}{2},$$

because then $x/2 \geq 1$ and $x \leq x^3 \leq x^3 \cdot x/2 = x^4/2$. So we can estimate for all $x \in [2, \infty)$:

$$\frac{x^4}{2} \leq x^4 + x\cos x + 3 \leq x^4 + x^4 + 3x^4 = 5x^4.$$

Similarly, we can estimate for all $x \in (0, 2]$:

$$1 \leq 3 - x \leq x^4 + x\cos(x) + 3 \leq 16 + 2 + 3 = 19.$$

So we can use for $x \in (0, 2]$ the comparison

$$x^\beta \leq \frac{x^\beta}{\sqrt{x^4 + x\cos(x) + 3}} \leq 19x^\beta$$

and conclude that the integral $\int_0^2 \frac{x^\beta}{\sqrt{x^4+x\cos(x)+3}}dx$ converges iff $\beta > -1$. For $x \in [2, \infty)$ we can use the comparison

$$\frac{1}{\sqrt{5}}x^{\beta-2} \leq \frac{x^\beta}{\sqrt{x^4 + x\cos(x) + 3}} \leq \sqrt{2}x^{\beta-2}$$

and conclude that the integral $\int_2^\infty \frac{x^\beta}{\sqrt{x^4+x\cos(x)+3}}dx$ converges iff $\beta - 2 < -1$. So we have convergence of the whole integral iff $-1 < \beta < 1$.

Uniform Convergence

Problem 42 Let $X \subset \mathbb{R}$ and $f_n \in C(X)$ and $f_n \to f$ uniformly on X. Show that also $f \in C(X)$, i.e., uniform convergence preserves continuity.

Solution: To prove continuity of f, the idea is to first choose a function f_n uniformly close to f and then to employ the continuity of the function f_n. So we start with a poijt $x \in X$ and a given $\epsilon > 0$. We need to find $\delta > 0$ such that, for all $y \in X$ with $|y - x| < \delta$:

$$|f(y) - f(x)| < \epsilon.$$

We first find a function f_n which is uniformly close to f with error smaller than $\epsilon/3$: The uniform convergence $f_n \to f$ implies that we can find $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|f(y) - f_n(y)| < \frac{\epsilon}{3}$$

for all $y \in X$. In particular, we have

$$|f(y) - f_N(y)| < \frac{\epsilon}{3}$$

for all $y \in X$. Now, we employ continuity of the function f_N at $x \in X$: There exists $\delta > 0$ such that for all $y \in X$ with $|y - x| < \delta$:

$$|f_N(y) - f_N(x)| < \frac{\epsilon}{3}.$$

Combining all these results leads to the following fact: For all $y \in X$ with $|y - x| < \delta$ we have

$$\begin{aligned}|f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \leq \\ &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

This proves continuity of f at $x \in X$.

Problem 43 Consider $f_n : (0, 1) \to \mathbb{R}$ with $f_n(x) = \frac{nx}{nx^2+1}$. Does (f_n) have a pointwise and a uniform limit?

Solution: Let us first check whether f_n converges pointwise. Let $x \in (0, 1)$. Then we have

$$f_n(x) = \frac{nx}{nx^2 + 1} = \frac{x}{x^2 + 1/n} \to \frac{x}{x^2} = \frac{1}{x} \quad \text{as } n \to \infty.$$

This shows that $f_n(x) \to f(x) = 1/x$ on $(0, 1)$ pointwise. Now we need to check whether this convergence is uniform. If $f_n \to f$ is uniform, there must be, for every $\epsilon > 0$, a start index $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in (0, 1)$, we have

$$|f(x) - f_n(x)| < \epsilon.$$

Let us look at the difference $f(x) - f_n(x)$:

$$f(x) - f_n(x) = \frac{1}{x} - \frac{nx}{nx^2 + 1} = \frac{nx^2 + 1 - nx^2}{x(nx^2 + 1)} = \frac{1}{x(nx^2 + 1)}.$$

For very small $x > 0$, $nx^2 + 1$ goes to infinity as $n \to \infty$, but we could choose x changing with n to compensate for that. So we try $x = 1/n \in (0, 1)$, and we have $nx^2 + 1 = 1 + 1/n$, so $\frac{1}{nx^2+1}$ would not go to zero, as $n \to \infty$. The extra factor x in the denominator makes things even worse and, for $x = 1/n$ we obtain

$$|f(1/n) - f_n(1/n)| = \frac{n}{1 + 1/n} = \frac{n^2}{n + 1} = \frac{(n + 1)^2 - 2(n + 1) + 1}{n + 1} = n - 1 + \frac{1}{n + 1} \geq n - 1.$$

This shows that we can never achieve

$$|f(1/n) - f_n(1/n)| < \epsilon$$

for all large enough $n \in \mathbb{N}$ and we do not have uniform convergence.

Problem 44 Formulate and prove Weierstrass' M-Test.

Solution: Weierstrass' M-Test reads as follows: Let $X \subset \mathbb{R}$ and $f_k : X \to \mathbb{R}$ be functions with $|f_k(x)| \leq M_k$ for all $x \in X$ with some suitable constants

$M_k \geq 0$. Let $g_n = \sum_{k=0}^{n} f_k$. Then the sequence (g_n) of functions converges uniformly to a limit function $g : X \to \mathbb{R}$ if we have $\sum_{k=0}^{\infty} M_k < \infty$.

Now we develop a proof for the Weierstrass M-Test: First of all, we need to construct a "pointwise" limit function $g : X \to \mathbb{R}$ of the sequence (g_n). For every fixed $x \in X$, we know that $\sum_{k=0}^{n} f_n(x)$ converges absolutely by comparison with $\sum_{k=0}^{n} M_k$, since $0 \leq |f_n(x)| \leq M_k$ and $\sum_{k=0}^{\infty} M_k < \infty$. Since absolute convergence implies ordinary convergence, we conclude that for every $x \in X$, we have convergence of $\sum_{k}^{\infty} f_k(x)$, and we denote the limit by $g(x)$. In this way, we construct the candidate for the uniform limit.

Now it remains to show that the so constructed function $g : X \to \mathbb{R}$ is, indeed, a uniform limit of $g_n = \sum_{k=0}^{n} f_n$. Let $\epsilon > 0$. Then we can find $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} M_k < \epsilon$ (since $\sum_{k=0}^{\infty} M_k < \infty$). Our aim is to show that we have for all $n \geq N$ and all $x \in (0, 1)$,

$$|g(x) - g_n(x)| < \epsilon.$$

Note that $g(x) = \sum_{k=0}^{\infty} f_k(x)$ and $g_n(x) = \sum_{k=0}^{n} f_k(x)$. This implies that, for all $n \geq N$,

$$|g(x) - g_n(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k \leq \sum_{k=N+1}^{\infty} M_k < \epsilon,$$

which finishes the proof.

Power Series

The following theorem is just for information. Using the concept of limes superior, it provides a formula which describes the radius of convergence of every power series. So far, we only had two methods to calculate the radius of convergence (based on the Ratio Test and the n-th Root Test), but there are still power series for which both methods fail to provide a definitive result for the radius of convergence. This was the case, since we did not have the concept of limes superior to state the complete result describing the radius of convergence for every possible power series. But as mentioned above, this result is just for curious students and not part of the official material covered in this course.

Theorem (Cauchy/Hadamard). *Let $\sum a_k z^k$ be a complex power series. If the sequence $(|a_k|^{1/k})$ is unbounded, then the radius of convergence is equal to zero: $R = 0$. Otherwise $(|a_k|^{1/k})$ is bounded and we have the limes superior $L = \limsup_{k\to\infty} |a_k|^{1/k} < \infty$. If $L = 0$ then the radius of convergence is equal to infinity: $R = \infty$. In the only remaining case $0 < L < \infty$, the radius of convergence is equal to $R = 1/L$.*

Problem 45 Calculate the first 5 terms (up to x^4) of the Taylor Series of $f(x) = e^{\sin(x)}$.

Solution: Recall that the Taylor series of a infinitely many times differentiable function f is given by $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. This formula is very important and should be known to everybody (by heart). It is not always the case that the Taylor Series represents the function, even not necessarily within its radius of convergence, and to check about this coincidence of function and its Taylor Series we need to investigate the so called remainder term, introduced in the

Calculus Course and to check whether the remainder term tends to zero for Taylor polynomials with increasing degrees. But this is another issue.

To calculate the first terms of the Taylor Series, we need to calculate the derivatives of f: We have

$$\begin{aligned}
f(x) &= e^{\sin(x)}, \\
f'(x) &= e^{\sin(x)}\cos(x), \\
f''(x) &= e^{\sin(x)}(\cos^2(x) - \sin(x)), \\
f^{(3)}(x) &= e^{\sin(x)}(\cos^3(x) - 3\sin(x)\cos(x) - \cos(x)), \\
f^{(4)}(x) &= e^{\sin(x)}(\cos^4(x) - 6\sin(x)\cos^2(x) + 3\sin^2(x) - 4\cos^2(x) + \sin(x)).
\end{aligned}$$

Plugging $x = 0$ into these expressions, we obtain

$$\begin{aligned}
f(0) &= e^0 = 1, \\
f'(0) &= e^0\cos(0) = 1, \\
f''(0) &= e^0(\cos^2(0) - \sin(0)) = 1, \\
f^{(3)}(0) &= e^0(\cos^3(0) - 3\sin(0)\cos(0) - \cos(0)) = 0, \\
f^{(4)}(0) &= e^0(\cos^4(0) - 6\sin(0)\cos^2(0) + 3\sin^2(0) - 4\cos^2(0) + \sin(0)) = -3.
\end{aligned}$$

So the first five terms of the Taylor series of $f(x) = e^{\sin(x)}$ are given by

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 \cdots$$

Problem 46 Evaluate by using well-known Taylor series:

$$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{4^n(2n+1)!}.$$

Solution: The give-away here is that we have alternating terms via $(-1)^n$ and the factorials $(2n+1)!$ in the denominator. This indicates that we should consider the Taylor series of $\sin(x)$, which is

$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

Note that the radius of convergence here is $R = \infty$ and that the Taylor series represents the function $\sin(x)$ on all of $\mathbb{R}$ (even on all of $\mathbb{C}$). The remaining factors $\pi/2$ and $\pi^{2n}/4^n$ can be combined to $(\pi/2)^{2n+1}$, and we end up with

$$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{4^n(2n+1)!} = \sin(\pi/2) = 1.$$