Analysis 1 Solutions (Michaelmas Term 2014)

1 Basic logic and sets

1. a) *True*: If the two odd numbers are $a = 2n + 1$ and $b = 2m + 1$ then their product is $ab = 2(2nm + n + m) + 1$ and is therefore again odd.

   b) *False:* We only need to find real numbers p, q such that $p^2 - 4q \geq 0$ and $x^2 + px + q = 0$ does not have two real solutions. Choose, for example $p = 2$ and $q = 1$, then $p^2 - 4q = 4 - 4 = 0$ and $x^2 + 2x + 1 = (x + 1)^2 = 0$ has only one real solution, namely $x = -1$.

   c) *True*: For $\epsilon > 0$ choose a natural number n such that $0 < \frac{1}{\epsilon} < n$. Then $\frac{1}{n} < \epsilon$.

   d) *False*: We have $1001 = 7 * 11 * 13$, so 1001 is not a prime number. Moreover, we have $|sin(x^2)| \leq 1$ for all real x, so

$$|\int_0^\pi \sin(x^2)dx| \leq \int_0^\pi |sin(x^2)|dx \leq \int_0^\pi 1dx = \pi < 4,$$

   i.e., "$\int_0^\pi sin(x^2)dx \geq 4$" is false. This shows that the combined statement is also false.

2. • Statement b) implies that Tom or Max is the lecturer.
   • We conclude from statement c) that today is Wednesday (since Anna cannot be the lecturer).
   • We conclude from statement e) that Tom must be lecturer (since Anna is not the lecturer).
   • We conclude from statement d) that Anna is the electrician.
   • Since Tom is the lecturer and Anna the electrician, Max must be the builder.

Of course, you need to check that with this solution all statements a)-g) are satisfied. The uniqueness of the solution follows from the fact that each conclusion above was obligatory.

3.

A	B	(not A) or B	A and B	not A	(A and B) or (not A)
false	false	true	false	true	true
false	true	true	false	true	true
true	false	false	false	false	false
true	true	true	true	false	true

This shows that both statements are equivalent.

4. a) We have

$$\begin{aligned} \text{not}\,(\text{not}\,(A)\,\text{and}\,(B\,\text{and}\,(\text{not}\,C))) &\Leftrightarrow \text{not}\,(\text{not}\,(A)\,\text{and}\,\text{not}\,(\text{not}\,(B)\,\text{or}\,C)) \\ &\Leftrightarrow A\,\text{or}\,(\text{not}\,(B))\,\text{or}\,C. \end{aligned}$$

b) We have

$$\begin{aligned} (A\,\text{and}\,\text{not}\,(B))\,\text{or}\,\text{not}\,(A\,\text{and}\,\text{not}\,(C)) &\Leftrightarrow (A\,\text{and}\,\text{not}\,(B))\,\text{or}\,(\text{not}\,(A)\,\text{or}\,C) \\ &\Leftrightarrow \text{not}\,(\text{not}\,(A)\,\text{or}\,B)\,\text{or}\,(\text{not}\,(A)\,\text{or}\,C). \end{aligned}$$

c) De Morgan's Law holds also for combining finitely many sets by " and". Therefore, we have

$$A\,\text{and}\,\text{not}\,(B)\,\text{and}\,\text{not}\,(C)\,\text{and}\,D \quad \Leftrightarrow \quad \text{not}\,(\text{not}\,(A)\,\text{or}\,B\,\text{or}\,C\,\text{or}\,\text{not}\,(D)).$$

5. a) At least one of the two statements "A", "not (A)" must be true. Therefore, combining them with " or" leads always to a true statement.

b) B is either true or false. If B is false, then "not B" is true and therefore "A or $(A$ or $B)$ or not (B)" is also true. If B is true, then "A or B" is also true and so is "A or $(A$ or $B)$ or not (B)". So this statement is a tautology.

c) "$(A$ and $B)$ or $(B$ and $C)$" can only be true if B is true, in which case "not B" is false and so is "$((A$ and $B)$ or $(B$ and $C))$ and (not B)".
If "$(A$ and $B)$ or $(B$ and $C)$" is false then so is "$((A$ and $B)$ or $(B$ and $C))$ and (not B)".
Therefore, "$((A$ and $B)$ or $(B$ and $C))$ and (not B)" is always false, and its negation is a tautology.

6. We know that "$A \Leftrightarrow B$" is only true if the statements A and B are either both true *or* both false. The statement "A and B" is only true if both statements A and B are true. Similarly, the statement "not (A) and not (B)" is only true if both statements A and B are false. Therefore we have

$$(A \Leftrightarrow B) \quad \Leftrightarrow \quad (A\,\text{and}\,B)\ \text{or}\ (\text{not}\,(A)\,\text{and}\,\text{not}\,(B)).$$

7. Let $X = \{x \in \mathbb{R} \mid x^2 - 9x + 14 = 0\}$ and $Y = \{y \in \mathbb{Z} \mid 3 \leq y < 10\}$. Since $x^2 - 9x + 14 = (x-7)(x-2)$, we have $X = \{2, 7\}$ and, therefore,

$$\begin{aligned} X \cup Y &= \{2,3,4,5,6,7,8,9\}, \\ X \cap Y &= \{7\}. \end{aligned}$$

8. There are different ways to describe this set. One expression for it is

$$((X \cap Z) \backslash Y) \cup (Y \backslash X).$$

Another expression is

$$(Y \cup (Z \cap X)) \backslash (X \cap Y).$$

9. Let $X = \{x \in \mathbb{R} \mid x \leq a\} \cap \{x \in \mathbb{R} \mid \min\{x, a\} \leq b\}$ and $Y = \{x \in \mathbb{R} \mid x \leq \min\{a, b\}\}$. To show that the two sets X and Y are equal, we have to prove two facts. Firstly, every element of X is also an element of Y. Secondly, every element of Y is also an element of X. Here are the arguments:

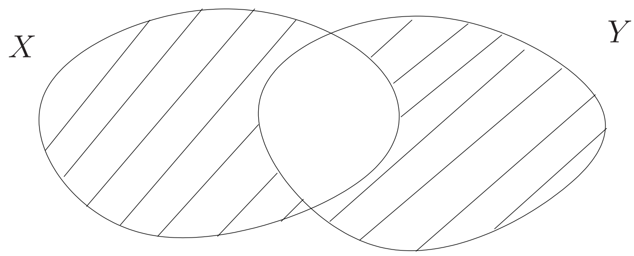
- If $x \in X$ then $x \in Y$: Note that $x \leq a$ and $\min\{x, a\} \leq b$ implies that $x = \min\{x, a\} \leq b$ and, therefore $x \leq \min\{a, b\}$.
- If $x \in Y$ then $x \in X$: We conclude from $x \leq \min\{a, b\}$ that $x \leq a$, i.e., $x = \min\{x, a\}$, and $x \leq b$. This implies $x \leq a$ and $x = \min\{x, a\} \leq b$.

10. a) Assume that $X \cup Y = Y$. Let $x \in X$. Then $x \in X \cup Y = Y$. This shows that $X \subset Y$.

    b) Assume that $X = X \cap Y$. Let $x \in X = X \cap Y$. Therefore, $x \in Y$. This shows that $X \subset Y$.

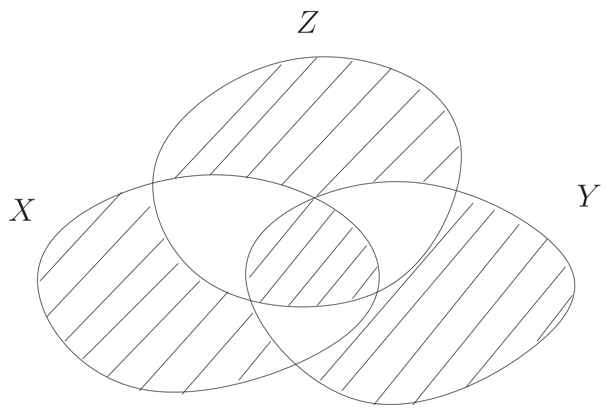
    c) We always have $Y \subset X \cup Y$, since if $x \in Y$ then x is also in the union of X and Y, i.e., $x \in X \cup Y$. Assume $X \subset Y$. It remains to show that then $X \cup Y \subset Y$. Let $x \in X \cup Y$. Then $x \in X$ or $x \in Y$. If $x \in X$, then $x \in Y$ because of $X \subset Y$. If $x \notin X$, then we must have $x \in Y$ for "$x \in X$ or $x \in Y$" to be true. So we have in either case that $x \in Y$. This shows that $X \cup Y \subset Y$.

11. a) The Venn Diagram looks as follows:



$X \Delta Y$

    b) The Venn Diagram for both sets looks the same:



$(X \Delta Y) \Delta Z = X \Delta (Y \Delta Z)$

Both sets are equal and can be described in words as follows: They consist of all elements which belong to only one of the three sets or lie in the intersection of all three sets X, Y, Z. Therefore, another way of describing these sets is

$$(X \cap Y^c \cap Z^c) \cup (X^c \cap Y \cap Z) \cup (X \cap Y \cap Z^c) \cup (X \cap Y \cap Z).$$

c) Let $x \in X\Delta Z$. Then x belongs to precisely one of the two sets X and Z. Now we have two cases to consider: The first case is $x \in Y$ and the second case is $x \notin Y$. One of these two cases is always fulfilled.

Firstly, assume that $x \in Y$. Since x belongs to precisely one of the two sets X and Z, it does not belong to either X or to Z. If x does not belong to X, then $x \in X\Delta Y$. If x does not belong to Z, then $x \in Y\Delta Z$. So we conclude in the first case that we always have $x \in (X\Delta Y) \cup (Y\Delta Z)$.

Secondly, assume that $x \notin Y$. Since x belongs to precisely one of the two sets X and Z, it belongs to either X or to Z. If x belongs to X, then $x \in X\Delta Y$. If x belongs to Z, then $x \in Y\Delta Z$. So we conclude in the second case that we always have $x \in (X\Delta Y) \cup (Y\Delta Z)$.

This shows that

$$x \in X\Delta Y \quad \Rightarrow \quad x \in (X\Delta Y) \cup (Y\Delta Z),$$

finishing the proof of the inclusion.

12. 1. The statement is true. We give names to the elements of the set X, i.e. $X = \{a_1, \ldots, a_n\}$. Now, every subset of X corresponds uniquely to n yes/no choices, deciding for each of the elements a_j whether it is in the subset or not. We have 2^n possibilities to make these choices, therefore $\mathcal{P}(X)$ has exactly 2^n elements.

    2. The statement is true.

       Let $U \in \mathcal{P}(Z)$. Then $U \subset Z$. Since $U \subset Z$ and $Z \subset X$ and $Z \subset Y$, we also have $U \subset X$ and $U \subset Y$, i.e., $U \in \mathcal{P}(X)$ and $X \in \mathcal{P}(Y)$. This shows that $U \in \mathcal{P}(X) \cap \mathcal{P}(Y)$.

       Conversely, let $U \in \mathcal{P}(X) \cap \mathcal{P}(Y)$. Then $U \subset X$ and $U \subset Y$, i.e., $U \subset X \cap Y = Z$. This shows that $U \in \mathcal{P}(Z)$.

    3. The statement is false. We only need to provide a counterexample. Let $X = \{a\}$ and $Y = \{b\}$. Then $Z = \{a, b\}$ and $Z \in \mathcal{P}(Z)$. But $Z \not\subset X$ and $Z \not\subset Y$, therefore $Z \notin \mathcal{P}(X) \cup \mathcal{P}(Y)$.

13. We show first that $X \subset Y$. Let $(x, y) = (\cos(t), \sin(t)) \in X$. Then we have

$$x^2 + y^2 = \cos^2(t) + sin^2(t) = 1.$$

This shows that $(x, y) \in Y$. The proof of $Y \subset X$ is more complicated. Let $(x, y) \in Y$, i.e., $x^2 + y^2 = 1$. Then $x \in [-1, 1]$ and there exists $t \in [0, \pi]$ such that $\cos(t) = x$. This implies that

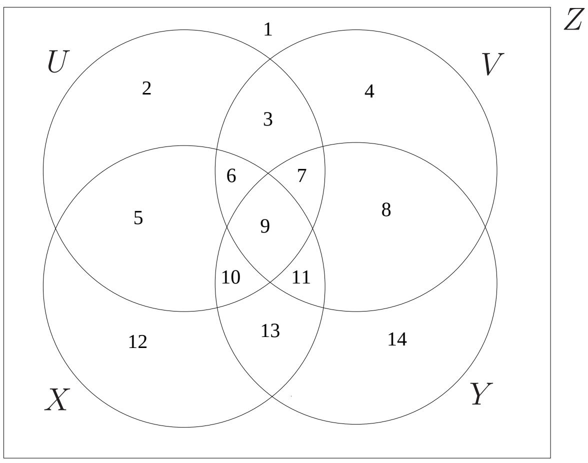
$$y^2 = 1 - x^2 = 1 - \cos^2(t) = \sin^2(t).$$

So we have $y = \pm\sin(t)$. In the case that $y = \sin(t)$, we have $(x, y) = (\cos(t), \sin(t))$ for some $t \in [0, \pi]$, which implies that $(x, y) \in X$. If $y = -\sin(t)$, then we have $s = 2\pi - t \in [\pi, 2\pi]$ and

$$\begin{aligned}(\cos(s), \sin(s)) = (\cos(2\pi - t), \sin(2\pi - t)) &= (\cos(-t), \sin(-t)) \\ &= (\cos(t), -\sin(t)) = (x, y),\end{aligned}$$

showing again that $(x, y) \in X$. In both cases we have that $(x, y) \in Y$ implies $(x, y) \in X$, i.e., $Y \subset X$.

14. Both Jack's Venn Diagram and his example are correct. The problem lies in the conclusion from the Venn Diagram. We enumerate the components of both Venn Diagrams from 1 to 14 as follows:



You can easily check that these components describe the following subsets of Z:

component	subset	component	subset
1	$U^c \cap V^c \cap X^c \cap Y^c$	8	$U^c \cap V \cap X^c \cap Y$
2	$U \cap V^c \cap X^c \cap Y^c$	9	$U \cap V \cap X \cap Y$
3	$U \cap V \cap X^c \cap Y^c$	10	$U \cap V^c \cap X \cap Y$
4	$U^c \cap V \cap X^c \cap Y^c$	11	$U^c \cap V \cap X \cap Y$
5	$U \cap V^c \cap X \cap Y^c$	12	$U^c \cap V^c \cap X \cap Y^c$
6	$U \cap V \cap X \cap Y^c$	13	$U^c \cap V^c \cap X \cap Y$
7	$U \cap V \cap X^c \cap Y$	14	$U^c \cap V^c \cap X^c \cap Y$

Now, there are 16 combinations $U^* \cap V^* \cap X^* \cap Y^*$, where $*$ is either no symbol or the complement symbol, so the Venn Diagram misses out the two combinations $U^c \cap V \cap X \cap Y^c$ and $U \cap V^c \cap X^c \cap Y$. In other words, the diagram identifies the set $U \cap V^c \cap X^c \cap Y$ with the empty set (i.e., there is no region representing this set). So in the Venn Diagram the sets $Y \cap (U^c \cap V^c \cap X^c)$ and

$$T := (Y \cap (U^c \cap V^c \cap X^c)) \cup (U \cap V^c \cap X^c \cap Y) \tag{1}$$

are indistiguishable, since the second set in the union (1) is represented as the empty set. Using the laws of commutativity, associativity and distributivity and, finally, De Morgan's Rule we transform the set (1) into the set $(V \cup X \cup Y^c)^c$:

$$\begin{aligned} T &= (U^c \cap (V^c \cap X^c \cap Y)) \cup (U \cap (V^c \cap X^c \cap Y)) \\ &= (U^c \cup U) \cap (V^c \cap X^c \cap Y) \\ &= Z \cap (V^c \cap X^c \cap Y) \\ &= (V^c \cap X^c \cap Y) \\ &= (V \cup X \cup Y^c)^c. \end{aligned}$$

Here, we see that we have to **be careful with Venn Diagrams**. While Venn Diagrams usually illustrate set relations correctly for operations on three sets, they cannot represent all 16 possibilities of intersection in the plane in the case of four sets. A remedy would be to draw Venn Diagrams with sets in $\mathbb{R}^3$, but this would be hard to imagine.

2 Numbers and Inequalities

15.
$$\frac{3x+4}{2} \leq \frac{6-x}{4} \Longleftrightarrow 6x+8 \leq 6-x$$
$$\Longleftrightarrow 7x \leq -2 \Longleftrightarrow x \leq -2/7.$$

16. $$x^2 - x < 2 \Longleftrightarrow (x-2)(x+1) < 0 \Longleftrightarrow -1 < x < 2$$

17.
$$\frac{-3}{x-4} \leq x \Longleftrightarrow \frac{-3}{x-4} - x \leq 0$$
$$\Longleftrightarrow \frac{-3-x^2+4x}{x-4} \leq 0 \Longleftrightarrow \frac{x^2-4x+3}{x-4} \geq 0$$
$$\Longleftrightarrow \frac{(x-3)(x-1)}{x-4} \geq 0$$
$$\Longleftrightarrow x > 4 \text{ or } 1 \leq x \leq 3.$$

Here the critical values where factors change sign are $1, 3, 4$. Moreover, we need $x \neq 4$ to have non-zero denominator. Therefore, we have to analyse the intervals $(-\infty, 1]$, $[1, 3]$, $[3, 4)$ and $(4, \infty)$ separately.

18.
$$\begin{aligned} \frac{3}{x-4} < -x \quad &\Longleftrightarrow \quad \frac{3}{x-4} + x < 0 \\ &\Longleftrightarrow \quad \frac{3+x^2-4x}{x-4} < 0 \\ &\Longleftrightarrow \quad \frac{(x-1)(x-3)}{(x-4)} < 0 \\ &\Longleftrightarrow \quad 3 < x < 4 \text{ or } x < 1. \end{aligned}$$

19. $|x^2+x-4| = 2 \Longleftrightarrow x^2+x-4 = \pm 2$
$\Longleftrightarrow (x+3)(x-2) = 0$ or $(x+2)(x-1) = 0 \Longleftrightarrow x = -3, -2, 1, 2.$

20.
$$|8x-9| < 7x-6 \Longleftrightarrow -7x+6 < 8x-9 < 7x-6$$
$$\Longleftrightarrow -14x+6 < x-9 < -6$$
$$\Longleftrightarrow 1 < x < 3.$$

Also, $|x-2| < 1 \Longleftrightarrow -1 < x-2 < 1 \Longleftrightarrow 1 < x < 3$. The result now follows.

21.

$$\begin{aligned}
|2x+1| < 3x &\iff -3x < 2x+1 < 3x \\
&\iff -3x < 2x+1 \text{ and } 2x+1 < 3x \\
&\iff x > -\frac{1}{5} \text{ and } x > 1 \\
&\iff x > 1.
\end{aligned}$$

22. $|2x+5| > 4 \iff 2x+5 > 4$ or $2x+5 < -4 \iff x > -1/2$ or $x < -9/2$.

23.

$$\begin{aligned}
|2x+1| \le |3x-6| &\iff (2x+1)^2 \le (3x-6)^2 \\
&\iff 4x^2+4x+1 \le 9x^2-36x+36 \\
&\iff 5x^2-40x+35 \ge 0 \\
&\iff (x-7)(x-1) \ge 0 \\
&\iff x \ge 7 \text{ or } x \le 1.
\end{aligned}$$

24. The points at which the absolute values change signs are $x = 1$ and $x = 2$. So we consider 3 cases:

- $x \le 1$. Then
$$|x-1| + |x-2| = (1-x) + (2-x) = 3-2x > 1,$$
i.e., $x < 1$. So this case yields the solution set $(-\infty, 1)$.
- $2 \ge x \ge 1$. Then
$$|x-1| + |x-2| = (x-1) + (2-x) = 1 > 1,$$
which is a contradiction. So this case yields the empty solution set.
- $x \ge 2$. Then
$$|x-1| + |x-2| = (x-1) + (x-2) = 2x-3 > 1,$$
i.e., $x > 2$. So this case yields the solution set $(2, \infty)$.

Therefore, we have $x < 1$ or $x > 2$.

25. The points at which the absolute values change signs are $x = -1$ and $x = 1$. So we consider 3 cases:

- $x \le -1$. Then
$$|x-1| + |x+1| = (1-x) + (-1-x) = -2x < 2,$$
i.e., $x > -1$. So this case yields the empty solution set.
- $1 \ge x \ge -1$. Then
$$|x-1| + |x+1| = (1-x) + (x+1) = 2 < 2,$$
which is a contradiction. So this case yields the empty solution set.

- $x \geq 1$. Then

$$|x-1| + |x+1| = (x-1) + (x+1) = 2x < 2,$$

i.e., $x < 1$. So this case yields also the empty solution set.

Therefore, there are no real numbers x satisfying this inequality.

26. We have

$$|a| = \left| \frac{a+b}{2} + \frac{a-b}{2} \right| = \frac{1}{2}|(a+b) + (a-b)| \leq \frac{1}{2}(|a+b| + |a-b|).$$

Analogously, we obtain

$$|b| \leq \frac{1}{2}(|a+b| + |a-b|).$$

Combining both inequalities, we end up with

$$|a| + |b| \leq |a+b| + |a-b|.$$

3 Basics about sequences and limits

In the following, we shall be making use of the following results: the Calculus of Limits Theorem (COLT), the continuity of $\sqrt{x}$ and $\log x$ for $x > 0$, the Squeezing Theorem, and the limits $\lim_{n\to\infty} \alpha^n = 0$ (for $|\alpha| < 1$), $\lim_{x\to 0} x^{-1} \sin x = 1$, $\lim_{n\to\infty}(1 + c/n)^n = \mathrm{e}^c$, $\lim_{n\to\infty} n^p \exp(-n) = 0$ and $\lim_{n\to\infty} n^{-p} \log(1+n) = 0$ for $p > 0$.

27. (a) $0 \leq |x_n| < 1/n \to 0$ as $n \to \infty$, so $x_n \to 0$ as $n \to \infty$.
    (b) $x_n = (3 + 1/n)^2/\sqrt{4 + 1/n^4} \to 9/2$ as $n \to \infty$.
    (c) $x_n = [1 + 1/(2n)]^n \to \mathrm{e}^{1/2} = \sqrt{\mathrm{e}}$ as $n \to \infty$.
    (d) Note that $(2n+1)/(n+1) \geq 3/2$ for $n \geq 1$; and $(3/2)^{2n} = (9/4)^n$ has no limit, since $9/4 > 1$. So the given sequence has no limit either.
    (e) $x_n > \sqrt{n/2}$, so no limit.
    (f) $\log x_n = \frac{10}{n} \log n + \frac{2}{n} \log(1 + \frac{\log n}{n^5}) \to 0$, so $x_n \to \mathrm{e}^0 = 1$ as $n \to \infty$.
    (g) $x_n = -2n^2/(n^2 - 1) = -2/(1 - 1/n^2) \to -2$ as $n \to \infty$.

28. (a) $(2n+1)^2(n^4+1)^{-1/2} = (2 + \frac{1}{n})^2(1 + \frac{1}{n^4})^{-1/2} \to 4$.
    (b) $n(\sqrt{1+n^2} - n) = n/(\sqrt{1+n^2} + n) = 1/(1 + \sqrt{1 + 1/n^2}) \to 1/2$.
    (c) $\log(n) - \log(n+1) = \log \frac{n}{n+1} = \log \frac{1}{1+1/n} \to \log \frac{1}{1} = 0$.
    (d)
    $$0 < \frac{n^2 + \mathrm{e}^{-n}}{\log(n) + 5n^3} \leq \frac{1+n^2}{5n^3} = \frac{1 + 1/n^2}{5n} \to 0, \quad \text{so} \quad x_n \to 0.$$
    (e)
    $$\frac{(n!)^2}{(n-2)!(n+2)!} = \frac{n(n-1)}{(n+2)(n+1)} = \frac{1 - \frac{1}{n}}{(1 + \frac{2}{n})(1 + \frac{1}{n})} \to 1.$$

(f) $0 < n!\,n^{-n} = \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\ldots\frac{n}{n} \le \frac{1}{n} \to 0$, so $x_n \to 0$.

(g)

$$0 < \frac{2^n}{n!} = \frac{2^n}{1.2.3.4.\ldots.n} \le \frac{8}{6}\frac{2^{n-3}}{4^{n-3}} = \frac{4}{3}\left(\frac{1}{2}\right)^{n-3} \to 0, \quad \text{so} \quad x_n \to 0.$$

(h) $n\sin(\pi/n) = \pi\frac{\sin(\pi/n)}{\pi/n} \to \pi$.

(i) Note that $0 \le \log x_n = \frac{1}{n}\log(1+n^2) \le \frac{1}{n}\log(2n^2) = \frac{1}{n}(2\log n+\log 2) \to 0$. So $x_n \to \mathrm{e}^0 = 1$.

(j)

$$\frac{(n+3)!}{n!\,n^3} = \frac{(n+3)(n+2)(n+1)}{n^3} = (1+\frac{3}{n})(1+\frac{2}{n})(1+\frac{1}{n}) \to 1.$$

(k)

$$n^2\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{n}{n+1} = \frac{1}{1+1/n} \to 1.$$

29. Let $\epsilon > 0$, and suppose that $n > 2/(5\epsilon)$. Then for $n \ge 1$,

$$|x_n| = \frac{1}{5n}\,\frac{1+1/(n^2\mathrm{e}^n)}{1+(\log n)/(5n^3)} < \frac{\epsilon}{2}\left(1+\frac{1}{n^2\mathrm{e}^n}\right) < \epsilon.$$

30. (a) $x_n = \frac{1+2(\log n)/n}{\sqrt{1+2/n^2}} \to \frac{1+0}{\sqrt{1+0}} = 1$ as $n \to \infty$, by COLT.

(b) $0 \le |x_n| < 1/\sqrt{n}$, and $1/\sqrt{n} \to 0$ as $n \to \infty$. So $x_n \to 0$ by squeezing.

31. (a) $\log x_n = n^{-1}\log\left[\mathrm{e}^n(1+n^2\mathrm{e}^{-n})\right] = 1+n^{-1}\log(1+n^2\mathrm{e}^{-n}) \to 1+0\times 0 = 1$, by COLT. So $x_n \to \mathrm{e}$.

(b) $x_n = \frac{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n-1}\right)\left(\sqrt{n+1}+\sqrt{n-1}\right)}{\sqrt{n+1}+\sqrt{n-1}} = \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{\sqrt{1+1/n}+\sqrt{1-1/n}}$ $\to \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = 1$, by COLT.

(c) $x_n = \left(1-\frac{2}{n+1}\right)^n = \left(1-\frac{2}{n+1}\right)^{n+1}/\left(1-\frac{2}{n+1}\right) \to \mathrm{e}^{-2}/(1-0) = \mathrm{e}^{-2}$, by COLT.

32. (a) $\log x_n = 2n^{-1}\log n + n^{-1}\log(1+1/n) \to 0$ as $n \to \infty$, by COLT. So $x_n \to \mathrm{e}^0 = 1$.

(b) $\sqrt{x_n} = \frac{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)} = \frac{\sqrt{n}(n+1-n)}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{1+1/n}+1} \to \frac{1}{2}$ as $n \to \infty$. Hence $x_n \to 1/4$. [Alternatively, multiply out and use a similar argument.]

33. Suppose $x^* > 0$, and set $\epsilon = x^*/2$. Find N such that $|x_n - x^*| < \epsilon$ for $n \ge N$. Then $x_N - x^* > -\epsilon = -x^*/2 \Rightarrow x_N > x^*/2 > 0$, which contradicts the condition that $x_n < 0$ for all n. Hence $x^* \le 0$. We could have $x^* = 0$, for example $x_n = -1/n$.

34. $[(n+1)^2-(n-1)^2]\,/\,(n+\sqrt{n}) = 4n/(n+\sqrt{n}) = 4/(1+1/\sqrt{n}) \to 4$ as $n \to \infty$, by COLT.

35. (a) $\log x_n = 2n^{-1}\log n + n^{-1}\log(1+2/n^2) \to 0$ as $n \to \infty$. So $x_n \to \mathrm{e}^0 = 1$.
(b) $x_n = \left[\left(1+\frac{1}{n+1}\right)^{n+1} / \left(1+\frac{1}{n+1}\right)\right]^2 \to (\mathrm{e}/1)^2 = \mathrm{e}^2$ as $n \to \infty$.
(c) $x_n = n\frac{(\sqrt{n^2+1}-\sqrt{n^2-1})(\sqrt{n^2+1}+\sqrt{n^2-1})}{(\sqrt{n^2+1}+\sqrt{n^2-1})} = \frac{2n}{\sqrt{n^2+1}+\sqrt{n^2-1}} = \frac{2}{\sqrt{1+n^{-2}}+\sqrt{1-n^{-2}}} \to 1$ as $n \to \infty$.

36. $[(n+1)^2 - n^2] / (n + \log n) = (2+n^{-1})/(1+n^{-1}\log n) \to 2$ as $n \to \infty$, by COLT.

37. (a) $x_n = 1/(1+1/n)^n \to 1/\mathrm{e}$ as $n \to \infty$.
(b) $x_n = -2/[(n^2-1)\sin(1/n^2)] = \left(\frac{-2}{1-1/n^2}\right)\left(\frac{1/n^2}{\sin(1/n^2)}\right) \to -2$ as $n \to \infty$. We have used COLT and the fact that $\sin(\theta)/\theta \to 1$ as $\theta \to 0$.
(c) $\frac{3n+2}{2n+1} = \frac{3}{2} + \frac{1}{4n+2} > \frac{3}{2}$, and $\left(\frac{3}{2}\right)^n \to \infty$ as $n \to \infty$, so x_n has no limit as $n \to \infty$.

38. Suppose that $x^* \neq x'$, and let $\epsilon = |x^* - x'|$. Then we can choose $n \in \mathbb{N}$ such that $|x_n - x^*| < \epsilon/2$ and $|x_n - x'| < \epsilon/2$. But the triangle inequality gives $|x^* - x'| \leq |x_n - x^*| + |x_n - x'| < \epsilon = |x^* - x'|$, a contradiction. Thus $x^* = x'$.

39. Let $\epsilon > 0$. Since $|x_n| \to 0$ as $n \to \infty$, we have $N \in \mathbb{N}$ with

$$||x_n| - 0| < \epsilon \quad \text{for all } n \geq N.$$

Since $||x_n| - 0| = |x_n| = |x_n - 0|$, this implies that

$$|x_n - 0| < \epsilon \quad \text{for all } n \geq N,$$

showing that we also have $x_n \to 0$ as $n \to \infty$.

40. a) We have

$$(1-c)x_n = 1 + c + c^2 + \cdots + c^{n-1} - (c + c^2 + c^3 + \cdots + c^n) = 1 - c^n.$$

Since $c \neq 1$, we can divide by $1-c$ and obtain

$$x_n = \frac{1-c^n}{1-c}.$$

b) We assume $|c| < 1$. Then we know from Theorem 3.11 that $c^n \to 0$ as $n \to \infty$ and, consequently, using the COLT Theorem,

$$\lim_{n\to\infty} x_n = \frac{1 - \lim_{n\to\infty} c^n}{1-c} = \frac{1}{1-c}.$$

41. Suppose that $x^* > b$, and take $\epsilon = x^* - b$. Then we can find $n \in \mathbb{N}$ such that $|x^* - x_n| < \epsilon$. But $x_n \leq b = x^* - \epsilon$, so that $x^* - x_n \geq \epsilon$, a contradiction. Thus $x^* \leq b$.

42. Since $\{x_n\}$ is bounded, we can choose K such that $K \geq |x_n|$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow |y_n| < \epsilon/K$. But then $n > N \Rightarrow |x_n y_n| < \epsilon$, so that $x_n y_n \to 0$.

43. (a) This is false. A counterexample is the sequence $x_n = 2^{1/n}$.
(b) We have then to prove that this one is true. Let N be such that $n > N \Rightarrow |x_n - r| < (1-r)/2$. Then, for $n > N$, $|x_n| \leq r + |x_n - r| < (1+r)/2 < 1$. Thus $|x_n|^n < ((1+r)/2)^n$, which tends to 0 as $n \to \infty$.

44. Write $x_n = (t + 1/n)^n = t^n[1 + 1/(tn)]^n$. Note that $[1 + 1/(tn)]^n \to \exp(1/t)$ as $n \to \infty$. If $t > 1$, then $t^n \to \infty$, so $x_n \to \infty$. If $t = 1$, then $x_n \to \mathrm{e}$. If $0 < t < 1$, then $t^n \to 0$, so $x_n \to 0$.

45. For $|x| \leq 1$, the limit is zero, since $|x^n/n| \leq 1/n \to 0$. For $|x| > 1$, the sequence has no limit. Proof: use $n^{-1}\exp(\alpha n) \to \infty$ as $n \to \infty$ if $\alpha > 0$, putting $\alpha = \log x$.

46. (a) $p \leq x_n \leq 2^{1/n}p$. Since $2^{1/n} \to 1$ as $n \to \infty$, $x_n \to p$.
(b) $x_n = 1 - \mathrm{e}^{-n^2} \to 1$.
(c) $\frac{1}{n}\log x_n = n\log(1+\frac{1}{n}) \geq n/(n+1)$. Thus $\log x_n \geq n^2/(n+1) \geq n/2$ and so the limit as $n \to \infty$ does not exist.

47. a) If $(|a_n|)$ is convergent then so is $(a_n^2 = |a_n|^2)$, by COLT. Conversely: Assume that $(|a_n|^2 = a_n^2)$ is convergent. Then $\lim_{n\to\infty} a_n^2 \geq 0$. Assume first that $\lim_{n\to\infty} a_n^2 = 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$|a_n^2| < \epsilon^2 \quad \text{for all } n \geq N.$$

This implies that

$$|a_n| < \epsilon \quad \text{for all } n \geq N,$$

i.e., (a_n) is also convergent. Now, assume that $\lim_{n\to\infty} a_n^2 > 0$. Then there exists $A > 0$ with $A^2 = \lim_{n\to\infty} a_n^2$. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$|a_n^2 - A^2| < A\epsilon \quad \text{for all } n \geq N.$$

This implies that

$$||a_n| - A| \cdot ||a_n| + A| < A\epsilon \quad \text{for all } n \geq N.$$

Note that $||a_n| + A| \geq A > 0$ and, therefore, we can divide by $||a_n| + A|$ to obtain

$$||a_n| - A| < \frac{A}{||a_n| + A|}\epsilon \leq \epsilon \quad \text{for all } n \geq N.$$

This shows that $(|a_n|)$ is also convergent and $\lim_{n\to\infty} |a_n| = A$.

b) Assume that (a_n) is convergent and $a^* = \lim_{n\to\infty} a_n$. Then we also have $a^* = \lim_{n\to\infty} a_{n+1}$ and, using COLT,

$$\lim_{n\to\infty} a_{n+1} - a_n = a^* - a^* = 0,$$

which means that $a_{n+1} - a_n \to 0$ as $n \to \infty$.

4 More logic: Quantifiers, negation and proof techniques

48. The negation of the fact is: $X \cap Y \subset Z$ and $x \in Y$ and $x \in X \backslash Z$. This implies that $x \in Y$ and $x \in X$ and $x \notin Z$. This implies, in turn, that $x \in X \cap Y$ and $x \notin Z$, in contradiction to $X \cap Y \subset Z$.

49. a) The long version is: "For all $C > 0$ there exists $n \in \mathbb{N}$ such that we have $x_n > C$." In own words: "The sequence (x_n) is not bounded above." Negation:
$$\exists\, C > 0 \quad \forall\, n \in \mathbb{N}: \quad x_n \leq C.$$
    b) The long version is: "For all $x \in \mathbb{R}$ and all $y \geq x$, we have $f(x) \leq f(y)$." In own words: "The function $f : \mathbb{R} \to \mathbb{R}$ is monotone increasing." Negation:
$$\exists\, x \in \mathbb{R} \quad \exists\, y \geq x: \quad f(x) > f(y).$$
    c) The long version is: "For all $y \in Y$ there exists $x \in X$ such that $y = g(x)$." In own words: "Any element of Y is an image value of g" or "$g : X \to Y$ is surjective". Negation:
$$\exists\, y \in Y \quad \forall\, x \in X: \quad y \neq g(x),$$
    or even shorter:
$$\exists\, y \in Y : y \notin g(X).$$

50. a) If a triangle is not right-angled then its side lengths a, b, c do not satisfy $a^2 + b^2 = c^2$.
    b) If there is a pair of opposite angles of a quadrilateral in the plane which do not add up to 180^o then its four vertices do not lie on a common circle.

51. We have
$$\begin{aligned}
X \backslash \left(\bigcap_{n \in \mathbb{N}} X_n \right) &= X \backslash \{x \in X \mid \forall\, n \in \mathbb{N} : x \in X_n\} \\
&= \{x \in X \mid \text{not } (\forall\, n \in \mathbb{N} : x \in X_n)\} \\
&= \{x \in X \mid \exists\, n \in \mathbb{N} : x \notin X_n\} \\
&= \{x \in X \mid \exists\, n \in \mathbb{N} : x \in (X \backslash X_n)\} \\
&= \bigcup_{n \in \mathbb{N}} (X \backslash X_n).
\end{aligned}$$
Analogously, we have
$$\begin{aligned}
X \backslash \left(\bigcup_{n \in \mathbb{N}} X_n \right) &= X \backslash \{x \in X \mid \exists\, n \in \mathbb{N} : x \in X_n\} \\
&= \{x \in X \mid \text{not } (\exists\, n \in \mathbb{N} : x \in X_n)\} \\
&= \{x \in X \mid \forall\, n \in \mathbb{N} : x \notin X_n\} \\
&= \{x \in X \mid \forall\, n \in \mathbb{N} : x \in (X \backslash X_n)\} \\
&= \bigcap_{n \in \mathbb{N}} (X \backslash X_n).
\end{aligned}$$

52. (i) $\bigcup_{n\in\mathbb{N}}[1/n,1) = (0,1)$. Since every set $[1/n,1)$ is contained in $(0,1)$, we have the inclusion "$\subset$". On the other hand, for every $x \in (0,1)$, we have $x > 0$ and there exists $n \in \mathbb{N}$ with $1/x < n$, i.e., $1/n < x$. This shows that $x \in [1/n,1)$ and, therefore, x lies in the union $\bigcup_{n\in\mathbb{N}}[1/n,1)$. This shows the inclusion "$\supset$". and both sets are equal.

(ii) $\bigcap_{n\in\mathbb{N}}(-1/n,2/n) = \{0\}$. Since $0 \in (-1/n,2/n)$ for all $n \in \mathbb{N}$, we have the inclusion "$\supset$". We show that there is no real $x \neq 0$ in this intersection. Let $x \neq 0$. Then we can find $n \in \mathbb{N}$ with $|x| > 2/n$. This implies that $x \notin (-1/n,2/n)$ and, therefore, $x \notin \bigcap_{n\in\mathbb{N}}(-1/n,2/n)$. This shows the inclusion "$\subset$", and both sets are equal.

(iii) $\bigcup_{n\in\mathbb{N}}[1,n) = [1,\infty)$. Obviously, we have $[1,n) \subset [1,\infty)$, which shows the inclusion "$\subset$". For every $x \in [1,\infty)$, there exists $n \in \mathbb{N}$ with $x < n$, and we see that $x \in \bigcup_{n\in\mathbb{N}}[1,n)$. This shows the inclusion "$\supset$", and both sets are equal.

53. (a) We know for $x \in [0,2\pi)$ that $0 < \sin(x) \leq 1$ is equivalent to $x \in (0,\pi)$. We also know that the sine function is 2π periodic, i.e., $\sin(x+2k\pi) = \sin(x)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Therefore, we can write

$$\{x \in \mathbb{R} \mid 0 < \sin(x) \leq 1\} = \bigcup_{k\in\mathbb{Z}}(2k\pi,(2k+1)\pi).$$

(b) Let us first express the set X of all natural numbers which are squares or cubes of primes:

$$X = \bigcup_{p \text{ prime}} \{p^2,p^3\}.$$

Now, let us apply De Morgan to find the complement of X:

$$\mathbb{N}\backslash X = \mathbb{N}\backslash \bigcup_{p \text{ prime}} \{p^2,p^3\} = \bigcap_{p \text{ prime}} \mathbb{N}\backslash\{p^2,p^3\}.$$

54. Statements (b) and (c) are plainly equivalent. To show that (a)$\Rightarrow$(c), suppose (a). Then if $\epsilon > 0$, $x+\epsilon > x \geq y$, so $x+\epsilon > y$. Thus (a) $\Rightarrow$ (c). Conversely, suppose (b). Then, if $x < y$, $(y-x) > 0$, so that $x > y-(y-x) = x$, Contradiction. Thus $x \geq y$, and so (b) $\Rightarrow$ (a).

55. i) Let $A(n)$ be "$1+2+3+\cdots+n = \frac{n(n+1)}{2}$".
(a) Start of Induction: We have $1 = \frac{1\cdot 2}{2}$.
(b) Induction Step: Assume that $A(n)$ holds. Then

$$1+2+3+\cdots+n+(n+1) = (1+2+3+\cdots+n)+(n+1) = \frac{n(n+1)}{2}+(n+1) = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2},$$

i.e., $A(n+1)$ is then also true.

- Let $A(n)$ be "$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{4}$".
(a) Start of Induction: We have $1^2 = \frac{1\cdot 2\cdot 3}{6}$.

(b) Induction Step. Assume that $A(n)$ holds. Then

$$\begin{aligned}1^2+2^2+\cdots+n^2+(n+1)^2 &= (1^2+2^2+\cdots+n^2)+(n+1)^2 = \\ \frac{n(n+1)(2n+1)}{6}+(n+1)^2 &= \frac{(n+1)(n(2n+1)+6(n+1)}{6} = \\ \frac{(n+1)(2n^2+7n+6)}{6} &= \frac{(n+1)(n+2)(2n+3)}{6} = \\ &\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6},\end{aligned}$$

i.e., $A(n+1)$ is then also true.

56. Let $x > -1$ and $A(n)$ be "$(1+x)^n \geq 1+nx$".
(a) Start of Induction: We have $(1+x)^1 \geq 1+1\cdot x$.
(b) Induction Step: Assume that $A(n)$ holds. Then $1+x > 0$ and

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+(n+1)x+nx^2 \geq 1+(n+1)x,$$

i.e., $A(n+1)$ is then also true.

57. a) If $c > 1$, we can set $c = 1+x$ with $x > 0$. Bernoulli's Inequality yields

$$c^n = (1+x)^n \geq 1+nx.$$

Given $K > 0$, we can then choose $N \in \mathbb{N}$ such that $1+Nx \geq K$. Then we have for all $n \geq N$:

$$c^n \geq 1+nx \geq 1+Nx \geq K.$$

b) If $0 < c < 1$, we can set $c = \frac{1}{1+x}$ with $x > 0$. Bernoulli's Inequality yields

$$c^n = \frac{1}{(1+x)^n} \leq \frac{1}{1+nx}.$$

Given $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that $1+Nx > 1/\epsilon$, i.e., $\epsilon > 1/(1+Nx)$. Then we have for all $n \geq N$:

$$0 < c^n \leq \frac{1}{1+nx} \leq \frac{1}{1+nX} < \epsilon.$$

This shows that $c^n \to 0$ as $n \to \infty$.

58. a) We need to show that $\sqrt{a_1a_2} \leq \frac{a_1+a_2}{2}$ for all $a_1, a_2 > 0$. Since all involved numbers are non-negative, we have

$$\begin{aligned}\sqrt{a_1a_2} \leq \frac{a_1+a_2}{2} \quad &\Leftrightarrow \quad 4a_1a_2 \leq (a_1+a_2)^2 \\ &\Leftrightarrow \quad 0 \leq (a_1+a_2)^2 - 4a_1a_2 = (a_1-a_2)^2.\end{aligned}$$

Since $0 \leq (a_1-a_2)^2$ is always true, we see that $S(2)$ is true.

b) Assume that $S(2)$ and $S(n)$ are both true. First observe that

$$G(a_1, \dots, a_{2n}) = \sqrt{G(a_1, \dots, a_n)G(a_{n+1}, \dots, a_{2n})}.$$

Applying the true statement $S(2)$ to the non-negative numbers $G(a_1, \dots, a_n)$ and $G(a_{n+1}, \dots, a_{2n})$, we then obtain

$$G(a_1, \dots, a_{2n}) \le \frac{G(a_1, \dots, a_n) + G(a_{n+1}, \dots, a_{2n})}{2}.$$

Applying the true statement $S(n)$ twice, we conclude that

$$G(a_1, \dots, a_{2n}) \le \frac{A(a_1, \dots, a_n) + A(a_{n+1}, \dots, a_{2n})}{2}.$$

it is easy to check that

$$\frac{A(a_1, \dots, a_n) + A(a_{n+1}, \dots, a_{2n})}{2} = A(a_1, \dots, a_{2n}0,$$

which implies that $S(2n)$ is also true.

c) Assume that $n \ge 2$ and $S(n+1)$ is true. Note first that we have

$$G(a_1, \dots, a_n, G(a_1, \dots, a_n)) = \sqrt[n+1]{a_1 \dots a_n \sqrt[n]{a_1 \dots a_n}} =$$
$$\sqrt[n+1]{(a_1 \dots a_n)^{1+1/n}} = \sqrt[n+1]{(a_1 \dots a_n)^{(n+1)/n}} = \sqrt[n]{a_1 \dots a_n} = G(a_1, \dots, a_n).$$

Applying the true statement $S(n+1)$ to the non-negative numbers $a_1, \dots, a_n$ and $G(a_1, \dots, a_n)$, we obtain

$$G(a_1, \dots, a_n) = G(a_1, \dots, a_n, G(a_1, \dots, a_n)) \le$$
$$A(a_1, \dots, a_n, G(a_1, \dots, a_n)) = \frac{1}{n+1}(a_1 + \dots + a_n) + \frac{1}{n+1} G(a_1, \dots, a_n).$$

This implies that

$$\left(1 - \frac{1}{n+1}\right) G(a_1, \dots, a_n) \le \frac{1}{n+1}(a_1 + \dots + a_n),$$

which simplifies to

$$nG(a_1, \dots, a_n) \le a_1 + \dots + a_n.$$

Division by n yields

$$G(a_1, \dots, a_n) \le A(a_1, \dots, a_n).$$

This shows that $S(n)$ is also true.

d) Let $n \ge 2$. Our aim is to show that $S(n)$ is true. Starting from the fact that $S(2)$ is true (proved in a)), we iteratively conclude (using b)) that $S(2^k)$ is true for every $k \in \mathbb{N}$. We can choose $k \in \mathbb{N}$ large enough that $2^k \ge n$. So we know that $S(2^k)$ is true. Then, using c) iteratively, we conclude that $S(m)$ is true for all $2 \le m \le 2^k$. Since $2 \le n \le 2^k$, we see that $S(n)$ must be true.

5 The Completeness Axiom for $\mathbb{R}$

59. (a) $|2x-1| < 11 \Leftrightarrow -11 < 2x-1 < 11 \Leftrightarrow x > -5$ and $x < 6 \Leftrightarrow x \in (-5, 6)$. Thus $\sup(S) = 6$ and $\inf(S) = -5$.
(b) For $x \geq 1$, $x + |x-1| = 2x-1$, so that the set is unbounded above. For $x < 1$, we have $x + |x-1| = 1$. Moreover, $\inf_{x\geq 1}(2x-1) = 1$, so the inf is 1.

60. (a) Note that $x^2 + x - 1 < 0$ is equivalent to $(-1-\sqrt{5})/2 < x < (-1+\sqrt{5})/2$. Therefore, we have

$$\{x \mid x < 0 \,\text{and}\, x^2 + x - 1 < 0\} = \left(\frac{-1-\sqrt{5}}{2}, 0\right)$$

and we conclude that the infimum is $(-1-\sqrt{5})/2$ and the supremum is 0.
(b) Let $x_n = 1/n + (-1)^n$ for $n \in \mathbb{N}\}$. We have

$$\{x_n \mid n \in \mathbb{N}\} = \{0, 3/2, -2/3, 5/4, -4/5, 7/6, -6/7, \ldots\}.$$

So we guess that the supremum is 3/2 and infimum is -1. Since

$$-1 \leq 1/n + (-1)^n,$$

we see that -1 is a lower bound. Since

$$\lim_{k\to\infty} x_{2k-1} = \lim_{k\to\infty} \frac{1}{2k-1} + (-1)^{2k-1} = -1,$$

the infimum is indeed -1. Moreover, we see that, for $n \geq 2$,

$$x_n = \frac{1}{n} + (-1)^n \leq \frac{1}{2} + 1 = \frac{3}{2},$$

and $x_1 = 0$. Therefore, 3/2 is an upper bound which is assumed by x_2, and we have that 3/2 is the supremum.

61. Note that $f'(x) = e^x/(1+e^x)^2 > 0$, so f is an increasing function on $\mathbb{R}$. Thus $\inf(f) = \lim_{x\to-\infty} f(x) = 0$ and $\sup(f) = \lim_{x\to\infty} f(x) = 1$.

62. Claim: $|f(x)| < 1$ for all x. This says $|x|/(1+|x|) < 1 \iff |x| < 1 + |x|$ which is clearly true. So 1 is an upper bound for f, and -1 is a lower bound. Also, $f(x) \to \pm 1$ as $x \to \pm\infty$, so these are the sup and the inf, respectively.

63. (a) Write $x_n = (n^2 - n)/(n^2+1)$ Note that $x_n = (1-1/n)/(1+1/n^2) \leq 1$, and also that $x_n \to 1$ as $n \to \infty$. Thus $\sup(X) = 1$. When $n = 1$, $x_n = 0$; otherwise, $x_n \geq 0$. Thus $\inf(X) = \min(X) = 0$.
(b) Note that $(2m+n)/(m+3n) = (2+n/m)/(1+3n/m)$. Write $r = n/m$. Then

$$\frac{2+r}{1+3r} = \begin{cases} 2 - \frac{5r}{1+3r} & \leq 2 \\ \frac{1}{3} + \frac{5}{3+9r} & \geq \frac{1}{3}. \end{cases}$$

Now r may be made as small or as large as we please, and hence both $\frac{5r}{1+3r}$ and $\frac{5}{1+3r}$ can be made arbitrarily small. Thus $\sup(Y) = 2$ and $\inf(Y) = 1/3$.

64. $g(x) = \frac{(x\cos^2 x+1)-3}{2(x\cos^2 x+1)} = \frac{1}{2} - \frac{3}{2(x\cos^2 x+1)}$. Thus $g(x) \leq \frac{1}{2}$ for all $x \geq 0$. Moreover, as n increases, the sequence of values $g(n\pi)$ can be made arbitrarily close to $\frac{1}{2}$. Thus $\sup_{x\geq 0} g(x) = \frac{1}{2}$.

65. (a) $\frac{1-x^2}{2+x^2} \leq f(x) \leq \frac{1+x^2}{2+x^2}$, with equality on the left when $x = (2n+1)\pi$, and on the right when $x = 2n\pi$. But $\frac{1-x^2}{2+x^2} = -1+\frac{3}{2+x^2} \geq -1$, and $\frac{1+x^2}{2+x^2} = 1-\frac{1}{2+x^2} \leq 1$. Thus choosing n large enough will make $f(2n\pi)$ arbitrarily close to 1, and $f((2n+1)\pi)$ arbitrarily close to -1, so that $\sup(f) = 1$ and $\inf(f) = -1$.
(b) Note first that $g(-x) = g(x)$, so we can restrict to $x \geq 0$. Observe that $g(x) \geq 0$ for all $x \geq 0$, and $g(0) = 0$, so $\inf(g) = 0$. Also, $g(x) \to 0$ as $x \to \infty$, so $\sup(g) = \max(g) = g(c)$, where c is some number such that $g'(c) = 0$. Now $g'(x) = 2x(1-x^2)\exp(-x^2)$, so $c = 1$ and $\sup(g) = g(1) = \mathrm{e}^{-1}$.

66. Because $\sup(g)$ is an upper bound for g, it must be an upper bound for f also, and so $\sup(g) \geq \sup(f)$. It is not necessarily true that $\sup(g) > \sup(f)$. Counterexample: consider the functions $f(x) = \tanh x$, $g(x) = 1$ on $\mathbb{R}$.

67. $f(x) \geq 0$ and $f(0) = 0$, so $\inf(f) = 0$. Clearly $f(x) < 1$ and $f(x) \to 1$ as $x \to \infty$, so $\sup(f) = 1$.

68. Note that $f(0) = 0$, $f(x) > 0$ for $x > 0$, and $f(x) \to 0$ as $x \to \infty$. So $\inf(f) = 0$. Also, $f'(x) = 0 \Leftrightarrow (2+x)/(2\sqrt{x}) - \sqrt{x} = 0 \Leftrightarrow x = 2$. So $\sup(f) = \max(f) = f(2) = \sqrt{2}/4$.

69. Note that $f(x) \to 0$ as $x \to 0$, and $f(x) > 0$ for all $x > 0$; so $\inf(f) = 0$. Since also $f(x) \to 0$ as $x \to \infty$, the supremum must occur where $f'(x) = 0$. Now $f'(x) = (1-x^2)/(x^2+1)^2$ vanishes iff $x = 1$, so $\sup(f) = f(1) = 1/2$.

70. The set $X = \{x \in [a,b] : f(x) \geq x\}$ is a subset of $[a,b]$ and, therefore, bounded. Moreover, X is not empty since $a \in X$. So $x^* = \sup(X)$ must exist, by the Completeness Axiom for $\mathbb{R}$. Moreover, b is an upper bound of X and we conclude that $x^* \leq b$. No number $< a$ can be an upper bound, so we also have $a \leq x^*$. This shows that $x^* \in [a,b]$. Now we need to show that $f(x^*) = x^*$.

    Assume that $f(x^*) < x^*$. Since $x^* \notin X$, there exists a sequence (x_n) with $x_n \in X$ for all $n \in \mathbb{N}$ and $x_n \to x^*$, by the definition of the supremum. Since $x_n \in X$, we have $x_n \leq x^*$. Let $\epsilon = x^* - f(x^*) > 0$. Since $x_n \to x^*$, there exists $N \in \mathbb{N}$ with $|x^* - x_N| < \epsilon$. Since $x_N \leq x^*$, we have $x^* - x_N < \epsilon$. Using that f is monotone increasing, we obtain

$$f(x_N) \leq f(x^*) = x^* - \epsilon < x_N.$$

    But this means that $x_N \notin X$, which is a contradiction.

    Assume that $f(x^*) > x^*$. Then we have $x^* \in X$ and $x^* < b$. Let $\epsilon = f(x^*) - x^*$. Choose $x \in [a,b]$ with $x > x^*$ and $x - x^* < \epsilon$. Then we have, by the monotonicity of f,

$$f(x) \geq f(x^*) > x^* + \epsilon > x,$$

    i.e., $x \in X$. Since x^* is an upper bound of X, we then have $x \leq x^*$, which is, again, a contradiction.

    Therefore, we must have $f(x^*) = x^*$.

6 More on limits of sequences

71. The fixed points x satisfy $x^2 = 6+x$, so that $x = 3$ or $x = -2$. Considering the starting point, and the fact that x_n is always going to be positive, we expect convergence to 3. From the iteration formula,

$$x_{n+1} - 3 = \sqrt{6 + x_n} - 3 = \frac{x_n - 3}{\sqrt{6 + x_n} + 3}.$$

Thus $|x_{n+1} - 3| \le |x_n - 3|/3$. So $|x_1 - 3| \le |x_0 - 3|/3 = 7/3$, $|x_2 - 3| \le 7/9$, $|x_3 - 3| \le 7/27$, and in general $|x_n - 3| \le 7/3^n \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} x_n = 3$.

72. a) We have for $n \in \mathbb{N}$:

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} \\
&= \left(\frac{(n+2)n}{(n+1)^2}\right)^n \left(\frac{n+2}{n+1}\right) \\
&= \left(\frac{n^2+2n}{n^2+2n+1}\right)^n \left(\frac{n+2}{n+1}\right) \\
&= \left(1 - \frac{1}{(n+1)^2}\right)^n \left(\frac{n+2}{n+1}\right).
\end{aligned}$$

b) Recall Bernoulli's inequality $(1+x)^n \ge 1 + nx$ for $x > -1$ and $n \in \mathbb{N}$. Choosing $x = -1/(n+1)^2 > -1$, we conclude that

$$\frac{x_{n+1}}{x_n} \ge \left(1 - \frac{n}{(n+1)^2}\right)\left(\frac{n+2}{n+1}\right) = \frac{((n+1)^2 - n)(n+2)}{(n+1)^3} =$$
$$\frac{(n+1)^2(n+2) - n(n+2)}{(n+1)^3} = \frac{(n+1)^2(n+2) - ((n+1)^2 - 1)}{(n+1)^3} =$$
$$\frac{1 + (n+1)^2(n+2-1)}{(n+1)^3} = \frac{1 + (n+1)^3}{(n+1)^3} > 1,$$

i.e., $x_n = (1 + 1/n)^n > 0$ is monotone increasing.

c) We have for $n \in \mathbb{N}$ using, again, Bernoulli's inequality:

$$\frac{y_{n+1}}{y_n} = \frac{\left(\frac{n+1}{n+2}\right)^{n+2}}{\left(\frac{n}{n+1}\right)^{n+1}} = \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+1}\left(\frac{n+1}{n+2}\right) =$$
$$\left(1 + \frac{1}{n(n+2)}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \ge \left(1 + \frac{n+1}{n(n+2)}\right)\left(\frac{n+1}{n+2}\right) =$$
$$\frac{(n^2+3n+1)(n+1)}{n(n+2)^2} = \frac{n(n+2)^2+1}{n(n+2)^2} = 1 + \frac{1}{n(n+2)^2} > 1,$$

i.e., y_n is monotone increasing, as well.

d) We have for $n \in \mathbb{N}$:

$$x_{n+1}y_n = \left(1 + \frac{1}{n+1}\right)^{n+1} \left(1 - \frac{1}{n+1}\right)^{n+1} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \leq 1.$$

We have $x_1 = 2$ and, for $n \geq 2$,

$$x_n \leq \frac{1}{y_{n-1}} \leq \frac{1}{y_1} = \frac{1}{(1-1/2)^2} = 4.$$

Therefore (x_n) is bounded above by 4.

e) Convergence of (x_n) follows now from the fact that (x_n) is bounded (since $2 \leq x_n \leq 4$) and monotone increasing (Theorem 6.2).

73. (a) Assume that $1 \leq a_n \leq (1+\sqrt{5})/2$. Then

$$1 \leq \sqrt{1} \leq a_{n+1} = \sqrt{a_n + 1} \leq \sqrt{\frac{3+\sqrt{5}}{2}} = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2} = \frac{1+\sqrt{5}}{2}.$$

(b) We know that $1 \leq a_n \leq (1+\sqrt{5})/2$ for all $n \in \mathbb{N}$. We need to show for every $n \in \mathbb{N}$ that $a_{n+1} \geq a_n$. This is equivalent to $a_n^2 - a_n + 1 \leq 0$. Factorisation leads to the equivalent statement

$$\left(a_n - \frac{1+\sqrt{5}}{2}\right)\left(a_n - \frac{\sqrt{5}-1}{2}\right) \leq 0,$$

which is true for all $n \in \mathbb{N}$, since $a_n \leq (1+\sqrt{5})/2$ and $(\sqrt{5}-1)/2 < 1 \leq a_n$.

(c) We know from Theorem 6.2 that (a_n) is convergent. Let $a^* = \lim_{n\to\infty} a_n$. We know from $a_n \geq 1$ that $a^* \geq 1$. Taking the limit on both sides of the recursion formula, the limit must satisfy

$$a^* = \sqrt{a^* + 1},$$

i.e., $(a^*)^2 - a^* + 1 = 0$. The two solutions are $(\sqrt{5} \pm 1)/2$, and $a^* \geq 1$ implies that $a^* = (1+\sqrt{5})/2$.

74. Property (a) means that the sequence (a_n) is monotone increasing and the sequence (b_n) is monotone decreasing and non-emptyness of the intervals means that $a_n \leq b_n$. Therefore, both sequences are also bounded and they have limits, by Theorem 6.2. Property (b) means that both limits agree:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c.$$

Moreover, we know from Theorem 6.2 that c is an upper bound of $\{a_n \in n \in \mathbb{N}\}$, i.e., $a_n \leq c$ for all $n \in \mathbb{N}$. Analogously, we have $b_n \geq c$ for all $n \in \mathbb{N}$. This shows that $c \in [a_n, b_n] = I_n$ for all $n \in \mathbb{N}$. Assume there would be $c' \neq c$ with $c' \in I_n$ for all $n \in \mathbb{N}$. Since $b_n - a_n \to 0$ as $n \to \infty$, there exists $n \in \mathbb{N}$ with $b_n - a_n < |c' - c|$. On the other hand, we assumed that $c, c' \in I_n = [a_n, b_n]$, therefore

$$a_n \leq c, c' \leq b_n,$$

which implies that $|c' - c| \leq b_n - a_n$, a contradiction.

75. Choose the intervals $I_n = (0, 1/n)$. These intervals are non-empty and we have both properties (a) and (b). Assume we have $c \in I_n$ for all $n \in \mathbb{N}$. Then we must have $c > 0$ and there exists $N \in \mathbb{N}$ with $c > 1/N$. But this shows that $c \notin I_N$, a contradiction.

76. (a) Let $\epsilon > 0$ be given. Since $x^* = \lim_{n\to\infty} x_n$, there exists $N \in \mathbb{N}$ such that we have
$$|x_n - x^*| < \epsilon \quad \forall\, n \geq N.$$
Note that we have $n_j \geq j$, and so we conclude that
$$|x_{n_j} - x^*| < \epsilon \quad \forall\, j \geq N.$$
This shows that $(x_{n_j})_{j\in\mathbb{N}}$ also convergent and
$$x^* = \lim_{j\to\infty} x_{n_j}.$$
(b) The contrapositive statement reads as follows: Let $(x_{n_j})_{j\in\mathbb{N}}$ be a subsequence of (x_n). If $(x_{n_j})_{j\to\mathbb{N}}$ is not convergent with limit $x^* \in \mathbb{R}$, then (x_n) is not convergent with limit $x^* \in \mathbb{R}$.

77. Let (x_n) be a Cauchy sequence. Choosing $\epsilon = 1$, we find $N \in \mathbb{N}$ such that
$$|x_n - x_m| < 1 \quad \forall\, n, m \geq N.$$
In particular, we have
$$|x_n - x_N| < 1 \quad \forall\, n \geq N.$$
This shows that we have for all $n \geq N$:
$$|x_n| \leq |x_n - x_N| + |x_N| < |x_N| + 1.$$
Choosing
$$M := \max\{|x_1|, \dots, |x_{N-1}|, |x_N| + 1\},$$
we have
$$|x_n| \leq M \quad \forall\, n \in \mathbb{N},$$
i.e., (x_n) is bounded.

78. Let (x_n) be convergent. Then there exists $x^* = \lim_{n\to\infty} x_n$. We need to show that (x_n) is Cauchy. Let $\epsilon > 0$. Since $x_n \to x^*$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that
$$|x^* - x_n| < \frac{\epsilon}{2} \quad \forall\, n \geq N.$$
This implies for all $n, m \geq N$ that
$$|x_n - x_M| \leq |x_n - x^*| + |x^* - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$
i.e., (x_n) is Cauchy.

79. (a) We have

$$|a_{n+2} - a_{n+1}| = \left| \frac{a_{n+1} + a_n}{2} - a_{n+1} \right| = \frac{1}{2} |a_{n+1} - a_n|$$

for all $n \geq 1$. This implies that

$$|a_{n+1} - a_n| = \frac{1}{2^{n-1}} |b - a|.$$

Using the triangle inequality and the geometric series, we conclude that

$$|a_{n+k} - a_n| \leq |a_{n+k} - a_{n+k-1}| + \cdots + |a_{n+1} - a_n| \leq \frac{|b-a|}{2^{n-1}} \left(\frac{1}{2^{k-1}} + \cdots + 1 \right) \leq \frac{|b-a|}{2^{n-2}}.$$

For given $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\frac{|b-a|}{2^{N-2}} < \epsilon.$$

Then we have for $n, m \geq N$,

$$|a_n - a_m| \leq \frac{|b-a|}{2^{N-2}} < \epsilon,$$

i.e., (a_n) is Cauchy.
(b) Here we may use a method introduced in Discrete Mathematics. Make the Ansatz $a_n = c^n$ with $c \neq 0$. Then the recursion formula $a_{n+2} = (a_{n+1} + a_n)/2$ leads to $c^{n+2} = (c^{n+1} + c^n)/2$, i.e., to the quadratic equation

$$2c^2 - c - 1 = 0.$$

The solutions of this equation are

$$c = \frac{1}{4}(1 \pm \sqrt{1+8}),$$

i.e., $c = 1$ and $c = -1/2$. So the sequence has the general form

$$a_n = A \cdot 1^n + B \cdot \left(-\frac{1}{2} \right)^n.$$

The start values $a_1 = a$ and $a_2 = b$ yield $A = (a + 2b)/3$ and $B = 4(b - a)/3$. So we obtain the explicit formula

$$a_n = \frac{a + 2b}{3} + \frac{4(b-a)}{3} \left(-\frac{1}{2} \right)^n,$$

and therefore

$$\lim_{n \to \infty} a_n = \frac{a + 2b}{3}.$$

80. a) Assume that we do not have $\lim_{n\to\infty} u_n = 0$. Since $u_n \geq 0$, there exists $\epsilon > 0$ such that we have $u_n \geq \epsilon$ for infinitely many $n \in \mathbb{N}$. On the other hand, we can find $N \in \mathbb{N}$ such that $1/N \leq \epsilon/4$. Then we have the following facts:

(i) For all $n \geq N$ with $u_n \geq \epsilon$,

$$0 \leq u_{n+1} \leq \frac{u_n}{2} + \frac{1}{n} \leq \frac{u_n}{2} + \frac{\epsilon}{4} \leq \frac{3}{4}u_n.$$

(ii) For all $n \geq N$ with $u_n < \epsilon$,

$$0 \leq u_{n+1} \leq \frac{u_n}{2} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon.$$

(ii) shows that if for some index $n_0 \geq N$ we have $u_{n_0} < \epsilon$, then $u_n < \epsilon$ for all $n \geq n_0$. This would imply that we would have $u_n \geq \epsilon$ for only finitely many $n \in \mathbb{N}$, in contradiction to our assumption. Therefore, we have $u_n \geq \epsilon$ for all $n \geq N$. But then (i) tells us for $n \geq N$ that

$$0 \leq u_{n+1} \leq \frac{3}{4}u_n.$$

Repeating this inequality and using the fact that $u_{n+j} \geq \epsilon$ for all $j \geq 1$ and $n \geq N$, we obtain

$$0 \leq u_{n+k} \leq \left(\frac{3}{4}\right)^k u_n.$$

Since $(3/4)^k \to 0$ as $k \to \infty$, this implies that $u_n \to 0$ as $n \to \infty$. But this is in contradiction to our original assumption.

- Note that $a_n \geq 1$ implies

$$a_{n+1} = \sqrt{a_n} + \frac{1}{n} \geq \sqrt{a_n} \geq 1.$$

Since $a_1 \geq 1$, this shows that $a_n \geq 1$ for all $n \in \mathbb{N}$. Therefore, $u_n = a_n - 1 \geq 0$ for all $n \in \mathbb{N}$. Next, we observe that for $u \geq 0$ we have

$$\sqrt{1+u} \leq 1 + \frac{u}{2}.$$

This follows immediately via squaring (which is here an equivalence since all numbers involved are non-negative). Therefore, we have for all $n \in \mathbb{N}$:

$$u_{n+1} = a_{n+1} - 1 = \sqrt{1+u_n} + \frac{1}{n} - 1 \leq 1 + \frac{u_n}{2} + \frac{1}{n} - 1 = \frac{u_n}{2} + \frac{1}{n}.$$

So we see that the conditions of a) are satisfied and we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 + u_n = 1.$$

81. We have
$$u_{2(k+1)-1} = u_{2k-1} - \frac{1}{2k-1} + \frac{1}{2k} < u_{2k-1},$$
which shows that (u_{2k-1}) is monotone decreasing. Similarly, we have
$$u_{2(k+1)} = u_{2k} + \frac{1}{2k} - \frac{1}{2k+1} > u_{2k},$$
which shows that (u_{2k}) is monotone increasing. Therefore, u_1 is an upper bound for (u_{2k-1}) and u_2 is a lower bound for (u_{2k}). Since $u_{2k} < u_{2k+1} = u_{2k} + 1/(2k)$, we see that $(u_{2k}$ is bounded below by u_2 and above by u_1. Using Theorem 6.2, we conclude that (u_{2k}) is convergent. Let $\lim_{k\to\infty} u_{2k} = u^*$. It remains to show that (u_n) is also convergent. Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that
$$|u_{2k} - u^*| < \frac{\epsilon}{2} \quad \forall\, k \geq N_1.$$
Then we have
$$|u_{2k+1} - u^*| \leq |u_{2k+1} - u_{2k}| + |u_{2k} - u^*| < \frac{1}{2k} + \frac{\epsilon}{2} \quad \forall\, k \geq N_1.$$
We choose $N_2 \in \mathbb{N}$ such that $1/(2k) < \epsilon/2$ for all $k \geq N_2$. Then we have, for all $k \geq N = \max\{N_1, N_2\}$:
$$|u_{2k} - u^*|, |u_{2K+1-u^*}| < \epsilon,$$
i.e.,
$$|u_n - u^*| < \epsilon \quad \forall\, n \geq 2N.$$
This proves convergence of (u_n).

82. We assume there exists $w_1 \in \mathbb{R}$ such that (w_n) is convergent. Let $w^* = \lim_{n\to\infty} w_n \in \mathbb{R}$. Taking the limits on both sides of the recursion formula, we must then have
$$w^* = (w^*)^2 + 1,$$
i.e., the limit $w^* \in \mathbb{R}$ must satisfy the quadratic equation $(w^*)^2 - w^* + 1 = 0$. The solutions of this equation are
$$w^* = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}.$$
But both solutions are not real. So we end up with a contradiction.

83. Let (u_n) be bounded and $c \in \mathbb{R}$. We assume that we have (i) for every convergent subsequence $(u_{n_j})_{j\in\mathbb{N}}$ that $\lim_{j\to\infty} u_{n_j} = c$ and (ii) (u_n) does not have the limit c. We conclude from (ii) that there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ we have $n \geq N$ with
$$|u_n - c| > \epsilon.$$
Therefore, choosing $N = 1$, we find $n_1 \geq 1$ such that
$$|u_{n_1} - c| > \epsilon.$$

Choosing $N = n_1 + 1$, we find $n_2 \geq n_1 + 1 > n_1$ such that

$$|u_{n_2} - c| > \epsilon.$$

Repeating this argument, we find a subsequence (u_{n_j}) with

$$|u_{n_j} - c| > \epsilon \quad \forall j \in \mathbb{N}. \tag{2}$$

The subsequence (u_{n_j}) is bounded, since (u_n) is bounded. By the Bolzano-Weierstrass Theorem, there must exists a convergent subsequence of (u_{n_j}) (which is, therefore, also a subsequence of (u_n)) and which cannot converge to c because of (2). But this is a contradiction to our assumption (i).

84. A sequence converging to $\sqrt{5}$ is given by $u_1 = 1$ and $u_{n+1} = (u_n + 5/u_n)/2$ for $n \geq 1$. The very rough error is given by

$$|u_n - \sqrt{5}| \leq \frac{1}{2^{n-3}}|u_3 - u_2|.$$

We obtain

$$u_2 = \frac{1}{2}\left(u_1 + \frac{5}{u_1}\right) = 3, \quad u_3 = \frac{1}{2}\left(u_2 + \frac{5}{u_2}\right) = \frac{7}{3}.$$

Therefore, we have

$$|u_n - \sqrt{5}| \leq \frac{1}{3 \times 2^{n-2}}.$$

We need to find the smallest $n \in \mathbb{N}$ such that $1/(3 \times 2^{n-2}) \leq 10^{-4}$, i.e. $2^{n-2} \geq \frac{10^4}{3}$. We have $2^{11} = 2048 < \frac{10^4}{3}$ and $2^{12} = 4096 > \frac{10^4}{3}$, therefore the element u_{14} has definitively the required accuracy. We obtain

n	u_n	n	u_n	n	u_n
4	2.238095238...	8	2.236067977...	12	2.236067977...
5	2.236068896...	9	2.236067977...	13	2.236067977...
6	2.236067977...	10	2.236067977...	14	2.236067977...
7	2.236067977...	11	2.236067977...		

In fact, the above error estimate is extremely crude. It turns out that we have

$$\begin{aligned}
|u_4 - \sqrt{5}| &\leq 3 \cdot 10^{-3}, \\
|u_5 - \sqrt{5}| &\leq 10^{-6}, \\
|u_6 - \sqrt{5}| &\leq 2 \cdot 10^{-13}, \\
|u_7 - \sqrt{5}| &\leq 8 \cdot 10^{-27}, \\
|u_8 - \sqrt{5}| &\leq 2 \cdot 10^{-53}, \\
|u_14 - \sqrt{5}| &\leq 4 \cdot 10^{-3424},
\end{aligned}$$

i.e., the number of correct digits roughly doubles at every step and we could have stopped the calculation at

$$u_5 = 2 + \frac{233}{987}.$$

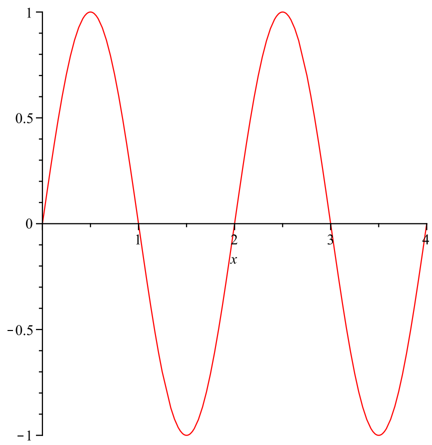
7 Functions, Limits and continuity

85. We have

$$\begin{aligned} f^{-1}(\{-1\}) &= \{\}, \\ f^{-1}(\{0\}) &= \{(0,0,0)\}, \\ f^{-1}(\{1\}) &= \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}, \\ f^{-1}([1,2]) &= \{(x,y,z) \in \mathbb{R}^3 \mid 1 \leq x^2+y^2+z^2 \leq 2\}. \end{aligned}$$

This means that $f^{-1}(\{-1\})$ is the empty set, $f^{-1}(\{0\})$ is the set containing just the origin of $\mathbb{R}^3$ and $f^{-1}(\{1\})$ is the Euclidean sphere around the origin of radius 1. Finally, $f^{-1}([1,2])$ is a closed Euclidean annulus, centered at the origin, with inner radius 1 and outer radius 2.

86. The graph of f looks as follows:



We have

$$\begin{aligned} f^{-1}([0,1)) = \{x \in [0,4] \mid 0 \leq \sin(\pi x) < 1\} \\ = [0,1/2) \cup (1/2,1] \cup [2,5/2) \cup (5/2,3] \cup \{4\}. \end{aligned}$$

87. (a) Let $y \in f(f^{-1}(Y_0))$. Then there exists $x \in f^{-1}(Y_0)$ with $y = f(x)$. But $x \in f^{-1}(Y_0)$ means that $f(x) \in Y_0$. So we conclude that there exists $x \in X$ such such $y = f(x) \in Y_0$. So we have

$$y \in f(f^{-1}(Y_0)) \quad \Rightarrow \quad y \in Y_0.$$

This shows that $f(f^{-1}(Y_0)) \subset Y_0$.

(b) Let $x \in X_0$. Then we have $f(x) \in f(X_0)$. Since $f(x) \in f(X_0)$, we conclude that $x \in f^{-1}(f(X_0))$. So we have

$$x \in X_0 \quad \Rightarrow \quad x \in f^{-1}(f(X_0)).$$

This shows that $X_0 \subset f^{-1}(f(X_0))$.

88. (a) $f(x) = 1/\sqrt{1+4/x^2} \to 1$ as $x \to \infty$, by COLT.
(b) $f(x) = (1+2x^{-1}\log x)/(3+2/x) \to 1/3$ as $x \to \infty$, by COLT.
(c) $f(x) = x[\sqrt{x^2+3}-x][\sqrt{x^2+3}+x]/[\sqrt{x^2+3}+x] = 3x/[\sqrt{x^2+3}+x]$ $= 3/[\sqrt{1+3/x^2}+1] \to 3/2$ as $x \to \infty$, by COLT.
(d) $f(n\pi) = n\pi$ for $n = 1, 2, 3, \ldots$; so there is no limit as $x \to \infty$.

89. We have

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{5x+5h+1}-\sqrt{5x+1}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{5x+5h+1}-\sqrt{5x+1}}{h}\frac{\sqrt{5x+5h+1}+\sqrt{5x+1}}{\sqrt{5x+5h+1}+\sqrt{5x+1}} \\
&= \lim_{h\to 0}\frac{(5x+5h+1)-(5x+1)}{h(\sqrt{5x+5h+1}+\sqrt{5x+1})} \\
&= \lim_{h\to 0}\frac{5}{\sqrt{5x+5h+1}+\sqrt{5x+1}} = \frac{5}{2\sqrt{5x+1}}.
\end{aligned}$$

90. We have

$$\begin{aligned}
\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2} &= \lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} \\
&= \lim_{x\to 1}\frac{(x-1)(\sqrt{x^2+3}+2)}{x^2-1} \\
&= \lim_{x\to 0}\frac{\sqrt{x^2+3}+2}{x+1} = 2.
\end{aligned}$$

91. (a) We have

$$\begin{aligned}
\lim_{x\to\infty}(x-\sqrt{x^2-1}) &= \lim_{x\to\infty}\frac{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}{(x+\sqrt{x^2-1})} \\
&= \lim_{x\to\infty}\frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} \\
&= \lim_{x\to\infty}\frac{1}{x+\sqrt{x^2-1}} = 0,
\end{aligned}$$

since $0\frac{1}{x+\sqrt{x^2-1}} \le \frac{1}{x}$ and $\frac{1}{x} \to 0$ as $x \to \infty$, by the Squeezing Theorem.

(b) Solving $x^2/a^2 - y^2/b^2 = 1$ for y yields $y = \pm b\sqrt{(x^2/a^2)-1}$ The hyperbola has two branches and we use $y = b\sqrt{(x^2/a^2)-1}$. We have to show that

$$\lim_{x\to\infty} b\left(\frac{x}{a}-\sqrt{\frac{x^2}{a^2}-1}\right) = 0.$$

We have

$$\begin{aligned}
\lim_{x\to\infty} b\left(\frac{x}{a}-\sqrt{\frac{x^2}{a^2}-1}\right) &= \frac{b}{a}\lim_{x\to\infty} x-\sqrt{x^2-a^2} \\
&= \frac{b}{a}\lim_{x\to\infty}\frac{x^2-(x^2-a^2)}{x+\sqrt{x^2-a^2}} \\
&= ab\lim_{x\to\infty}\frac{1}{x+\sqrt{x^2-a^2}} = 0.
\end{aligned}$$

92. We have

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x} &= \lim_{x\to 0}\frac{(\sqrt{x+3}-\sqrt{3})(\sqrt{x+3}+\sqrt{3})}{x(\sqrt{x+3}+\sqrt{3})}\\ &= \lim_{x\to 0}\frac{x}{x(\sqrt{x+3}+\sqrt{3})}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{x+3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.\end{aligned}$$

93. Plugging $x = 1$ into $2x^4 - 6x^3 + x^2 + 3$ yields $2 - 6 + 1 + 3 = 0$. Therefore, $x - 1$ must be a factor of $2x^4 - 6x^3 + x^2 + 3$. Polynomial division yields

$$2x^4 - 6x^3 + x^2 + 3 : x - 1 \;=\; 2x^3 - 4x^2 - 3x - 3,$$

and we obtain

$$\lim_{x\to 1}\frac{2x^4 - 6x^3 + x^2 + 3}{x-1} = \lim_{x\to 1} 2x^3 - 4x^2 - 3x - 3 = 2 - 4 - 3 - 3 = -8.$$

94. We have

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} &= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\\ &= \lim_{h\to 0}\frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}.\end{aligned}$$

95. (a) We have $8 - x^3 = (2 - x)(x^2 + 2x + 4)$. So we obtain

$$\begin{aligned}\lim_{x\to 2}\left(\frac{1}{2-x} - \frac{12}{8-x^3}\right) = \lim_{x\to 2}\frac{x^2+2x-8}{(2-x)(x^2+2x+4)} &= \lim_{x\to 2}\frac{(2-x)(-4-x)}{(2-x)(x^2+2x+4)}\\ &= -\lim_{x\to 2}\frac{4+x}{x^2+2x+4} = -\frac{6}{12} = -\frac{1}{2}.\end{aligned}$$

(b) We have

$$\lim_{x\to 0+}\frac{x}{|x|} = \lim_{x\to 0+}\frac{x}{x} = 1$$

and

$$\lim_{x\to 0-}\frac{x}{|x|} = \lim_{x\to 0-}\frac{x}{-x} = -1.$$

Since the one-sided limits do not coincide, the limit of $\frac{x}{|x|}$ as $x \to 0$ does not exist.

96. (a) We have

$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{x}-2}{4-x} &= \lim_{x\to 4}\frac{\sqrt{x}-2}{4-x}\frac{\sqrt{x}+2}{\sqrt{x}+2}\\ &= \lim_{x\to 4}\frac{x-4}{(4-x)(\sqrt{x}+2)} = \frac{-1}{2+2} = -\frac{1}{4}.\end{aligned}$$

(b) We have

$$\lim_{h\to 0}\frac{(2+h)^4-16}{h} = \lim_{h\to 0}\frac{2^4+4\cdot 2^3h+6\cdot 2^2h^2+4\cdot 2h^3+h^4-16}{h} = \lim_{h\to 0}\frac{h(32+24h+8h^2)}{h} = 32.$$

97. If $x \geq 0$ we have

$$f(x) = \frac{3x+x}{7x-5x} = \frac{4}{2} = 2.$$

Therefore, we have $\lim_{x\to\infty} f(x) = \lim_{x\to 0+} f(x) = 2$. If $x \leq 0$ we have

$$f(x) = \frac{3x-x}{7x+5x} = \frac{2}{12} = \frac{1}{6}.$$

Therefore, we have $\lim_{x\to-\infty} f(x) = \lim_{x\to 0-} f(x) = \frac{1}{6}$.

98. (a) We have

$$\lim_{x\to\infty}\left(\frac{3x}{x-1}-\frac{2x}{x+1}\right) = \lim_{x\to\infty}\frac{3x(x+1)-2x(x-1)}{x^2-1} = \lim_{x\to\infty}\frac{x^2+5x}{x^2-1} = \lim_{x\to\infty}\frac{1+5/x}{1-1/x^2} = 1.$$

(b) We have

$$\lim_{x\to 1}\frac{1}{x-1}\left(\frac{1}{x+3}-\frac{2x}{3x+5}\right) = \lim_{x\to 1}\frac{1}{x-1}\frac{3x+5-2x(x+3)}{(x+3)(3x+5)} = \lim_{x\to 1}\frac{1}{x-1}\frac{-2x^2-3x+5}{(x+3)(3x+5)} = \lim_{x\to 1}\frac{1}{x-1}\frac{(x-1)(-2x-5)}{(x+3)(3x+5)} = \lim_{x\to 1}-\frac{2x+5}{(x+3)(3x+5)} = -\frac{7}{4\cdot 8} = -\frac{7}{32}.$$

99. Later!

100. Later!

101. The contrapositive statement is: "If f is not continuous at c then there exists a convergent sequence (x_n) with $\lim_{n\to\infty} x_n = c$ and $(f(x_n))$ does not have $f(c)$ as a limit".

So we need to prove this. If f is not continuous at c then there exists $\epsilon > 0$ such that, for all $\delta > 0$, there exists $x' \in (c-\delta, c+\delta)$ with $|f(x)-f(x')| \geq \epsilon$. Choosing $\delta = 1/n$ for $n \in \mathbb{N}$, we find therefore $x_n \in (c-1/n, c+1/n)$ with $f(x)-f(x_n)| \geq \epsilon$. This means that there exists $\epsilon > 0$ and a sequence (x_n) with $c-1/n \leq x_n \leq c+1/n$ (i.e., $x_n \to c$ as $n\to\infty$) such that

$$|f(x)-f(x_n)| \geq \epsilon.$$

But this implies that the sequence $f(x_n)$ does not converge to $f(c)$, finishing the proof.

102. Let $c \in \mathbb{R}$. Given $\epsilon > 0$, we choose $\delta = \epsilon/3 > 0$ and we obtain

$$|f(x) - f(c)| = 3|x - c| < \epsilon \quad \forall\, |x - c| < \delta = \frac{\epsilon}{3}.$$

This shows that f is continous. Let $c > 0$. Then we have for $x > 0$,

$$|g(x) - g(c)| = \frac{|x - c|}{xc}.$$

IF we assume that $x > c/2$, we obtain

$$|g(x) - g(c)| < \frac{2}{c^2}|x - c|.$$

This suggests that we choose, for given $\epsilon > 0$, $\delta = \min\{c/2, c^2\epsilon/2\} > 0$. Then we have for all $|x - c| < \delta$:

$$|g(x) - g(c)| < \frac{2}{c^2}|x - c| < \frac{2\delta}{c^2} \leq \epsilon.$$

This shows that g is continuous.

103. We know that f is continuous at $n \in \mathbb{N}$ if we have

$$\forall\, \epsilon\, \exists\, \delta : |f(m) - f(n)| < \epsilon \quad \forall\, m \in \mathbb{N} \,\text{with}\, |m - n| < \delta.$$

Now, we choose $\delta = 1/2$. Then the condition $|m - n| < \delta$ means that $m = n$ and, therefore, $|f(m) - f(n)| = 0 < \epsilon$ for any $\epsilon > 0$. This shows that f is continuous at n.

104. We choose $\epsilon = |A|/2 > 0$. Then there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{|A|}{2} \quad \forall\, x \in (c - \delta, c + \delta) \cap [a, b].$$

Since $c \in (a, b)$, we can make $\delta > 0$ smaller,if necessary such that $a \leq c - \delta$ and $c + \delta \leq b$. Then we have

$$|f(x) - f(c)| < \frac{|A|}{2} \quad \forall\, x \in (c - \delta, c + \delta).$$

But this means that, for all $x \in (c - \delta, c + \delta)$,

$$|A| = |f(c)| \leq |f(x)| + |f(c) - f(x)| < |f(x)| + \frac{|A|}{2},$$

i.e.,

$$|f(x)| > \frac{|A|}{2}.$$

105. $f(x)g(x) - f(a)g(a) = f(x)[g(x) - g(a)] + [f(x) - f(a)]g(a)$. We show that given $\epsilon > 0$ we can make both terms smaller than $\epsilon/2$.
First, we can restrict the size of $f(x)$ by being close enough to a: since f is continuous, there is a $\delta_1 > 0$ such that $|x-a| < \delta_1 \Rightarrow |f(x)-f(a)| < 1$. Then by the triangle inequality, $|f(x)| \leq |f(a)| + |f(x) - f(a)| \leq |f(a)| + 1$ for such x.
Secondly, we can now ensure that the first term is small enough: since g is continuous, there is a $\delta_2 > 0$ such that $|x - a| < \delta_2 \Rightarrow |g(x) - g(a)| < \epsilon/2(|f(a)| + 1)$. Then if $|x - a| < \min\{\delta_1, \delta_2\}$, we have $|f(x)(g(x) - g(a))| \leq (|f(a)| + 1)|g(x) - g(a)| < \epsilon/2$.
The second term is if anything simpler. If $g(a) = 0$ we have nothing to do. Otherwise, we can find δ_3 such that $|x - a| < \delta_3 \Rightarrow |f(x) - f(a)| < \epsilon/2|g(a)|$. Then for such x, we have $|(f(x) - f(a))g(a)| < \epsilon/2$.
Bringing all of this together, set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$; then $|x - a| < \delta \Rightarrow |f(x)g(x) - f(a)g(a)| < \epsilon$.

106. We make use of the identities $\min\{a, b\} = (a+b-|a-b|)/2$ and $\max\{a, b\} = (a+b+|a-b|)/2$. Then if f and g are continuous, M and m will be continuous if $|f - g|$ is continuous. But $|x|$ is a continuous function of x, and so $|f - g|$ is just obtained by subtracting two continuous functions and composing the result with another continuous function. The result is another continuous function.

107. Let $c \in [a, b]$. Plainly $g(c) \leq h(c)$, since $h(c)$ is the supremum of a set containing $g(c)$. Since g is continuous, given any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in N(c, \delta) = (c - \delta, c + \delta) \cap [a, b]$, $|g(x) - g(c)| < \epsilon$.

If $h(c) = g(c)$, then we proceed as follows. Given arbitrary $\epsilon > 0$, and δ as above, we take $x \in N(c, \delta)$. If $x \leq c$, then (making use of the fact that h is increasing) $g(x) \leq h(x) \leq h(c) = g(c)$. But $|g(x)-g(c)| < \epsilon$, so $|h(x)-h(c)| < \epsilon$. If $x > c$, then $h(x) = \sup g([a, x]) = \max\{h(c), \sup g([c, x])\} = \sup g([c, x])$ (using $h(c) = g(c)$). Then $h(x) - h(c) = h(x) - g(c) = \sup g([c, x]) - g(c) = g(y) - g(c)$ for some $y \in [c, x]$ (since g is continuous). But $|g(y) - g(c)| < \epsilon$ for all $y \in [c, x]$, so $|h(x) - h(c)| < \epsilon$. Thus h is continuous at c.

If $h(c) < g(c)$, then we take $\epsilon = h(c) - g(c)$, and with δ chosen as above, we find that $g(x) < h(c)$ for all $x \in N(c, \delta)$. But $h(c) = \sup g([a, c]) = g(y)$ for some $y \in [a, c]$. Then $y \notin N(c, \delta)$ (since, if so, $g(y) < h(c)$). It follows that $h(x) = g(y)$ for all $x \in N(c, \delta)$, i.e. h is constant on $N(c, \delta)$, and is therefore continuous at c.

108. We have to show that the function $f(x) - 1 = 2x^3 - 3x^2 + 7x - 10$ has a zero in $(1, 2)$. We have

$$\begin{aligned} f(1) &= 2 - 3 + 7 - 10 = -4, \\ f(2) &= 16 - 12 + 14 - 10 = 8. \end{aligned}$$

Therefore, we must have $c \in (1, 2)$ with $f(c) - 1 = 0$, by the Intermediate Value Theorem.

109. $f(x) = 0$ is equivalent to $\cos(x) = -e^{-x}$. If $x < 0$, we have $e^{-x} > 1$ and $|\cos(x)| \leq 1$, so we cannot have zeros for $x < 0$. Also $x = 0$ is not a zero since $f(0) = 2$. This shows that there are no zeros of f in $(-\infty, 0]$. Choosing $x_n = n\pi$ for $n \in \mathbb{N}$ we obtain

$$f(x_n) = 1 + (-1)^n e^{n\pi}.$$

Since $e^{n\pi} > 1$, we conclude that

$$f(x_n) \begin{cases} > 0, & \text{if } n \in \mathbb{N} \text{ even,} \\ < 0, & \text{if } n \in \mathbb{N} \text{ odd.} \end{cases}$$

This means that we have a zero of f in every interval $(n\pi, (n+1)\pi)$ for $n \in \mathbb{N}$, by the Intermediate Value Theorem.

110. (a) Assume that n is odd and $a_0 > 0$. Let $A = \max\{a_{n-1}, \ldots, a_1\}$. Let $C > \max\{1, nA\}$. Then we have for $x = -C < 0$:

$$f(x) \leq -C^n + A(C^{n-1} + C^{n-2} + \cdots + C + 1) \leq -C^n + nAC^{n-1} < 0.$$

Since $f(0) = a_0 > 0$, we must have $c \in (-C, 0)$ with $f(c) = 0$ by the Intermediate Value Theorem.

(b) We assume that $a_0 < 0$. We choose, again, $C > \max\{1, nA\}$ and obtain for $x = C > 0$:

$$f(x) \geq C^n - A(C^{n-1} + C^{n-2} + \cdots + C + 1) \geq C^n - nAC^{n-1} > 0.$$

Since $f(0) = a_0 < 0$, we must have $c \in (0, C)$ with $f(c) = 0$ by the Intermediate Value Theorem.

(c) Assume that n is even and $a_0 < 0$. We choose, again, $C > \max\{1, na\}$ and obtain for $x = -C < 0$:

$$f(x) \geq C^n - A(C^{n-1} + C^{n-2} + \cdots + C + 1) \geq C^n - nAC^{n-1} > 0.$$

Since $f(0) = a_0 < 0$, we must have $c \in (-C, 0)$ with $f(c) = 0$ by the Intermediate Value Theorem.