Analysis 1 Solutions (Epiphany Term 2015)

8 Differentiable functions

111. We have $f(x) = f(c) + (x-c)f_1(x)$ with f_1 continuous at c. Since f is differentiable at $x = c$, f is also continuous at $x = c$, i.e.,

$$f(x) \to f(c) \neq 0 \quad \text{for } x \to c,$$

and, therefore, for x near c we have $f(x) \neq 0$. This implies that

$$\frac{1}{f(x)} - \frac{1}{f(c)} = \frac{f(c) - f(x)}{f(x)f(c)} = \frac{1}{f(x)f(c)}(c-x)f_1(x).$$

Therefore, we have

$$\frac{1}{f(x)} = \frac{1}{f(c)} + (x-c)\left(-\frac{f_1(x)}{f(x)f(c)}\right) = \frac{1}{f(c)} + (x-c)f_2(x)$$

with $f_2(x) = -f_1(x)/(f(x)f(c))$. Then f_2 is continuous at $x = c$ as expression of continuous functions at $x = c$ and since $f(c) \neq 0$, which implies that $1/f(x)$ is differentiable at $x = c$ with derivative

$$f_2(c) = -\frac{f_1(c)}{f^2(c)} = -\frac{f'(c)}{f^2(c)}.$$

112. Problems Class, 30 January 2015

113. Since $\sin x$ is bounded, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} h\sin(1/h) = 0.$$

Therefore, the derivative of f is given by

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

If f' were continuous at $x = 0$, we would need to have

$$\lim_{x\to 0} 2x\sin(1/x) - \cos(1/x) = 0.$$

While we have $x\sin(1/x) \to)$ as $x \to 0$, $\cos(1/x)$ is not convergent (choose sequences $x_n \to 0$ having different constant values $\cos(1/x_n)$). Therefore, $f'(x)$ is not continuous at $x = 0$.

114. Let $f(x) = e^{-x} - \sin x$ and $a, b \in \mathbb{R}$ with $a < b$ be two real solutions of $e^x \sin x = 1$. This means that we have $f(a) = f(b) = 0$. Since f is differentiable, we can apply Rolle's Theorem and find $c \in (a, b)$ with $0 = f'(c) = -e^{-c} - \cos c$. Rewriting this equation yields $e^c \cos c = -1$.

115. It suffices to prove that $f_n^{(n)}$ has precisely n pairwise different zeroes in $(-1, 1)$. Firstly, we prove that $f_n^{(k)}$ has at least k pairwise different zeroes in $(-1, 1)$ for $k \in \{0, 1, 2, \ldots, n\}$. In the case $k = 0$ there is nothing to prove. Assume we have already shown that $f_n^{(k)}$ has at least k pairwise different zeroes $x_1 < x_2 < \cdots < x_k$ in $(-1, 1)$ for some $0 \le k \le n-1$. Note that $x^2 - 1$ divides $f_n^{(k)}$, so $f_n^{(k)}$ has zeroes

$$-1 = x_0 < x_1 < x_2 < \cdots < x_k < x_{k+1} = 1.$$

Applying Rolle's Theorem to every interval $[x_{i-1}, x_i]$ with $i = 1, 2, \ldots, k+1$, we obtain $k+1$ pairwise different zeroes $x_i' \in (x_{i-1}, x_i)$ of $f_n^{(k+1)}$. This shows that $f_n^{(n)}$ has at least n pairwise different zeroes in $(-1, 1)$. Since f_n is a nonzero polynomial of order $2n$, $f_n^{(n)}$ is a nonzero polynomial of order n and can have at most n pairwise different real roots. Combining both facts proves that p_n has precisely n pairwise different zeroes in $(-1, 1)$.

116. We have $f(2) = 4$, $f(5) = 25$ and $f'(c) = 4c - 7$. Then the Mean Value Theorem claims the existence of $c \in (2, 5)$ satisfying $4c - 7 = (25-4)/(5-2) = 7$. The solution of $4c - 7 = 7$ is $c = 3.5$ which lies in the interval $(2, 5)$, confirming the Mean Value Theorem in this case.

117. (a) Applying the classical Mean Value Theorem to $f(x) = \log(x)$, we obtain for some $c \in (1, b/a)$,

$$f(b/a) - f(1) = \log\left(\frac{b}{a}\right) - 0 = \log\left(\frac{b}{a}\right) = (b/a - 1)f'(c) = \frac{b-a}{ac}.$$

Since $1 < c < b/a$, we have $a/b < 1/c < 1$ and, therefore,

$$1 - \frac{a}{b} = \frac{b-a}{b} = \frac{a}{b}\frac{b-a}{a} < \frac{b-a}{ac} = \log\left(\frac{b}{a}\right) < \frac{b-a}{a} = \frac{b}{a} - 1.$$

(b) Choose $a = 5$ and $b = 6$ to obtain

$$\frac{1}{6} = 1 - \frac{5}{6} < \log\left(\frac{6}{5}\right) = \log(1.2) < \frac{6}{5} - 1 = \frac{1}{5}.$$

118. Let $a \le x < y \le b$. Then by the classical Mean Value Theorem there exists $z \in (x, y)$ such that $f'(z) = (f(y) - f(x))/(y - x)$.
(a) Suppose that $f' \equiv 0$ on (a, b). Then $f'(z) = 0$, so $f(x) = f(y)$: i.e. f is constant on (a, b).
(b) Suppose that $f' > 0$ on (a, b). Then $f'(z) > 0$, and so $f(y) > f(x)$. I.e. f is increasing.
(c) Now suppose that $t \le f' \le T$ on (a, b). Then again $t \le f'(z) \le T$, and the result follows.

119. (a) We first check that

$$\sinh'(x) = \frac{e^x - (-1)e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

and

$$\cosh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} =$$

$$1 + \frac{e^{2x} - 2 + e^{-2x}}{4} = 1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1 + \sinh^2(x).$$

Since $\cosh(x) = (e^x + e^{-x})/2 > 0$, we know that $\sinh(x)$ is strictly monotone increasing. Let $y = \sinh(x)$. This implies that $2y = e^x - e^{-x}$ and, multiplying by e^x:

$$e^{2x} - 2ye^x - 1 = 0.$$

Let $c = e^x > 0$. Solving $c^2 - 2yc - 1 = 0$ leads to

$$c = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}.$$

Since $c > 0$, the only solution is

$$e^x = c = y + \sqrt{y^2 + 1},$$

i.e.,

$$x = \log(y + \sqrt{y^2 + 1}).$$

This shows that $\operatorname{Ar}\sinh(y) = \log(y + \sqrt{y^2 + 1})$. Now we differentiate and obtain

$$\operatorname{Ar}\sinh'(y) = \frac{1}{y + \sqrt{y^2 + 1}} \left(1 + \frac{2y}{2\sqrt{y^2 + 1}}\right) =$$

$$\frac{1}{y + \sqrt{y^2 + 1}} \left(1 + \frac{y}{\sqrt{y^2 + 1}}\right) = \frac{1}{y + \sqrt{y^2 + 1}} \frac{\sqrt{y^2 + 1} + y}{\sqrt{1 + y^2}} = \frac{1}{\sqrt{1 + y^2}}.$$

(b) Using (1) in Exercise 112 and $\cosh(x) = \sqrt{1 + \sinh^2(x)}$ yields

$$\operatorname{Ar}\sinh'(y) = \frac{1}{\cosh(\operatorname{Ar}\sinh(y))} = \frac{1}{\sqrt{1 + \sinh^2(\operatorname{Ar}\sinh(y))}} = \frac{1}{\sqrt{1 + y^2}}.$$

120. (a) Using the classical Mean Value Theorem, we obtain for $0 < a < b$ and some $c \in (a, b)$:

$$\arctan(b) - \arctan(a) = \frac{(b - a)}{1 + c^2}.$$

Since $1 + a^2 < 1 + c^2 < 1 + b^2$, we conclude that

$$\frac{b - a}{1 + b^2} < \arctan(b) - \arctan(a) < \frac{b - a}{1 + a^2}.$$

(b) Choosing $0 < a = 1 < b = 4/3$, we obtain

$$\frac{1/3}{1+16/9} < \arctan(4/3) - \arctan(1) < \frac{1/3}{2}.$$

Since $\arctan(1) = \pi/4$, we end up with

$$\frac{3}{25} = \frac{1}{3+16/3} < \arctan(4/3) - \frac{\pi}{4} < \frac{1}{6}.$$

121. We assume that $f, g : (a, b) \to \mathbb{R}$ are differentiable, $c \in (a, b)$, $f(c) = g(c) = 0$ and that $\lim_{x\to c} f'(x)/g'(x)$ exists. Using the formula, we find some $\xi \in (x, c)$ (if $x < c$) or $\xi \in (c, x)$ (if $c < x$) such that

$$g(x)f'(\xi) = (g(x) - g(c))f'(\xi) = (f(x) - f(c))g'(\xi) = f(x)g'(\xi). \qquad (1)$$

The assumption that $\lim_{x\to c} f'(x)/g'(x)$ exists implies that we have for all $x \neq c$, sufficiently close to c, $g'(x) \neq 0$. Applying (1) to those x, we also have $g'(\xi) \neq 0$, since $\xi \neq c$ is even closer to c than x. Moreover, using the classical Mean Value Theorem, we have

$$g(x) = g(x) - g(c) = (x - c)g'(\eta) \neq 0$$

for some η strictly between x and c, and we can therefore divide (1) by $g(x)g'(\xi) \neq 0$ and obtain

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}.$$

Now, if $x \to c$, $x \neq c$, we also have $\xi \to c$, $\xi \neq c$, and therefore,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{\xi\to c} \frac{f'(\xi)}{g'(\xi)},$$

showing that the limit $\lim_{xtoc} f(x)/g(x)$ must exist and must agree with the well-defined limit $\lim_{x\to c} f'(x)/g'(x)$.

122. Let $f(x) = 1 + \cos(\pi x)$ and $g(x) = x^2 - 2x + 1$. Then $f(1) = g(1) = 0$ and $f'(x) = -\pi \sin(\pi x)$ and $g'(x) = 2x - 2$. Then $f'(1) = g'(1) = 0$ and $f''(x) = -\pi^2 \cos(\pi x)$ and $g''(x) = 2$. Then

$$\lim_{x\to 1} \frac{f''(x)}{g''(x)} = \lim_{x\to 1} \frac{-\pi^2 \cos(\pi x)}{2} = \frac{\pi^2}{2}.$$

Applying L'Hoṗital twice, we obtain

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{f''(x)}{g''(x)} = \frac{\pi^2}{2}.$$

123. Let $f(x) = x - \sin x$ and $g(x) = x^3$. Then $f(0) = 0 = g(0) = 0$ and $f'(x) = 1 - \cos x$ and $g'(x) = 3x^2$. Then $f'(0) = g'(0) = 0$ and $f''(x) = \sin x$ and $g''(x) = 6x$. Then $f''(0) = g''(x) = 0$ and $f^{(3)}(x) = \cos x$ and $g^{(3)}(x) = 6$. Then

$$\lim_{x\to 0} \frac{f(3)(x)}{g^{(3)}(x)} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

Applying L'Hopital three times, we obtain

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)} = \lim_{x\to 0}\frac{f(3)(x)}{g^{(3)}(x)} = \frac{1}{6}.$$

124. We have

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{g(x)}{x^2}.$$

Let $h(x) = x^2$. Then $g(0) = h(0) = 0$ and $h'(x) = 2x$. Then $g'(0) = h'(0) = 0$ and $h''(x) = 2$. Applying L'Hôpital twice, we obtain

$$f'(0) = \lim_{x\to 0}\frac{g(x)}{h(x)} = \lim_{x\to 0}\frac{g'(x)}{h'(x)} = \lim_{x\to 0}\frac{g''(x)}{h''(x)} = \frac{17}{2}.$$

125. Let $f(x) = 5\sin x - 4x$. Then $f'(x) = 5\cos x - 4$ and Newton's iteration is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{5\sin x - 4x}{5\cos x - 4}.$$

We start with $x_1 = 1$ and obtain successively

$$\begin{aligned} x_2 &= 1 - \frac{5\sin(1)-4}{5\cos(1)-4} = 1.15969\ldots, \\ x_3 &= 1 - \frac{5\sin(x_2)-4x_2}{5\cos(x_2)-4} = 1.13203\ldots, \\ x_4 &= 1 - \frac{5\sin(x_3)-4x_3}{5\cos(x_3)-4} = 1.13110\ldots. \end{aligned}$$

We check that

$$f(1.131) = 0.000192\ldots \quad \text{and } f(1.132) = -0.001682\ldots,$$

which means that there must be a zero within the interval $(1.131, 1.132)$ by the Intermediate Value Theorem.

9 Infinite series

126. $(2+n)/\sqrt{4n^4-1} > n/\sqrt{4n^4} = (2n)^{-1}$, and $\Sigma(2n)^{-1}$ diverges; so the given series diverges, by comparison.

127. $\sqrt{n}/(n^3+1) < n^{-5/2}$, and $\Sigma n^{-5/2}$ converges; so given series converges, by comparison.

128. $|\sin(2^n)/2^n| \le 2^{-n}$ and $\sum_{n=1}^{\infty} 2^{-n}$ converges, so the given series converges absolutely, by comparison.

129. Write $x_n = (n-3)(2+9n^6)^{-1/2}$. Note that $0 \le x_n < n/\sqrt{9n^6} = 1/(3n^2)$, and $\sum_{n=1}^{\infty} 1/(3n^2)$ converges; so the given series converges, by comparison.

130. (a) $0 \le x_n \le 1/n^2$, so the series converges.
(b) $x_n \ge \frac{1}{2n}$, so the series diverges.
(c) For $n > 2$, we have $|x_n| \le n^{-9/2}$, so series converges absolutely.
(d) $x_n = \frac{n^2}{(n+1)(n+2)(n+3)} \ge \frac{1}{n}\left(\frac{n}{n+3}\right)^3 \ge \frac{1}{n}\frac{1}{4^3}$, so the series is divergent.
(e) Since $x^8 \exp(-x) \to 0$ as $x \to \infty$, the set $\{n^4 \exp(-\sqrt{n})\}$ is bounded above, say by K. So $0 < x_n < K/n^2$. Thus the given series converges, by comparison with the convergent series $\sum K/n^2$.
(f) $|x_n| \le n^{-2}$, so the series is absolutely convergent.
(g) $\sin\theta < \theta$ for $\theta > 0$, so $0 < x_n < n^{-2}$ for $n \ge 1$. Since $\sum n^{-2}$ converges, so does $\sum x_n$, by comparison.
(h) Since $n^{-1/2}(\log n)^4 \to 0$ as $n \to \infty$, the set $\{n^{-1/2}(\log n)^4\}$ is bounded above, say by K. So $0 < x_n < K/n^{3/2}$. Thus the given series converges, by comparison with the convergent series $\sum K/n^{3/2}$.
(i) $x_n = 1/(\sqrt{1+n^2}+n) \ge 1/(n+\sqrt{2n^2}) \ge 1/n(1+\sqrt{2})$, so the series is divergent.

131. (a) $n\log(1+\frac{1}{n}) \to 1$ as $n \to \infty$, so there exists K such that $x_n = (n^2+1)^{-\alpha}\log(1+\frac{1}{n}) \le Kn^{-2\alpha-1}$; hence the series is convergent for $\alpha > 0$, by comparison with $\sum n^{-2\alpha-1}$. For $\alpha \le 0$, we can say that for n large enough, $x_n > \frac{1}{2}\frac{1}{n(1+n^2)^\alpha} > \frac{1}{2^{1+2\alpha}}\frac{1}{n^{1+2\alpha}} \ge \frac{1}{2^{1+2\alpha}}\frac{1}{n}$;
so the series is divergent, by comparison with $\sum 1/n$.
(b) $x_n = n^\alpha\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \frac{n^\alpha(\sqrt{n+1}-\sqrt{n})}{\sqrt{n(n+1)}} = \frac{n^\alpha}{\sqrt{n(n+1)}(\sqrt{n+1}+\sqrt{n})}$.
Now $\frac{n^{3/2}}{\sqrt{n(n+1)}(\sqrt{n+1}+\sqrt{n})} \to \frac{1}{2}$ as $n \to \infty$, so that, by comparison, the series converges for $\alpha - \frac{3}{2} < -1$, i.e. for $\alpha < \frac{1}{2}$.

132. The series has partial sums $x_1 - x_2$, $x_1 - x_3$, $x_1 - x_4$, $\dots$, and the result follows.

133. Define the partial sums $X_n = \sum_{k=1}^n x_k$ and $Y_n = \sum_{k=1}^n y_k$. Then $X_n \to s$ as $n \to \infty$. But $Y_n = \frac{1}{2}X_n + \frac{1}{2}(X_{n+1} - x_1)$, so $Y_n \to s - x_1/2$.

134. Since $\sum x_n$ converges, $x_n \to 0$ as $n \to \infty$, and so there exists K such that $|x_n| \le K$ for all n. But then $|x_n y_n| \le K|y_n|$, so that $\sum x_n y_n$ converges absolutely by comparison with $\sum |y_n|$. Conditional convergence of $\sum y_n$ is not enough. For example, consider $x_n = y_n = (-1)^n n^{-1/2}$. Then $\sum x_n$ and $\sum y_n$ are convergent, by the alternating series test, but $\sum x_n y_n$ is the harmonic series and is divergent.

135. (a) The tan function is increasing on $[0, \pi/2)$, so $\{\tan(\pi/n)\}$ is a decreasing sequence for $n \ge 3$; its limit is $\tan 0 = 0$. Also $\cos(n\pi) = (-1)^n$ — so by the Alternating Sign Test, the series converges.
(b) Write $f(x) = 1/[x(\log x)^3]$ on $[2, \infty)$. Then f is a positive decreasing function, and $\int_2^M f(x)\,dx = -\frac{1}{2}\left[(\log x)^{-2}\right]_2^M = \frac{1}{2}(\log 2)^{-2} - \frac{1}{2}(\log M)^{-2} \to \frac{1}{2}(\log 2)^{-2}$
as $n \to \infty$. Hence $\sum_{n=2}^\infty f(n)$ converges, by the Integral Test.
(c) Write $x_n = (2n)!\,5^{-n}(n!)^{-2}$. Then

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(2n+2)(2n+1)}{5(n+1)^2} \to \frac{4}{5} \quad \text{as} \quad n \to \infty.$$

Since $4/5 < 1$, we conclude that $\sum_{n=1}^{\infty} x_n$ converges, by the Ratio Test.

136. (a) Write $x_n = 1/[\sqrt{n}\tanh(n)]$. Both $\sqrt{n}$ and $\tanh(n)$ are increasing sequences, so $\{x_n\}$ is decreasing. Also, $x_n \to 0$ as $n \to \infty$. So by the Alternating Sign Test, the given series converges.
(b) Write $x_n = (2/9)^n(2n)!/(n!)^2$. Then

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{2(2n+2)(2n+1)}{9(n+1)^2} \to \frac{8}{9}$$

as $n \to \infty$. So by the Ratio Test, $\sum x_n$ converges.
(c)

$$0 \le \frac{n-1}{(n^2+2)(n^2+1)^{1/4}} < \frac{n}{n^2 n^{1/2}} = \frac{1}{n^{3/2}}$$

and $\sum n^{-3/2}$ converges, so the given series converges by comparison.

137. (a) The ratio of successive terms is $\frac{((n+1)!)^2(2n)!}{(n!)^2(2n+2)!} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{4n+2} \to \frac{1}{4}$ as $n \to \infty$, so convergent by Ratio Test.
(b) $\sum_{n=1}^{\infty} x_n$ is the same series as $\sum_{n=2}^{\infty} \frac{1}{n\log n}$ Since $f(x) = 1/(x\log x)$ is decreasing on $[2,\infty)$, and $\int_2^M f = \log\log M - \log\log 2$ is unbounded as $M \to \infty$, the series diverges (Integral Test).
(c) $\cos(\pi n) = (-1)^n$, so that we have an alternating series. Thus the Alternating Sign Test tells us that for convergence it is sufficient to have $|x_n| \to 0$ monotonically as $n \to \infty$, which certainly is the case here.

138. (a) Ratio test: $|x_{n+1}/x_n| = |\alpha|(1+1/n)^\alpha \to |\alpha|$ as $n \to \infty$. So series converges if $|\alpha| < 1$ and diverges if $|\alpha| > 1$. If $\alpha = 1$ then $x_n = n$ clearly divergent; while if $\alpha = -1$ then $x_n = (-1)^n/n$ which gives an alternating series which converges since $\{1/n\}$ is a decreasing sequence tending to zero. So we have convergence iff $-1 \le \alpha < 1$.

(b) The terms of the series vanish as $n \to \infty$ (and so the series can converge) only for $|\alpha| \le 3$. When $\alpha = 3$, the series is a harmonic series and diverges. When $\alpha = -3$ the series converges by the Alternating Sign Test. When $|\alpha| < 3$, the series is absolutely convergent by comparison with the convergent geometric series $\sum(\alpha/3)^n$.

(c) By the comparison test, $\sum x_n$ converges if and only if $\sum(n+1)^{-1}(\log(n+1))^{-\alpha}$ does. Since $f(x) = (x+1)^{-1}(\log(x+1))^{-\alpha}$ is decreasing on $[1,\infty)$ for all α, we can apply the Integral Test. The $\alpha = 1$ case was covered in Problem 137(b); for $\alpha \ne 1$ we have $(1-\alpha)\int_1^M f(x)\,dx = [\log(M+1)]^{1-\alpha} - [\log 2]^{1-\alpha}$. This has a limit as $M \to \infty$, and hence $\sum x_n$ converges, if and only if $\alpha > 1$.

(d) $|x_{n+1}/x_n| = (n+1)|\alpha|$; if $\alpha \ne 0$, then this ratio tends to infinity as $n \to \infty$, so the series diverges by the Ratio Test. If $\alpha = 0$, then the series clearly converges.
(e) We have $x_n = (\alpha/2)^n/(3 - 1/n)$ and since $2 \le 3 - 1/n \le 3$, the series converges if and only if the geometric series $\sum(\alpha/2)^n$ converges (by the Comparison Test), and this converges for $|\alpha| < 2$.

139. (a) The series is absolutely convergent for any z by the Ratio Test.
(b) The series is absolutely convergent for any z by the Ratio Test.
(c) The series is a geometric series and is convergent if and only if $|zc| < 1$.
(d) The ratio test implies that the series is absolutely convergent when $|z| < 1$, and the vanishing condition implies that it is divergent otherwise.
(e) Since $\alpha^n/n! \to 0$ as $n \to \infty$ for any $\alpha \in \mathbb{R}$, the terms of this series do not vanish for any $z \neq 0$, and so the series is divergent for all $z \neq 0$.

140. (a) Write $x_n = n^2\, 2^{-n}$. Then $|x_{n+1}/x_n| = (1 + 1/n)^2/2 \to 1/2$ as $n \to \infty$. So the series converges, by the Ratio Test.
(b) Write $x_n = [1+\exp(-n)]/[(n+1)^2-(n-1)^2]$. Then $x_n = (1+\mathrm{e}^{-n})/(4n) > 1/(4n)$, and $\Sigma(4n)^{-1}$ diverges; hence the given series diverges, by comparison.
(c) Write $x_n = n^{-2}\log n$. Since $n^{-1/2}\log n \to 0$ as $n \to \infty$, there exists a number K such that $\log n \le K\sqrt{n}$ for all n. Thus $0 \le n^{-2}\log n \le Kn^{-3/2}$, and $\Sigma K n^{-3/2}$ converges; so the given series converges by comparison.
(d) Write $x_n = n!\, 2^n\, n^{-n}$. Then $|x_{n+1}/x_n| = 2[n/(n+1)]^n = 2/(1 + 1/n)^n \to 2/\mathrm{e}$ as $n \to \infty$. Since $2/\mathrm{e} < 1$, the Ratio Test says that Σx_n converges.

141. We use the n^{th} Root Test. Let

$$a_n = \left[n^4 \sin^2\left(\frac{2n}{3n^3 - 2n^2 + 5}\right)\right]^n.$$

Then we have

$$|a_n|^{1/n} = n^4 \sin^2\left(\frac{2n}{3n^3 - 2n^2 + 5}\right).$$

Note that $(2n)/(3n^3 - 2n^2 + 5) \to 0$ as $n \to \infty$, which implies that

$$\lim_{n\to\infty} \frac{\sin^2((2n)/(3n^3 - 2n^2 + 5))}{(2n)^2/(3n^3 - 2n^2 + 5)^2} = 1,$$

using $\sin(x)/x \to 1$ as $x \to 0$. This means we obtain

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n^4(2n)^2}{(3n^3 - 2n^2 + 5)^2} \frac{\sin^2((2n)/(3n^3 - 2n^2 + 5))}{(2n)/(3n^3 - 2n^2 + 5)} = \frac{4}{9} < 1.$$

The n^{th} root test tells us that the series converges.

142. We consider the series $\sum(3n - 1)!/(3n)!$ and $\sum 4^{n+1}/(3n)!$ separately. The first series $\sum 1/(3n)$ is equal to 1/3 times the harmonic series, which diverges. We apply the Ratio Test to the second series $\sum 4^{n+1}/(3n)!$:

$$\frac{4^{n+2}\cdot(3n)!}{(3n+3)!\cdot 4^{n+1}} = \frac{4}{(3n+1)(3n+2)(3n+3)} \to 0 \quad \text{as } n \to \infty.$$

This shows that the second series is convergent. If the original series were convergent, then the series $\sum(3n-1)!/(3n)!$ were also convergent as the sum of the original series and the series $\sum 4^{n+1}/(3n)!$, by COLT. But $\sum(3n-1)!/(3n)!$ is divergent. Therefore this series is divergent.

143. Let $s_N = \sum_{n=2}^{N} \frac{(-1)^n}{n+(-1)^n}$. Note that we have

$$s_{2N+1} = \sum_{n=2}^{2N} \frac{(-1)^n}{n+(-1)^n} = \sum_{k=1}^{N} \frac{(-1)^{2k}}{2k+(-1)^{2k}} + \frac{(-1)^{2k+1}}{2k+1+(-1)^{2k+1}} = -\sum_{k=1}^{N} \frac{1}{2k(2k+1)}.$$

Therefore, the partial sums s_{2N+1} converge, by Comparison with the convergent series $\sum_k 1/(4k^2)$. Let $s^* = \lim_{N\to\infty} s_{2N+1}$. Then we also have

$$\lim_{N\to\infty} s_{2N} = \lim_{N\to\infty} s_{2N+1} + \frac{1}{2N} = s^*,$$

and the sequence (s_n) of all partial sums converges. This shows convergence of the series.

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146. Assume that $\sum a_n^+$ contains only finitely many nonzero elements. Then this sum is convergent and also absolutely convergent, since it only contains nonnegative elements. Applying COLT to $\sum a_n - \sum a_n^+$ would then show that also $\sum a_n^-$ is convergent and, therefore, also absolutely convergent, since it only contains nonpositive elements. But then also the sum $\sum a_n = \sum a_n^+ + \sum a_n^-$ would be absolutely convergent, in contradiction to the assumption that $\sum a_n$ is only conditionally convergent. This shows that $\sum a_n^+$ contains infinitely many nonzero elements and a similar reasoning shows that also the $\sum a_n^-$ has infinitely many nonzero elements. Assume that at least one of the sums $\sum a_k^+, \sum a_k^-$ were convergent. Let $\sum a_k^+$ be convergent. Then $\sum a_k^+$ is also absolutely convergent (only nonnegative terms) and then also $\sum a_k^- = \sum a_k - \sum a_k^+$ is also convergent, by COLT. But then $\sum a_k^-$ would be also absolutely convergent (only nonpositive terms) and we would, again, obtain that $\sum a_k = \sum a_k^+ + \sum a_k^-$ were absolute convergent, which is again a contradiction. So both series $\sum a_k^+$ and $\sum a_k^-$ must be divergent and, therefore, the partial sums must be unbounded.

147. The crucial point that we can establish the inequality $U_1 \geq s^*$ is that $\sum_{k\geq 1} a_k^+$ is monotone increasing and unbounded above. The crucial point that we can then establish the inequality $U_1 + L_1 < s^*$ is that $\sum_{k\geq 1} a_k^-$ is monotone decreasing an unbounded below. Next, we can find a smallest index n_2 such that $U_1 + L_1 + \sum_{k=n_1+1}^{n_2} a_k^+ \geq s^*$, since $\sum_{k\geq n_1+1} a_k^+$ is still unbounded above. We define

$$U_2 = a_{n_1+1}^+ + a_{n_1+2}^+ + \cdots + a_{n_2}^+.$$

Next, we can find a smallest index m_2 such that $U_1+L_1+U_2+\sum_{k=m_1+1}^{m_2} a_k^- < s^*$, since $\sum_{k\geq m_1+1} a_k^-$ is still unbounded below. We define

$$L_2 = a_{m_1+1}^- + a_{m_1+2}^- + \cdots + a_{m_2}^-.$$

It is clear how this method proceeds and that the process never stops, since we have always unbounded series $\sum_{k\geq n_j+1} a_k^+$ and $\sum_{k\geq m_j+1} a_k^-$ left. Note also that, by construction, we have

$$|s^* - (U_1 + L_1 + \cdots + U_k)| \leq a_{n_k}^+$$

and

$$|s^* - (U_1 + L_1 + \cdots + U_k + L_k)| \le |a^-_{m_k}|.$$

Since $\sum a_n$ is convergent, we have $a_n \to 0$ and this implies that also $a_n^+ \to 0$ and $a_n^- \to 0$. This final fact shows that we have convergence $s_k^U \to s^*$ and $s_k^L \to s^*$.

148. We know that the series $\sum \frac{(-1)^k}{\sqrt{k+1}}$ is convergent by the Alternating Sign Test. Since $1/\sqrt{k+1} \ge 1/(k+1)$, divergence of $\sum \frac{1}{\sqrt{k+1}}$ follows from Comparison with the harmonic series. This shows that $\sum \frac{(-1)^k}{\sqrt{k+1}}$ is only conditionally convergent. For the Cauchy product, we have to consider the terms

$$c_k = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

It is easy to see that we have $\sqrt{(k+1)(n-k+1)} \le n+1$ and, therefore,

$$|c_k| \ge \sum_{k=0}^{n} \frac{1}{n+1} = 1.$$

So $\sum c_k$ cannot converge since then we would have $c_k \to 0$ in contrast to $|c_k| \ge 1$.

149. (a) $|z_n| = 1/\sqrt{n^4+1} \to 0$ as $n \to \infty$, so $z_n \to 0$.
(b) $|z_n| = n^2 \exp(-n) \to 0$ as $n \to \infty$, so $z_n \to 0$.
(c) By COLT, $z_n \to \exp(i\pi/4)\sqrt{2} = 1 + i$ as $n \to \infty$.
(d) $z_n = (-1)^n x_n$, where $x_n = 2n/(n+i) \to 2$ as $n \to \infty$, so $\{z_n\}$ has no limit (but is bounded).

150. (a) $\mathrm{Re}(z_n) = n/(n^2+1) \ge (2n)^{-1}$, and $\Sigma(2n)^{-1}$ diverges, so $\Sigma\,\mathrm{Re}(z_n)$ diverges by comparison, and hence Σz_n diverges.
(b) $|z_n| = 1/\sqrt{n^4+1} < n^{-2}$, and Σn^{-2} converges, so Σz_n converges absolutely, by comparison.
(c) $|z_{n+1}/z_n| = \sqrt{29}/(n+1) \to 0$ as $n \to \infty$, so Σz_n converges absolutely, by the Ratio Test.
(d) $n^2|z_n| = n^2(n^2+4)^2 \exp(-n) \to 0$ as $n \to \infty$, so there exists K such that $0 < n^2|z_n| < K$ for all n. Hence Σz_n converges absolutely, by comparison with the convergent series $\Sigma K n^{-2}$.

10 Integrals

151. (a) Since f is decreasing on $[0,1]$, we have $U(f,\mathcal{P}_n) = n^{-1}\left(1 + \mathrm{e}^{-1/n} + \mathrm{e}^{-2/n} + \cdots + \mathrm{e}^{-(n-1)/n}\right)$ and $L(f,\mathcal{P}_n) = n^{-1}\left(\mathrm{e}^{-1/n} + \mathrm{e}^{-2/n} + \cdots + \mathrm{e}^{-1}\right)$.
(b) Then $U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = n^{-1}(1 - \mathrm{e}^{-1})$, and this $\to 0$ as $n \to \infty$, so f is Riemann integrable.
(c) $\int_0^1 \mathrm{e}^{-x}\,dx = 1 - \mathrm{e}^{-1}$. $L(f,\mathcal{P}_n) = \alpha n^{-1}(1 + \alpha + \cdots + \alpha^{n-1})$, where $\alpha = \exp(-1/n)$, so $L(f,\mathcal{P}_n) = [\alpha(1-\alpha^n)]/[n(1-\alpha)] = [\alpha(1-\mathrm{e}^{-1})]/[n(1-\alpha)] = [(1-\mathrm{e}^{-1})]/[n(\mathrm{e}^{1/n}-1)]$. The result follows.

152. $U(f,\mathcal{P}_n) = \frac{1}{n}\left(\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \cdots + \log(2)\right)$, and $L(f,\mathcal{P}_n) = \frac{1}{n}\left(0 + \log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \cdots + \log\left(1+\frac{n-1}{n}\right)\right)$. Then $U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = \frac{\log 2}{n} \to 0$ as $n \to \infty$, so that f is Riemann integrable on $[1,2]$.
Now the integral is $I = \int_1^2 f(x)\,dx = 2\log 2 - 1$. Moreover, $L(f,\mathcal{P}_n) \leq I \leq U(f,\mathcal{P}_n) = L(f,\mathcal{P}_n) + \frac{\log 2}{n}$, so that $I - \frac{\log 2}{n} \leq L(f,\mathcal{P}_n) \leq I$, and then $\lim_{n\to\infty} L(f,\mathcal{P}_n) = I$ by the Squeezing Theorem. The final result follows by taking the exponential of both sides: $\exp(L(f,\mathcal{P}_n)) = \left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n-1}{n}\right)\right)^{1/n}$, and $\exp(I) = 4/\mathrm{e}$.

153. We have $f(x) = 1/x$. $U(f,\mathcal{P}_2) = \frac{1}{2}(1+\frac{2}{3}) = \frac{5}{6}$, and $L(f,\mathcal{P}_2) = \frac{1}{2}(\frac{2}{3}+\frac{1}{2}) = \frac{7}{12}$. $U(f,\mathcal{P}_4) = \frac{1}{4}(1+\frac{4}{5}+\frac{2}{3}+\frac{4}{7}) = \frac{319}{420}$, and $L(f,\mathcal{P}_4) = \frac{1}{4}(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}+\frac{1}{2}) = \frac{533}{840}$. Expressing the results to 4 decimal places, we have

$I - L(f,\mathcal{P}_2)$	-0.1402
$I - U(f,\mathcal{P}_2)$	0.1098
$I - L(f,\mathcal{P}_4)$	0.0586
$I - U(f,\mathcal{P}_4)$	-0.0664

154. Let $\mathcal{P}_n$ be the partition of $[0,\pi/2]$ into n subintervals of equal length. Then we can write

$$\frac{\pi}{2n}\left(\sin(\frac{\pi}{2n}) + \sin(\frac{2\pi}{2n}) + \sin(\frac{3\pi}{2n}) + \cdots + \sin(\frac{n\pi}{2n})\right) = U(f,\mathcal{P}_n)$$

with $f(x) = \sin(x)$. Note that

$$L(f,\mathcal{P}_n) = \frac{\pi}{2n}\left(\sin(\frac{\pi}{2n}) + \sin(\frac{2\pi}{2n}) + \sin(\frac{3\pi}{2n}) + \cdots + \sin(\frac{(n-1)\pi}{2n}\right) = U(f,\mathcal{P}_n) - \frac{\pi}{2n}\sin(\frac{n\pi}{2n}),$$

i.e.,

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = \frac{\pi}{2n}\sin(\frac{\pi}{2}) = \frac{\pi}{2n} \to 0.$$

Therefore, we have

$$\lim_{n\to\infty}\frac{\pi}{2n}\left(\sin(\frac{\pi}{2n}) + \sin(\frac{2\pi}{2n}) + \sin(\frac{3\pi}{2n}) + \cdots + \sin(\frac{n\pi}{2n})\right) = \lim_{n\to\infty} U(f,\mathcal{P}_n) =$$
$$\int_0^{\pi/2}\sin(x)dx = [-\cos x]_0^{\pi/2} = 1,$$

which implies

$$\lim_{n\to\infty}\frac{1}{n}\left(\sin(\frac{\pi}{2n}) + \sin(\frac{2\pi}{2n}) + \sin(\frac{3\pi}{2n}) + \cdots + \sin(\frac{n\pi}{2n})\right) = \frac{2}{\pi}.$$

155. Problems Class, 12 February 2015

156. We use the criterion given in Theorem 10.4. First of all, every uniformly continuous function $f : [a,b] \to \mathbb{R}$ is obviously continuous and, therefore,

bounded. Let $\epsilon > 0$. Then we can find $\delta > 0$ such that we have, for all $x, y \in [a, b]$ with $|y - x| < \delta$,

$$|f(y) - f(x)| < \frac{\epsilon}{b-a}.$$

Now we choose $n \in \mathbb{N}$ large enough such that $(b-a)/n < \delta$. Let $\mathcal{P}_n$ be the partition of the interval $[a, b]$ into n subintervals of equal length. Then we have

$$L(f, \mathcal{P}_n) = \frac{b-a}{n} \sum_{i=1}^{n} m_i$$

and

$$U(f, \mathcal{P}_n) = \frac{b-a}{n} \sum_{i=1}^{n} M_i,$$

with

$$m_i = \inf\{f(x) \mid a + (i-1)\frac{b-a}{n} \leq x \leq a + i\frac{b-a}{n}\} = f(\xi_i)$$

and

$$M_i = \sup\{f(x) \mid a + (i-1)\frac{b-a}{n} \leq x \leq a + i\frac{b-a}{n}\} = f(\eta_i).$$

We obviously have $\xi_i, \eta_i \in [a, b]$ and $|\eta_i - \xi_i| \leq (b-a)/n < \delta$. Therefore, we conclude that

$$M_i - m_i = |f(\eta_i) - f(\xi_i)| < \frac{\epsilon}{b-a},$$

i.e.,

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = \frac{b-a}{n} \sum_{i=1}^{n} M_i - m_i < \frac{b-a}{n} \cdot n \cdot \frac{\epsilon}{b-a} = \epsilon.$$

But this implies that f is Riemann integrable.

157. Problems Class, 12 February 2015

158. Using for $a < b$ that $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$, we obtain

$$\left| \int_0^{2\pi} \frac{\sin(kx)}{x^2 + k^2} dx \right| \leq \int_0^{2\pi} \left| \frac{\sin(kx)}{x^2 + k^2} \right| dx \leq \int_0^{2\pi} \frac{1}{k^2} dx = \frac{2\pi}{k^2} \to 0.$$

159. Using for $a < b$ that $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$, we obtain

$$\left| \int_1^{\sqrt{3}} \frac{e^{-x}\sin(x)}{x^2 + 1} dx \right| \leq \int_1^{\sqrt{3}} \frac{e^{-x}}{1 + x^2} dx \leq e^{-1} \int_1^{\sqrt{3}} \frac{dx}{1 + x^2} =$$

$$\frac{1}{e}(\arctan(\sqrt{3}) - \arctan(1)) = \frac{1}{e}(\pi/3 - \pi/4) = \frac{1}{12e}\pi.$$

160. (a) Let $r = x_i/x_{i-1}$. Then we have $x_k/x_0 = r^k$ and, therefore, $r^n = b/a$. Let $c = r^n$. Then

$$x_i = x_0 \cdot \frac{x_i}{x_0} = a \cdot r^i = ac^{i/n}.$$

(b) Note that we have

$$U(f,\mathcal{P}_n) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

and

$$L(f,\mathcal{P}_n) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Moreover, we have

$$x_i - x_{i-1} = ac^{i/n} - ac^{(i-1)/n} = ac^{(i-1)/n}(c^{1/n} - 1).$$

Using $f(x_i) = (x_i)^p = a^p c^{ip/n}$, this implies that

$$U(f,\mathcal{P}_n) = \sum_{i=1}^{n} a^p c^{ip/n} ac^{(i-1)/n}(c^{1/n} - 1) = a^{p+1}(1 - c^{-1/n}) \sum_{i=1}^{n} \left(c^{(p+1)/n}\right)^i .$$

Now we use the formula for the geometric series $\sum_{i=1}^{n} \alpha^i = \alpha \frac{1-\alpha^n}{1-\alpha}$ and obtain

$$U(f,\mathcal{P}_n) = a^{p+1}(1 - c^{-1/n})c^{(p+1)/n} \frac{1 - c^{p+1}}{1 - c^{(p+1)/n}} =$$

$$a^{p+1}(1 - c^{p+1})c^{(p+1)/n} \frac{1 - c^{-1/n}}{1 - c^{(p+1)/n}} = (a^{p+1} - b^{p+1})c^{(p+1)/n} \frac{1 - c^{-1/n}}{1 - c^{(p+1)/n}} =$$

$$(b^{p+1} - a^{p+1})c^{p/n} \frac{1 - c^{1/n}}{1 - c^{(p+1)/n}}.$$

Using the formula for the geometric series $\sum_{i=0}^{p} \alpha^i = \frac{1-\alpha}{1-\alpha^{p+1}}$ again yields

$$U(f,\mathcal{P}_n) = (b^{p+1} - a^{p+1})c^{p/n} \frac{1}{1 + c^{1/n} + c^{2/n} + \cdots + c^{p/n}}.$$

For $L(f,\mathcal{P}_n)$ we obtain

$$L(f,\mathcal{P}_n) = \sum_{i=1}^{n} a^p c^{(i-1)p/n} ac^{(i-1)/n}(c^{1/n} - 1) = c^{-p/n} U(f,\mathcal{P}_n) =$$

$$(b^{p+1} - a^{p+1}) \frac{1}{1 + c^{1/n} + c^{2/n} + \cdots + c^{p/n}}.$$

(c) Since $c^{j/n} \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} U(f,\mathcal{P}_n) = \frac{b^{p+1} - a^{p+1}}{p+1}$$

and also

$$\lim_{n\to\infty} L(f,\mathcal{P}_n) = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

This shows that $f(x) = x^p$ is Riemann integrable over $[a,b]$ and we have

$$\in_a^b x^p x^p dx = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

161. Let $f(x) = \sin(\pi x)$ and $g(x) = \frac{1}{1+x^2}$. Then both functions are continuous and, therefore, Riemann integrable over $[0,1]$. Moreover, we have $f, g \geq 0$ on $[0,1]$. So we can apply the Mean Value Theorem for integrals in two different ways to obtain on the one hand

$$\int_0^1 f(x)g(x)dx = g(\xi_1)\int_0^1 \sin(\pi x)dx = g(\xi_1)\frac{\cos(0)-\cos(\pi)}{\pi} = \frac{2}{\pi}g(\xi_1) = \frac{2}{\pi(\xi_1^2+1)},$$

and on the other hand

$$\int_0^1 f(x)g(x)dx = f(\xi_2)\int_0^1 \frac{dx}{1+x^2} = f(\xi_2)\arctan(1) = f(\xi_2)\frac{\pi}{4} = \frac{\pi\sin(\pi\xi_2)}{4}.$$

162. (a) We choose $g(x) = 1$. Then $g \geq 0$ and we can apply the Mean Value Theorem for Integrals to obtain

$$\int_a^b f(x)dx = \int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx = f(c)\int_a^b dx = f(c)(b-a).$$

(b) Since f is continuous on $[a,b]$, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.
Firstly, let $c \in (a,b)$ and $h > 0$ such that $c+h \in [a,b]$. Then we have with (a):

$$|F(c+h) - F(c)| = \left|\int_c^{c+h} f(x)dx\right| = h|f(\xi)| \leq hM.$$

with some $\xi \in (c, c+h)$. This shows that

$$\lim_{h\to 0+} F(c+h) - F(c) = 0.$$

A similar argument applies for $h < 0$. If we consider the case $c = a$ and $c = b$, we have to restrict to one-sided limits.

163. Let $c \in (a,b)$ and $h > 0$ such that $c+h \in (a,b)$. Then we have, using the results of the previous problem

$$\frac{F(c+h)-F(c)}{h} = \frac{1}{h}\int_c^{c+h} f(x)dx = f(\xi)$$

for some $\xi \in [c, c+h]$. If $h \to 0+$ we have $\xi \to c$, which implies using continuity of f

$$F'(c) = \lim_{h\to 0}\frac{F(c+h)-F(c)}{h} = \lim_{\xi\to c} f(\xi) = f(c).$$

We have tacitly assumed here that $h > 0$, but the arguments can be modified easily to cover also the case $h < 0$.

164. We have

$$\lim_{c\to 0}\left|\int_0^c \sin(x^3)dx\right| \leq \lim_{c\to 0}\int_0^c \left|\sin(x^3)\right| dx \leq \lim_{c\to 0}\int_0^c dx = \lim_{c\to 0} c = 0.$$

So we can try to apply L'Hopital. Let $f(c) = \int_0^c \sin(x^3)dx$ and $g(c) = c^4$. Then we have $f(0) = g(0) = 0$ and $f'(c) = \sin(c^3)$ and $g'(c) = 4c^3$. Then we have $f'(0) = g'(0) = 0$ and $f''(c) = 3c^2\cos(c^3)$ and $g''(c) = 12c^2$. Here we can calculate the limit:

$$\lim_{c\to 0}\frac{f''(c)}{g''(c)} = \lim_{c\to 0}\frac{3c^2\cos(c^3)}{12c^2} = \lim_{c\to 0}\frac{\cos(c^3)}{4} = \frac{1}{4}.$$

Applying L'Hopital twice yields

$$\lim_{c\to 0}\frac{f(c)}{g(c)} = \lim_{c\to 0}\frac{f'(c)}{g'(c)} = \lim_{c\to 0}\frac{f''(c)}{g''(c)} = \frac{1}{4}.$$

165. Let $f(x) = ex^2/\pi - 2\pi/4 + \int_x^{\pi/2} e^{\sin t}dt$ and $g(x) = 1+\cos(2x)$. We easily check that $f(\pi/2) = g(\pi/2) = 0$, so we can try to apply L'Hopital. We have $f'(x) = 2xe/\pi - e^{\sin x}$ and $g'(x) = -2\sin(2x)$. Then we still have $f'(\pi/2) = g'(\pi/2) = 0$ and we differentiate again: $f''(x) = 2e/pi - \cos x e^{\sin x}$ and $g''(x) = -4\cos(2x)$. Here we can take the limit and, using continuity of f'' and g'', we obtain

$$\lim_{x\to\pi/2}\frac{f''(x)}{g''(x)} = \frac{f''(\pi/2}{g''(\pi/2)} = \frac{2e/\pi}{4} = \frac{e}{2\pi}.$$

Applying L'Hopital twice yields

$$\lim_{x\to\pi/2}\frac{f(x)}{g(x)} = \lim_{x\to\pi/2}\frac{f'(x)}{g'(x)} = \lim_{x\to\pi/2}\frac{f''(x)}{g''(x)} = \frac{e}{2\pi}.$$

166. Problems Class, 27 February 2015

167. (a) Since $(f(x)+\lambda g(x))^2 \geq 0$, we conclude from Monotonicity of the Integral that, for all $\lambda\in\mathbb{R}$,

$$\int_a^b (f(x)+\lambda g(x))^2dx \geq 0.$$

This implies that

$$B\lambda^2 + 2C\lambda + A \geq 0.$$

Since $B \neq 0$, this is a quadratic polynomial in λ which is non-negative for all choices of $\lambda\in\mathbb{R}$. Therefore, we must have

$$(4C)^2 - 4BA = 4(C^2 - AB) \leq 0.$$

(b) We proved in (a) that $C^2 \leq AB$. Replacing A, B, C by the expressions they represent, we obtain

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b (f(x))^2dx \int_a^b (g(x))^2dx.$$

168. Since g is continuous and not identically zero, we have $B \neq 0$. Since equality in (3) implies that $C^2 - AB = 0$, the quadratic equation

$$B\lambda^2 + 2C\lambda + A = 0$$

has a solution $\lambda_0 \in \mathbb{R}$. This means that we have

$$\int_a^b (f(x) + \lambda_0 g(x))^2 dx = 0.$$

Since $(f+\lambda_0 g)^2$ is continuous and non-negative, this means that $(f+\lambda_0 g) = 0$, i.e., $f = -\lambda_0 g$.

169. (a) We have $|\cos x/(x+\mathrm{e}^x)| \leq \mathrm{e}^{-x}$, and $\int_0^\infty \mathrm{e}^{-x}\,dx$ converges. Thus $\int_0^\infty (\cos x)/(x+\mathrm{e}^x)\,dx$ converges absolutely, by comparison.
(b) $(x+\sqrt{x})^{-1} \geq 1/(2x)$, and $\int_1^\infty (2x)^{-1}\,dx$ diverges. Thus $\int_1^\infty (x+\sqrt{x})^{-1}\,dx$ diverges by comparison.
(c) $\sqrt{(6+x)/(1+x^6)} \leq \sqrt{7x/x^6} = \sqrt{7}x^{-5/2}$, and so $\int_1^\infty \sqrt{(6+x)/(1+x^6)}\,dx$ converges by comparison with $\sqrt{7}\int_1^\infty x^{-5/2}\,dx$.
(d) $\int_0^R x^2\mathrm{e}^{-x}\,dx = -R^2\mathrm{e}^{-R} - 2R\mathrm{e}^{-R} - 2\mathrm{e}^{-R} + 2 \to 2$ as $R \to \infty$. So the integral converges. Alternatively, use $x^4\mathrm{e}^{-x} \to 0$ as $x \to \infty$, and comparison with $\int_1^R x^{-2}\,dx$.
(e) $0 \leq (1+x^3)^{-1/2} \leq x^{-3/2}$, and so the integral converges by comparison with $\int_1^\infty x^{-3/2}\,dx$.
(f) On $(0,1]$, $x^{-3/2}\mathrm{e}^{-x} > x^{-3/2}/\mathrm{e}$, and $\int_0^1 x^{-3/2}\,dx$ diverges, so the given integral diverges by comparison.
(g) $0 < \mathrm{e}^{-x}/\sqrt{x} < 1/\sqrt{x}$ for $x > 0$, and $\int_0^1 dx/\sqrt{x}$ converges, so the given integral converges by comparison.
(h) $\int_0^c x/\sqrt{1-x^2}\,dx = 1 - \sqrt{1-c^2}$ for $0 \leq c < 1$; and this has a finite limit (namely 1) as $c \to 1$. So the integral converges, by definition.
(i) Write $f(x) = x^{-1/3}\cos x$. For $0 < x \leq 1$, we have $0 < f(x) < x^{-1/3}$. Since $\int_0^1 x^{-1/3}\,dx$ converges, we deduce that the given integral converges by comparison.
(j) For $0 < x \leq 1$, we have $0 \leq \sqrt{x-x^2}/x = \sqrt{1-x}/\sqrt{x} < 1/\sqrt{x}$; and $\int_0^1 x^{-1/2}\,dx$ converges, so the given integral converges by comparison.

170. If $L > 0$, we can say that there is a number $R > 0$ such that $|L - f(x)| < L/2$ (say) for all $x > R$. But then we can deduce that the integral $\int_R^\infty f(x)\,dx$ is divergent by comparison with the divergent integral $\int_R^\infty L/2\,dx$, and so $\int_0^\infty f(x)\,dx = \int_0^R f(x)\,dx + \int_R^\infty f(x)\,dx$ is divergent. If $L < 0$ the same argument can be applied to $-f$. Thus, if the integral converges, we must have $L = 0$.

171. Integrating by parts on $[0,R]$ gives $\int_0^R xf'(x)\,dx = Rf(R) - 0f(0) - \int_0^R f(x)\,dx = Rf(R) - \int_0^R f(x)\,dx$. This has a limit as $R \to \infty$ if $\int_0^\infty f(x)\,dx$ converges and if $\lim_{R\to\infty} Rf(R) = L$ (finite). (Note that, by an argument similar to that of the previous problem, L in fact has be zero.)

172. (a) $\int_0^{2-c} x(16-x^4)^{-1/2}\,dx = \int_0^{(2-c)^2} (16-u^2)^{-1/2}\,du/2 \to \pi/4$ as $c \to 0$ (It's a $\sin^{-1}$.) Thus the integral converges.

(b) $16-x^4=(4+x^2)(2-x)(2+x)$. Then $x(16-x^4)^{-1/2} \le 2(8(2-x))^{-1/2}$ on $[0,2]$, and so the integral converges by comparison with the convergent integral $2^{-1/2}\int_0^2 (2-x)^{-1/2}\,dx$.

173. (a) $\int_a^1 (\log x)^2\,dx = -a(\log a)^2 + 2a\log a + 2(1-a) \to 2$ as $a \to 0$. Thus the integral converges.
(b) Since $x^{1/4}\log x \to 0$ as $x \to 0$, there is a number K such that $0 \le (\log x)^2 \le K/\sqrt{x}$ for $x \in (0,1]$. Now $\int_0^1 K\,dx/\sqrt{x}$ converges, therefore so does the given integral, by comparison.

174. $\tan x$ becomes unbounded as x approaches $\pi/2$, so we consider $\int_0^a \tan^3 x\,dx$ for $a < \pi/2$. Writing $\tan^3 x = -\tan x + \tan x \sec^2 x$, we see that $\tan^3 x = d[\log\cos x + (\sec^2 x)/2]/dx$ on $[0,a]$. Thus $\int_0^a \tan^3 x\,dx = \log\cos a + (\sec^2 a - 1)/2$, which has no limit as $a \to \pi/2$: the integral diverges.

175. Parts (a) and (c) in Problems Class, 27 February 2015 (b) $(x+1/x)^\alpha = x^{-\alpha}(1+x^2)^\alpha$. Thus $\min\{1,2^\alpha\}x^{-\alpha} \le (x+1/x)^\alpha \le \max\{1,2^\alpha\}x^{-\alpha}$ on $[0,1]$. By comparison with $\int_0^1 x^{-\alpha}\,dx$, the integral is convergent for $\alpha < 1$ and divergent otherwise.
(d) As in part (c), there are positive numbers c and C such that $cx^{1-\alpha} \le x^{-\alpha}\sin x \le Cx^{1-\alpha}$. Thus by comparison with $\int_0^1 x^{1-\alpha}\,dx$, the integral is convergent for $\alpha < 2$ and divergent otherwise.
(e) We split the integral into two components: $A = \int_0^1 \frac{x^{\alpha-1}}{1+x}\,dx$ and $B = \int_1^\infty \frac{x^{\alpha-1}}{1+x}\,dx$. Since $\frac{1}{2} \le \frac{1}{1+x} \le 1$ on $[0,1]$, A converges if and only if $\int_0^1 x^{\alpha-1}\,dx$ converges (by comparison), i.e. when $\alpha > 0$.
As for B, $\frac{1}{2}x^{\alpha-2} \le \frac{x^{\alpha-1}}{1+x} \le x^{\alpha-2}$ for $x \ge 1$, so B converges if and only if $\int_1^\infty x^{\alpha-2}\,dx$ converges (again by comparison), i.e. when $\alpha < 1$. The integral converges if and only if both A and B converge, i.e. for $0 < \alpha < 1$.

176. Write $f(x) = x^{-4/3}\sin x$. For $x \ge 1$, we have $0 < |f(x)| < x^{-4/3}$; and $\int_1^\infty x^{-4/3}\,dx$ converges, so $\int_1^\infty f(x)\,dx$ converges absolutely, by comparison. For $0 < x < 1$, we have $|x^{-1}\sin x| < 1$; and $\int_0^1 x^{-1/3}\,dx$ converges, so $\int_0^1 f(x)\,dx$ converges absolutely, by comparison. Hence $\int_0^\infty f(x)\,dx$ converges.

177. Write $f(x) = x^c/\sqrt{x^2+x} = x^{c-1/2}/\sqrt{x+1}$. For $x \ge 1$, we have $2^{-1/2}x^{c-1} < f(x) < x^{c-1}$, and $\int_1^\infty x^{c-1}\,dx$ converges iff $c-1 < -1$, that is iff $c < 0$. Next, for $0 < x < 1$, we have $2^{-1/2}x^{c-1/2} < f(x) < x^{c-1/2}$, and $\int_0^1 x^{c-1/2}\,dx$ converges iff $c - 1/2 > -1$, that is iff $c > -1/2$. So by comparison, $\int_0^\infty f(x)\,dx$ converges iff $-1/2 < c < 0$.

178. Problems Class, 27 February 2015

179. Write $f(x) = (x+x^2)^{-p}$. For $x \ge 1$, we have $\frac{1}{2x^2} < \frac{1}{x+x^2} < \frac{1}{x^2}$, and $\int_1^\infty x^{-2p}\,dx$ converges iff $2p > 1$; so $\int_1^\infty f(x)\,dx$ converges iff $p > 1/2$, by comparison. Next, for $0 < x \le 1$, we have $\frac{1}{2x} < \frac{1}{x+x^2} < \frac{1}{x}$, and $\int_0^1 x^{-p}\,dx$ converges iff $p < 1$; so $\int_0^1 f(x)\,dx$ converges iff $p < 1$, by comparison. Thus $\int_0^\infty f(x)\,dx$ converges iff $1/2 < p < 1$.

180. Problems Class, 27 February 2015

11 Sequences of functions and uniform convergence

181. The pointwise limit is the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 0$ since, for every $x \in \mathbb{R}$, there exists $N \in \mathbb{N}$ with $x \leq N$ and we have $f_n(x) = 0$ for all $n \geq N$. The convergence is not uniform, since we have $f_n(n+1) - f(n+1) = 1$. (If $f_n \to f$ were uniform, we could find for $\epsilon = 1$ an index $N \in \mathbb{N}$ with $|f_n(x) - f(x)| < 1$ for all $n \geq N$ and $x \in \mathbb{R}$.)

182. The pointwise limit is the function $f : (1, \infty) \to \mathbb{R}$, given by $f(x) = 0$ since, for every $x \in (1, \infty)$, $x^n \to \infty$ as $n \to \infty$. The convergence is not uniform since every function f_n is unbounded (recall that $\lim_x \to \infty \frac{x^n}{e^x} = 0$) but the limit function is bounded.

183. Note that $\lim_{c\to\infty} e^{-c} = 0$. This implies that we have, for every $x \in [-1, 1]$, $x \neq 0$,
$$\lim_{n\to\infty} e^{-nx^2} = 0.$$
At $x = 0$, we always have $f_n(0) = e^0 = 1$, so the limit function is
$$f(x) = \begin{cases} 1 & \text{if } 0 < |x| \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$
The convergence cannot be uniform, since all the functions f_n are continuous on $[-1, 1]$ but the pointwise limit function f is discontinuous at $x = 0$.

184. Note that $e^{-x^2} \leq 1$ for all $x \in \mathbb{R}$. Therefore, we have for all $x \in \mathbb{R}$,
$$1 - \frac{1}{n} \leq f_n(x) \leq 1.$$
Here we have uniform convergence to $f(x) = 1$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ with $1/N < epsilon$ and we have, for all $n \geq N$ and all $x \in \mathbb{R}$,
$$|f(x) - f_n(x)| \leq \frac{1}{n} < \epsilon.$$

185. The pointwise limit of x^n on $[0, 1]$ is
$$f(x) = \begin{cases} 0, & \text{if } x \in [0, 1), \\ 1, & \text{if } x = 1. \end{cases}$$
Since x^{2n} is a subsequence, its pointwise limit is the same function f, so the difference converges pointwise to the function $g(x) = 0$ on $[0, 1]$. Let us determine
$$\max_{x\in[0,1]} f_n(x) - g(x) = \max_{x\in[0,1]} x^n - x^{2n}.$$
Obviously, we have $f_n(0) = f_n(1) = 0$ and $x^n \geq x^{2n}$ on $[0, 1]$, so if $f_n(x_0)$ with $x_0 \in (0, 1)$ is a positive maximum, we must have $f_n'(x_0) = 0$. This leads to $f_n'(x_0) = nx_0^{n-1} - 2nx_0^{2n-1} = 0$, which yields $x_0^n = 1/2$, i.e., $x_0 = (1/2)^n$. There we obtain
$$f_n((1/2)^n) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$
So we obtain a contradiction to uniform convergence by choosing $\epsilon < 1/4$.

186. We have $f_n(0) = 0$, and for any fixed $x > 0$ we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n+x} = x.$$

Therefore, the pointwise limit function is given by $f(x) = x$. Now we consider

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+n+x} - x\right| = \left|\frac{nx - x - nx - x^2}{1+n+x}\right| = x\frac{1+x}{1+n+x}.$$

Choosing $x = n$, we see that

$$|f_n(x) - f(x)| = n\frac{1+n}{1+2n} \geq n\frac{1+n}{2+2n} = \frac{n}{2}.$$

This expression becomes arbitrarily large as $n \to \infty$, so we cannot have uniform convergence.

187. For every $x \in [0, \infty)$ we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n^2}} = \sqrt{x^2} = x.$$

So the pointwise limit function is $f(x) = x$. Now we calculate $|f_n(x) - f(x)|$:

$$\left|\sqrt{x^2 + \frac{1}{n^2}} - x\right| = \frac{(x^2 + 1/n^2) - x^2}{\sqrt{x^2 + \frac{1}{n^2}} + x} = \frac{1}{n^2 x + n^2\sqrt{x^2 + 1/n^2}}.$$

Since $n = n^2\sqrt{1/n^2} \leq n^2 x + n^2\sqrt{x^2 + 1/n^2}$, we obtain

$$\left|\sqrt{x^2 + \frac{1}{n^2}} - x\right| \leq \frac{1}{n},$$

188. First of all, we know that the limit function $f : [a, b] \to \mathbb{R}$ is again continuous and, therefore, all functions f_n, f are Riemann integrable on $[a, c]$.

Let $\epsilon > 0$. Then we know that there exists $N \in \mathbb{N}$ such that

$$f(x) - \epsilon \leq f_n(x) \leq f(x) + \epsilon \quad \text{for all } n \geq N.$$

By Monotonicity of the Integral, we conclude that for all $n \geq N$,

$$\int_a^c (f(x) - \epsilon)dx \leq \int_a^c f_n(x)dx \leq \int_a^c (f(x) + \epsilon)dx.$$

Observe that

$$\int_a^c (f(x) \pm \epsilon)dx = \int_a^c f(x)dx \pm \epsilon \int_a^c dx = \int_a^c f(x)dx \pm (c-a)\epsilon.$$

This shows that we have for all $n \geq N$,

$$\left|\int_a^c f(x)dx - \int_a^c f_n(x)dx\right| < (c-a)\epsilon \leq (b-a)\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\lim_{n\to\infty} \int_a^c f_n(x)dx = \int_a^c f(x)dx.$$

189. (a) If $f(x) = 0$ for all $x \in [a,b]$, we obviously have $\|f\|_\infty = 0$. Now let $\|f\|_\infty = 0$. If we had $f(x) \neq 0$ for some $x \in [a,b]$, we also had $|f(x)| > 0$, which would imply $\|f\|_\infty = \sup |f(x)| > 0$. This shows the converse direction.

(b) We have

$$\|\lambda f\|_\infty = \sup_{x \in [a,b]} |\lambda f(x)| = |\lambda| \sup_{x \in [a,b]} |f(x)| = |\lambda| \cdot \|f\|_\infty.$$

(c) Note that continuity of $|f|$ implies that there exists $x_0 \in [a,b]$ with $\|f\|_\infty = |f(x_0)|$. So we have $x_0, y_0 \in [a,b]$ with $\|f\|_\infty = |f(x_0)|$ and $\|g\|_\infty = |g(y_0)|$. This means that we have $|f(x)| \leq |f(x_0)|$ and $|g(x)| \leq |g(y_0)|$ for all $x \in [a,b]$, i.e.,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq |f(x_0)| + |g(y_0)| \quad \text{for all } x \in [a,b].$$

So $|f(x_0)| + |g(y_0)|$ is an upper bound of $\{|f(x) + g(x)| \mid x \in [a,b]\}$ and we have

$$\|f + g\|_\infty = \sup_{x \in [a,b]} |f(x) + g(x)| \leq |f(x_0)| + |g(y_0)| = \|f\|_\infty + \|g\|_\infty.$$

190. Let $f_n \in C([a,b])$ be a Cauchy sequence. Let us first show that the sequence $f_n : [a,b] \to \mathbb{R}$ of continuous functions has a pointwise limit function $f : [a,b] \to \mathbb{R}$. Let $x \in [a,b]$ and $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon$$

for all $n, m \geq N$. This means that the sequence $(f_n(x))$ of real numbers is a Cauchy sequence and, therefore, has a limit, which we denote by $f(x)$:

$$f(x) = \lim_{n \to \infty} f_n(x).$$

So we showed that there exists $f : [a,b] \to \mathbb{R}$ such that $f_n \to f$ pointwise. This function f is the candidate for the limit. We first show that the convergence is not only pointwise, but uniform. Let $\epsilon > 0$ be given. f_n being a Cauchy sequence means that we have a start index $N \in \mathbb{N}$ such that for all $x \in [a,b]$ and all $n, m \geq N$

$$|f_n(x) - f_m(x)| < \epsilon.$$

Letting $m \to \infty$, we conclude that

$$|f_n(x) - f(x)| \leq \epsilon \tag{2}$$

for all $n \geq N$ and all $x \in [a,b]$. This shows that $f_n \to f$ uniformly. Therefore, the limit function $f : [a,b] \to \mathbb{R}$ is continuous and we have $f \in C([a,b])$. But (2) means also that for all $n \geq N$,

$$\|f_n - f\|_\infty = \sup_{x \in [a,b]} |f_n(x) - f(x)| \leq \epsilon,$$

i.e., we have convergence $f_n \to f$ in $C([a,b])$, finishing the proof.

12 Power series and Taylor series

191. $\Sigma a_n z^{2n} = \Sigma a_n (z^2)^n$, which converges for $|z^2| < R \Leftrightarrow |z| < \sqrt{R}$ and diverges for $|z^2| > R \Leftrightarrow |z| > \sqrt{R}$.

192. Parts (b) and (c) in Problems Class, 13 March 2015 (a) $|a_{n+1}/a_n| = \frac{(2n+2)(2n+1)}{(n+1)^2} \to 4$ as $n \to \infty$, so $R = 1/4$.
(d) $|a_{n+1}/a_n| = \frac{(3n+3)(3n+2)(3n+1)}{2(n+1)^3} \to \frac{27}{2}$ as $n \to \infty$, so $R = 2/27$.
(e) $|a_{n+1}/a_n| = \frac{(n+1)^2}{3n^2} \to \frac{1}{3}$ as $n \to \infty$, so $R = 3$.
(f) $|a_{n+1}/a_n| = \frac{2^{10}}{n+1} \to 0$ as $n \to \infty$, so R is infinite.
(g) $|a_{n+1}/a_n| = \frac{2(3^n+1)}{3^{n+1}+1} \to \frac{2}{3}$ as $n \to \infty$, so $R = 3/2$.

193. $|a_{n+1}/a_n| = 1/2$, so $R = \sqrt{2}$ by question 191.

194. $|a_{n+1}/a_n| = 1/2^{2n+1} \to 0$ as $n \to \infty$, so R is infinite.

195. Let $a_n = n!/n^n$. We need to find $\lim |a_n|^{1/n}$. We have

$$|a_n|^{1/n} = \frac{(n!)^{1/n}}{n},$$

and therefore

$$(2\pi n)^{1/2n}\frac{1}{e} < |a_n|^{1/n} < (2\pi n)^{1/2n}\frac{1}{e}e^{1/(12n^2)}.$$

Note for $a > 0$ that

$$\log((an)^{1/2n}) = \frac{\log(an)}{2n} \to 0,$$

which implies that $(an)^{1/2n} \to 1$. So we conclude that

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{e},$$

and, therefore, $R = 1/(1/e) = e$.

196. Let a_n as in the problem. Let $n = k!$. Then we have

$$|a_n|^{1/n} = (2^k)^{1/k!} = 2^{1/(k-1)!} \to 1 \quad \text{as } k \to \infty.$$

This becomes clear from the fact that $2^l \to 1$ for $l \to \infty$. If n is not a factorial, we have trivially $|a_n|^{1/n} = 1^{1/n} = 1$, so we have

$$\lim_{n\to\infty} |a_n|^{1/n} = 1,$$

and the radius of convergence is $R = 1$.

197. Let R be the radius of convergence of $\sum b_n z^n$. If $R = 0$ there ios nothing to show. Assume $R > 0$. Then we only have to convince ourselves that $\sum a_n z^n$ converges for all $|z| < R$, then the radius of convergence of $\sum a_n z^n$ must be $\geq R$. Let $z \in \mathbb{C}$ with $|z| < R$. Then we can find $r \in (|z|, R)$ and $\sum b_n r^n$ is convergent. By Lemma 12.2, $\sum b_n z^n$ is absolutely convergent. But then also $\sum |a_n z^n|$ is convergent, by comparison. Since $\sum a_n z^n$ is absolutely convergent, it is also convergent, which we wanted to show.

198. Using $\Sigma_{n=1}^{\infty} t^n = t/(1-t)$ for $|t| < 1$, we get $f(x) = x$ for all $x \neq 0$. Clearly $f(0) = 0$, so we have $f(x) = x$ for all x. Hence $df/dx = 1$, whereas $\Sigma_{n=1}^{\infty} u_n'(0) = \Sigma\, 0 = 0$: the two quantities are not equal.

199. The kth partial sum is $S_k(x) = kx\exp(-kx^2)$, so $f(x) = \lim_{k\to\infty} S_k(x) = 0$ for all x. Thus $\int_0^1 f(x)\,dx = 0$. On the other hand, $\Sigma_{n=1}^{k} \int_0^1 u_n(x)\,dx = \int_0^1 \Sigma_{n=1}^{k} u_n(x)\,dx = \int_0^1 S_k(x)\,dx = (1-\mathrm{e}^{-k})/2 \to 1/2$ as $k \to \infty$. So $\Sigma_{n=1}^{\infty} \int_0^1 u_n(x)\,dx = 1/2$: the two quantities are not equal.

200. Using the geometric series, we find

$$\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \cdots\right) = \frac{1}{(n+1)!} \cdot \frac{1}{1-1/(n+2)} = \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1}.$$

This implies that

$$0 < e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\right) = \frac{1}{5!} + \frac{1}{6!} + \cdots < \frac{1}{5!}\frac{6}{5} = \frac{6}{5\cdot 120} = \frac{1}{100}.$$

Now we have

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{48+12+4+1}{24} = \frac{65}{24} = 2.708333\ldots,$$

which yields the required result.

201. Assume that $e = p/q$ with natural numbers p, q. Then

$$N = eq! - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!}\right) q!$$

is a natural number and (4) implies that

$$N = q!\left(\frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \cdots\right) < \frac{q!}{(q+1)!} \cdot \frac{q+2}{q+1} = \frac{q+2}{(q+1)^2}.$$

But q is a natural number and

$$\frac{q+2}{(q+1)^2} \leq \frac{1}{2}\frac{q+2}{q+1} = \frac{1}{2}\left(1 + \frac{1}{q+1}\right) \leq \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4},$$

which is a contradiction.

202. (a) Note that $\mathrm{R}=\frac{1}{\sqrt{\pi}} > 0.56$.
(b) $|\sin(n|x|)| < 1$, and $\sum_1^{\infty} \frac{1}{n^2}$ converges.
(c) $|x^n| \leq 1$; moreover, $\frac{n}{n^3+|x|} \leq \frac{1}{n^2}$; hence have convergence.

203. Problems Class, 13 March 2013

204. Let $g(x) = nx/(1+n^4x^2)$. Then $g : [0,\infty) \to \mathbb{R}$ is continuous, non-negative and $\lim_{x\to\infty} g(x) = 0$ and $g(0) = 0$. We have

$$g'(x) = \frac{n(1+n^4x^2) - 2n^5x^2}{(1+n^4x^2)^2} = n\frac{1-n^4x^2}{(1+n^4x^2)^2},$$

and $g'(x) = 0$ leads to $x = 1/n^2$. Note that $g'(x) < 0$ for all $x \geq 1/n^2$, i.e., g is monotone decreasing on $[1/n^2, \infty)$. For given $a > 0$, we can find $N \in \mathbb{N}$ with $a > 1/N^2$. Then each term in the series

$$\sum_{n=N}^{\infty} \frac{nx}{1+n^4x^2}$$

can be estimated from above by $(na)/(1+n^4a^2)$. Since

$$\sum \frac{na}{1+n^4a^2} \leq \sum \frac{na}{n^4a^2} = \frac{1}{a}\sum \frac{1}{n^3}$$

is convergent, the original series is uniformly convergent, by the Weierstrass M-test.

205. Let $x \geq 0$. Then we have

$$f(x) = \sum_{n=0}^{\infty} \frac{nx}{1+n^4x^2} \geq \sum_{n=N}^{\infty} \frac{nx}{1+n^4x^2}.$$

We have for $n \geq N$ that $n^4/N^4 \geq 1$ and choosing $x = 1/N^2 \geq 0$ leads to

$$f(1/N^2) \geq \sum_{n=N}^{\infty} \frac{n/N^2}{1+n^4/N^4} \geq \sum_{n=N}^{\infty} \frac{n/N^2}{2n^4/N^4} = \frac{N^2}{2}\sum_{n=N}^{\infty} \frac{1}{n^3}.$$

Moreover, we have

$$\sum_{n=N}^{\infty} \frac{1}{n^3} \geq \int_N^{\infty} \frac{dx}{x^3} = [-x^{-2}/2]_{x=N}^{x=\infty} = 1/(2N^2).$$

Combining both results leads to

$$f(1/N^2) \geq \frac{N^2}{2}\frac{1}{2N^2} = \frac{1}{4}.$$

If the convergence were uniform on $\mathbb{R}$, we could conclude that $f : \mathbb{R} \to \mathbb{R}$ is continuous since the partial sums are continuous. This would imply that

$$f(0) = \lim_{N\to\infty} f(1/N^2) \geq \frac{1}{4}.$$

But the pointwise limit at $x = 0$ is $f(0) = 0$. Therefore, we cannot have uniform convergence on $\mathbb{R}$.

206. (a) $\cos^2 x = [1+\cos(2x)]/2 = 1 - x^2 + x^4/3 - \ldots$.
(b) $\sin(x^2) = x^2 - x^6/6 + x^{10}/120 - \ldots$.
(c) $\mathrm{e}^x \sin x = (1 + x + x^2/2 + x^3/6 + \ldots)(x - x^3/6 + \ldots) = x + x^2 + x^3/3 + \ldots$
(d) $1/(1+x^2) = 1 - x^2 + x^4 - \ldots$
(e) $x/(1+x^3) = x - x^4 + x^7 - \ldots$
(f) $(1+x^2)^{-2} = 1 - 2x^2 + 3x^4 - \ldots$
(g) $[\exp(x^4) - 1]/x^3 = x + x^5/2 + x^9/6 + \ldots$
(h) $(1-x)^{-3} = 1 + 3x + 6x^2 + \ldots$
(i) $\exp(x^2)\sin(x^2) = x^2 + x^4 + x^6/3 + \ldots$ [from (c)]
(j) $\exp[1/(1-2x)] = \mathrm{e}(1 + 2x + 6x^2 + \ldots)$
(k) $\exp(\exp x) = \mathrm{e}(1 + x + x^2 + \ldots)$
(l) $\log(1+2x^2) = 2x^2 - 2x^4 + 8x^6/3 - \ldots$
(m) $[\log(1+x)]^2 = (x - x/2 + x^3/3 - \ldots)^2 = x^2 - x^3 + 11x^4/12 + \ldots$

207. We prove by Induction that, for $x \neq 0$,

$$f^{(k)}(x) = p_k(1/x)e^{1/x^2},$$

where p_k is a polynomial of degree $3k$. For $k = 0$ there is nothing to prove. Given this fact holds for k, then we obtain

$$\begin{aligned} f^{(k+1)}(x) &= p_k'(1/x)(-1/x^2)e^{-1/x^2} + p_k(1/x)\frac{2}{x^3}e^{-1/x^2} \\ &= \left(p_k'(1/x)(-1/x^2) + p_k(1/x)\frac{2}{x^3}\right)e^{-1/x^2}, \end{aligned}$$

which shows that we need to choose $p_{k+1}(y) = -y^2 p_k'(y) + 2y^3 p_k(y)$, which has degree $3k+3$. This completes the induction proof.

Now we consider the derivatives $f^{(k)}(0)$. Again we use Induction. We start with $f^{(0)}(0) = f(0) = 0$. Assuming that $f^{(k-1)}(0)$ exists and is equal to zero, we obtain

$$\frac{f^{(k-1)}(x) - f^{(k-1)}(0)}{x} = \frac{1}{x}p_{k-1}(1/x)e^{-1/x^2}.$$

This implies that

$$\lim_{xto0+} \frac{f^{(k-1)}(x) - f^{(k-1)}(0)}{x} = \lim_{y\to\infty} y p_{k-1}(y)e^{-y^2} = 0.$$

The same argument applis for the elft hand limit. Therefore, $f^{(k)}(0)$ exists and is also zero.

Since $f^{(k)}(0) = 0$ for all $k \in \mathbb{N} \cup \{0\}$, the Taylor polynomial of f is trivial and converges to $f(x)$ only if $x = 0$.

208. Parts (b) and (c) Problems Class, 13 March 2015

(a) We have $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. So the sum is $\cos(2\pi) = 1$.

209. We have $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $\cos x = \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l}}{(2l)!}$, which converge absolutely for any choice of $x \in \mathbb{C}$. So we can apply the Cauchy product and obtain

$$(\sin x)(\cos x) = \sum_{n=0}^{\infty} c_n x^n$$

with

$$c_n = \sum_{k+l=n} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!} \frac{x^{2l}}{(2l)!} = (-1)^n \sum_{k=0}^{n} \frac{x^{2n+1}}{(2k+1)!((2n+1)-(2k+1))!} =$$

$$(-1)^n \sum_{k=0}^{n} \binom{2n+1}{2k+1} \frac{x^{2n+1}}{(2n+1)!}.$$

Now we use

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} = 2^{2n} \tag{3}$$

and conclude that

$$c_n = \frac{(-1)^n}{2} 2^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \frac{(-1)^n}{2} \frac{(2x)^{2n+1}}{(2n+1)!},$$

i.e.,

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \frac{1}{2} \sin(2x).$$

Now it remains to prove (3), using $(1+c)^{2n+1} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} c^k$. Choosing $c = -1$ and $c = 1$, we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k, \\ 2^{2n+1} &= \sum_{k=0}^{2n+1} \binom{2n+1}{k}. \end{aligned}$$

Adding the two equations kills all even k-terms and we obtain

$$2^{2n+1} = 2 \sum_{l=0}^{n} \binom{2n+1}{2l+1},$$

i.e.,

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} = 2^{2n}.$$

210. Problems Class, 13 March 2015