Analysis 1 (Epiphany Term 2015)

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Lectures (all in CLC013): Mon 14 (week 11 is Collection), Tues 16, Fri 13 (Fri weeks 13,15,17,19 are Problem Classes)

Contents

Some Books

a) *Mathematical Analysis, a straightforward approach*, K. G. Binmore.

b) *Calculus*, Michael Spivak.

c) *Limits, Limits Everywhere, The Tools of Mathematical Analysis*, David Applebaum.

d) *Calculus*, Schaum's Outlines, F. Ayres and E. Mendelson.

e) *Advanced Calculus*, Schaum's Outlines, R. Wrede and M. Spiegel.

f) *How to Think Like a Mathematician*, Kevin Houston.

g) *How to Read and Do Proofs*, Daniel Solow.

The books a)-c) are good introductions into material of Analysis. Don't be fooled by the title of b), this book is a highly recommendable Analysis book. However, c) falls short on the concepts of continuity, differentiation and integration. d) and e) contain a lot of solved problems and is a good exercise source. Finally, the books f) and g) cover the logic as well as proof techniques which are important in the study of Analysis.

This lecture notes are not meant to be complete, but they are a useful additional source of information!

The symbol  at the margin of the page informs you about pitfalls to be avoided.

At the end of each chapter you find a box with important points which are useful for you to check whether you understood the crucial concepts and can apply the methods introduced in this chapter.

8 Differentiable functions

8.1 Basics on differentiable functions

Let us start with the definition of differentiability.

Definition 8.1. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ be a function. f is* differentiable at $c \in X$ *if* $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ *exists. We denote this limit by $f'(c)$ and call $f'(c)$ the* derivative *of f at c. f is called a* differentiable function *if f is differentiable at all points $c \in X$.*

differentiable at c
derivative $f'(c)$
differentiable function

Remark. (a) The expression $\frac{f(x)-f(c)}{x-c}$ has an important geometric interpretation: It is the slope of the straight line passing through the two points $(c, f(c)) \in \mathbb{R}^2$ and $(x, f(x)) \in \mathbb{R}^2$ of the graph of f. As $x \to c$, the second point $(x, f(x))$ approaches the first point $(c, f(c))$, and the limit describes the slope of the *tangent* of the graph of f at the point $(c, f(c)) \in \mathbb{R}^2$.

(b) Differentiability at $c \in X$ can also be described in the (ϵ, δ)-formalism. f is differentiable at $c \in X$ if there exists a number $L \in \mathbb{R}$ (the derivative $f'(c)$) such that there exists for every $\epsilon > 0$ a positive number $\delta > 0$ such that

$$\left| \frac{f(c)-f(x)}{c-x} - L \right| < \epsilon \qquad \forall\, x \in X \text{ with } |x-c| < \delta.$$

(c) Another equivalent formulation for differentiability of f at $c \in X$ is the following: There exists a function $f_1 : X \to \mathbb{R}$ such that

$$f(x) = f(c) + (x-c)f_1(x)$$

and f_1 is continuous at c. Namely, we choose

$$f_1(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & \text{if } x \neq c, \\ f'(c) & \text{if } x = c. \end{cases}$$

Example. Let us prove differentiability of $f(x) = 1/x$ on $(0, \infty)$. Let $c \in (0, \infty)$. Then we have

$$\frac{f(x)-f(c)}{x-c} = \frac{c-x}{xc(x-c)} = -\frac{1}{xc},$$

which implies that

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = -\lim_{x\to c} \frac{1}{xc} = \frac{1}{c^2}.$$

This shows that $f'(c) = -1/c^2$.

Differentiability is a stronger property than continuity:

Theorem 8.2. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. If f is differentiable at $c \in X$ then f is also continuous at c.*

Proof. We have

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} \to 0 \cdot f'(c) = 0 \quad \text{as } x \to c.$$

This shows that $\lim_{x \to c} f(x) = f(c)$, in other words, f is continuous at c. □

Remark. There are continuous functions which are not differentiable. The easiest example is the function $f(x) = |x|$, which is continuous on $\mathbb{R}$ but not differentiable at $c = 0$. However, the construction of a function $f : \mathbb{R} \to \mathbb{R}$ which is everywhere continuous but nowhere differentiable is a much more difficult task. Historically, there was the longstanding belief that every continuous function should be differentiable everywhere except for a set of isolated points. This misconception was eradicated by Karl Weierstrass, who published in 1872 an everywhere continuous and nowhere differentiable function (see also the Wikipedia article "Weierstrass function").

Theorem 8.3. *(a) Let f, g be two functions which are differentiable at c. Then their sum $f + g$ is also differentiable at c and we have*

$$(f + g)'(c) = f'(c) + g'(c).$$

Similarly, their product fg is also differentiable at c with

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c). \tag{1}$$

(b) Let f, g be two functions such that g is differentiable at c and f is differentiable at $g(c)$. Then the composition $f \circ g$ is also differentiable at c and we have

$$(f \circ g)'(c) = f'(g(c))\, g'(c). \tag{2}$$

(c) Let f be a function which is differentiable at c and we have $f(c) \neq 0$. Then $1/f$ is also differentiable at c and we have

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f^2(c)}.$$

Proof. We only prove (1) and (2) of the Theorem via the equivalent formulation given in (c) of the above remark.

(a) We have $f(x) = f(c)+(x-c)f_1(x)$ and $g(x) = g(c)+(x-c)g_1(x)$. This implies that

$$(fg)(x) = (fg)(c) + (x-c)h(x).$$

with $h(x) = f(c)g_1(x) + f_1(x)g(c) + (x-c)f_1(x)g_1(x)$. It is easy to see that h is continuous at c and, therefore, fg is differentiable at c with derivative

$$(fg)'(c) = \lim_{x\to c} h(x) = f(c)g_1(c)+f_1(c)g(c) = f(c)g'(c)+f'(c)g(c).$$

(b) We can write $f(y) = f(g(c)) + (y - g(c))f_1(y)$ and $g(x) = g(c) + (x-c)g_1(x)$ with f_1 continuous at $g(c)$ and g_1 continuous at c. Then we have

$$\begin{aligned}(f\circ g)(x) = f(g(x)) = f(g(c)) + (g(x) - g(c))f_1(g(x)) = \\ f(g(c)) + (x-c)g_1(c)f_1(g(x)) = (f\circ g)(c) + (x-c)h(x)\end{aligned}$$

with $h(x) = (f_1 \circ g)(x))g_1(c)$. Since f_1 is continuous at $g(c)$ and g is continuous at c by Theorem 8.2, the composition $f_1 \circ g$ is continuous at c (see Theorem 7.8(iv)). Using again the equivalent formulation for differentiability, we conclude that $f \circ g$ is differentiable in c and we have

$$(f\circ g)'(c) = \lim_{x\to c} h(x) = f_1(g(c))g_1(c) = f'(g(c))g'(c).$$

□

Reader's Task. Prove part (c) of Theorem 8.3.

8.2 All types of Mean Value Theorems

You have already seen the following fact and its proof in the Calculus course:

Theorem 8.4. *If f is differentiable at c and has a local maximum or minimum at c then $f'(c) = 0$.*

You have also seen *Rolle's Theorem* and the *Mean Value Theorem* and their proofs in the Calculus course, so we just recall their statements:

Rolle's Theorem

Theorem 8.5 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b) and suppose that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Mean Value Theorem

Theorem 8.6 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rolle's Theorem implies an even stronger mean value theorem:

Cauchy's Mean Value Theorem

Theorem 8.7 (Cauchy's Generalised Mean Value Theorem)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

CAUTION. You may wonder why this is not a simple corollary of the Mean Value Theorem. But applying Theorem 8.6 to both functions f and g would lead to

$$\frac{f'(c_1)}{g'(c_2)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

The important point of Theorem 8.7 is that we can choose $c_1 = c_2$ on the left hand side.

Proof. Consider the function

$$h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x).$$

Then we have

$$h(a) = g(b)f(a) - f(b)g(a) = h(b).$$

h is obviously continuous on $[a, b]$ and differentiable on (a, b), and we can apply Rolle's Theorem to h. Therefore, there exists $c \in (a, b)$ such that

$$0 = h'(c) = (g(b) - g(a))f'(c) - (f(b) - f(a))g'(c). \tag{3}$$

Since $g'(x) \neq 0$ for all $x \in (a, b)$, we conclude from Theorem 8.6 that $g(b) - g(a) \neq 0$, and we can rewrite (3) as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

□

L'Hôpital's Rule

Theorem 8.8 (L'Hôpital's Rule)**.** *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable and $c \in (a, b)$. Assume that $f(c) = 0$ and $g(c) = 0$. Assume further that $\lim_{x\to c} f'(x)/g'(x)$ exists. Then also $\lim_{x\to c} f(x)/g(x)$ exists and we have*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(c)}{g'(c)}.$$

There is a simple proof of L'Hôpital's Rule[1] in the special case that f' and g' are continuous at c, which you may have seen in the Calculus course. The proof of the general case uses (3), which holds without the assumption $g'(x) \neq 0$ for all $x \in (a, b)$ (see, e.g., Spivak's Calculus Book).

Reader's Task. Prove Theorem 8.8.

Example. Show that $\lim\limits_{x\to 0} \dfrac{\cos^2(x) - 1}{x^2}$ exists and calculate it. Let $f(x) = \cos^2(x) - 1$ and $g(x) = x^2$. Then f and g are differentiable and we have $f(0) = g(0) = 0$, so the first assumption of Theorem 8.8 is satisfied. But we still do not know whether $\lim_{x\to 0} f'(x)/g'(x)$ exists, since $\lim_{x\to 0} f'(x) = \lim_{x\to 0} g'(x) = 0$. But the first assumption of Theorem 8.8 is still satisfied for the differentiable functions $f'(x) = -2\sin(x)\cos(x)$ and $g'(x) = 2x$, since $f'(0) = 0$ and $g'(0) = 0$, and we see that

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \sin^2(x) - \cos^2(x) = -1$$

exists. Arguing backwards, this shows that $\lim_{x\to 0} f'(x)/g'(x)$ exists and also that $\lim_{x\to 0} f(x)/g(x)$ exists with

$$\lim_{x\to 0} \frac{\cos^2(x) - 1}{x^2} = \lim_{x\to 0} \frac{-\sin(x)\cos(x)}{x} = \lim_{x\to 0} \sin^2(x) - \cos^2(x) = -1.$$

CAUTION. Be careful that all conditions of L'Hôpital's Rule are satisfied, in particular the condition that f, g are differentiable and that $f(c) = g(c) = 0$. Without checking these conditions, L'Hôpital's Rule may lead to **absolutely wrong results**, like the following:

$$\lim_{x\to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x\to 1} \frac{6x}{2} = 3.$$

[1] Marquis Guillaume de l'Hôpital(1661-1704) was a French mathematician, who presented this rule for the first time in his 1696 book on differential calculus. The discoverer of this rule is, however, believed to be the mathematician Johann Bernoulli(1667-1748), who was a member of a family which produced a dynasty of many famous Swiss mathematicians.

In fact, the correct limit is -4:

$$\lim_{x\to 1} \frac{x^3+x-2}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x^2+x+2)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{x^2+x+2}{x-2} = -4.$$

8.3 The Newton method revisited

We begin with the following definition.

Definition 8.9. *Let $-\infty < a < b < \infty$ and $I = (a,b)$ or $I = [a,b]$. Then $C(I)$ stands for the set of all continuous functions $f : I \to \mathbb{R}$ (in the case $I = [a,b]$, we only require one-sided continuity of f at $x = a$ and $x = b$)[2].* $C((a,b))$ and $C([a,b])$
Let $I = (a,b)$ or $I = [a,b]$ and $k \geq 1$. Then $C^k(I)$ stands for the set of all functions $f : I \to \mathbb{R}$ which have k continuous derivatives[3], i.e., the k-th derivative $f^{(k)} : I \to \mathbb{R}$ exists and is continuous[4]. $C^k((a,b))$ and $C^k([a,b])$

We recall the following fact (Taylor's Theorem with Lagrange remainder) from the Calculus course for functions $f \in C^2((a,b))$ and for all $c, x \in (a,b)$

$$f(c) = f(x) + f'(x)(c-x) + \frac{f''(\eta)}{2}(c-x)^2 \tag{4}$$

for a suitable point η between c and x. This is crucial in the proof of the following fact, which is a very effective and fundamental numerical method to find the zeros of functions (already mentioned in Section 6.2).

Newton iteration

Theorem 8.10 (Newton iteration)**.** *Let $f \in C^2((a,b))$ and $f(c) = 0$ for some $c \in (a,b)$. Assume that $f' > 0$ (i.e., f is strictly monotone increasing[5]) and $f'' > 0$ (i.e., f is strictly convex[6]). Assume further that $f(x_1) > 0$ for some $x_1 \in (a,b)$ and define the sequence $(x_n)_{n\in\mathbb{N}}$ recursively by*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{5}$$

for all $n \in \mathbb{N}$. Then the sequence (x_n) is convergent to c.

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[2] In fact, $C(I)$ is an *abstract real vector space* as introduced in Linear Algebra, since we can multiply functions in $C(I)$ with scalars in $\mathbb{R}$ and add them to obtain new functions in $C(I)$.

[3] Also $C^k((a,b))$ carries the additional structure of an abstract real vector space.

[4] If the boundary points a or b belong to I, then we have to take one-sided derivatives at these points

[5] This means that $f(x_1) < f(x_2)$ for all $a < x_1 < x_2 < b$.

[6] This means that $f(\alpha x_1 + \beta x_2) < \alpha f(x_1) + \beta f(x_2)$ for all $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and $a < x_1 < x_2 < b$, in other words, $f()$ lies strictly below the straight line segment connecting $f(x_1)$ and $f(x_2)$.

Moreover, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_{n+1} - c| \leq \left(\frac{f''(c)}{2f'(c)} + \epsilon \right) |x_n - c|^2. \tag{6}$$

Remark. (a) The geometric meaning of (5) is the following: Draw a tangent to the graph of f at the point $(x_n, f(x_n))$. Since $f' > 0$, this tangent is not horizontal and must, therefore, intersect with the x-axis at a unique point. This unique point is x_{n+1}. In other words, Newton's method is based on the approximation of the actual function f by a linear function with the same tangent.

(b) The estimate (6) provides important information about the convergence speed of the Newton iteration, namely, that this iteration converges quadratically near the zero c. This means in practical terms that the number of correct decimals eventually doubles with every iteration step.

Proof. Since f is strictly monotone increasing on (a, b) and $f(c) = 0$ and $f(x_1) > 0$, we can conclude that $x_1 > c$. We show first that we have

$$c \leq \cdots \leq x_4 \leq x_3 \leq x_2 \leq x_1, \tag{7}$$

i.e., (x_n) is monotone decreasing with lower bound c. Therefore we only need to show the following fact: If $c \leq x_n$ and $f(x_n) \geq 0$ then also $c \leq x_{n+1} \leq x_n$ and $f(x_{n+1}) \geq 0$. Then (7) follows by induction. Assume that $c \leq x_n$ and $f(x_n) \geq 0$. Since $f' > 0$, we conclude that

$$x_{n+1} = x_n - \underbrace{\frac{f(x_n)}{f'(x_n)}}_{\geq 0} \leq x_n.$$

Manipulating the recursion formula we obtain

$$f(x_n) - f'(x_n)x_n = -f'(x_n)x_{n+1}. \tag{8}$$

Employing (4) yields

$$f(c) = f(x_n) + f'(x_n)(c - x_n) + \frac{f''(\eta_n)}{2}(c - x_n)^2$$

for a suitable point $\eta_n \in [c, x_n]$. Plugging (8) into this Taylor formula leads to

$$0 = f(c) = f'(x_n)c - f'(x_n)x_{n+1} + \frac{f''(\eta_n)}{2}(c - x_n)^2,$$

i.e.,

$$0 = f'(x_n)(c - x_{n+1}) + \frac{f''(\eta_n)}{2}(c - x_n)^2. \tag{9}$$

Since the second summand is ≥ 0 by assumption, the first summand must be ≤ 0 and, using $f' > 0$, we conclude that $c - x_{n+1} \leq 0$, i.e., $c \leq x_{n+1}$. So we proved that $c \leq x_{n+1} \leq x_n$. Then strict monotonicity of f implies that $f(x_{n+1}) \geq f(c) = 0$, i.e., $f(x_{n+1}) \geq 0$, finishing the induction proof of our first statement.
Then we conclude from (7) with Theorem 6.2 that (x_n) is convergent. Let $x^* = \lim_{n\to\infty} x_n \geq c$ be its limit. Taking the limit in the recursion formula and using continuity of f and f', we conclude that

$$x^* = x^* - \frac{f(x^*)}{f'(x^*)},$$

i.e., $f(x^*) = 0$. Strict monotonicity of f implies that f can have at most one zero, leading to $x^* = c$. This proves the convergence.
For the convergence rate (6), we revisit equation (9) above:

$$0 = f'(x_n)(c - x_{n+1}) + \frac{f''(\eta_n)}{2}(c - x_n)^2. \tag{10}$$

Since $x_n \to c$ and f' is continuous, we can find for every $\epsilon > 0$ an index $N \in \mathbb{N}$ such that

$$\frac{f''(\eta_n)}{2f'(x_n)} \leq \frac{f''(\eta_n)}{2f'(\eta_n)} + \epsilon \quad \forall\, n \geq N.$$

Plugging this into (10), we obtain for all $n \geq N$

$$|x_{n+1} - c| = \frac{f''(\eta_n)}{2f'(x_n)}(c - x_n)^2 \leq \left(\frac{f''(\eta_n)}{2f'(\eta_n)} + \epsilon\right) |x_n - c|^2,$$

finishing the proof. □

Important concepts/typical problems in this chapter that you should try without looking anything up:

- Let $f : (a,b) \to \mathbb{R}$, $c \in (a,b)$ and $f(x) = f(c) + (x-c)f_1(x)$. Which property of f_1 is equivalent to the differentiability of f at c? In case of differentiability of f at c, how can we express $f'(c)$ in terms of f_1?
- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

  Show that f is differentiable at $x = 0$.
- Evaluate $\lim\limits_{x\to 1} \dfrac{1+\cos(\pi x)}{x^2 - 2x + 1}$.

9 Infinite series

9.1 Fundamental notions and properties of series

An (infinite) series is based on summing the terms of a sequence (a_n). More precisely, we consider the new sequence of numbers

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and call (s_n) a *series*.

Example (Geometric Series). A particularly important example is the *geometric series*, where $a_k = q^{k-1}$ for some value $q \in \mathbb{R}$. In an exercise, we have already seen that we have the identity *geometric series*

$$s_n = \sum_{k=1}^{n} q^{k-1} = \sum_{j=0}^{n-1} q^j = \frac{1-q^n}{1-q},$$

which describes s_n by an explicit formula. It is natural to ask for convergence of this new sequence (s_n). In the case of the geometric series, we have convergence of (s_n) if $|q| < 1$, since in this case

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1}{1-q}$$

and unboundedness of (s_n) if $|q| > 1$, since in this case $|q|^n \to \infty$. In the case $q = -1$, we have

$$s_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even,} \end{cases}$$

so (s_n) is bounded but not convergent. In the case $q = 1$, we have

$$s_n = \sum_{k=1}^{n} 1 = n,$$

so (s_n) is, again, unbounded.

Here is the precise definition of this new notion.

Definition 9.1. *Let (a_n) be a sequence of numbers. Then the new sequence (s_n), defined by*

$$s_n = \sum_{k=1}^{n} a_k$$

is called an (infinite) series. *The terms s_n of this new sequence are called the* partial sums *of this series. If the sequence (s_n) of partial sums is convergent, we say that the series $\sum_{k=1}^{\infty} a_k$ is* convergent, *and we write*

(infinite) series

partial sums

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n.$$

Otherwise we say that the series $\sum_{k=1}^{\infty} a_k$ is divergent.

A necessary condition for a series $\sum_{k=1}^{\infty} a_k$ to converge is that the summands a_k converge to 0:

Lemma 9.2. *If $\sum_{k=1}^{\infty} a_k$ is convergent, then $a_k \to 0$.*

Proof. Let $s_n = \sum_{k=1}^{n} a_k$. Since the series is convergent, we have $s_n \to s^*$. Note that $a_k = s_k - s_{k-1}$. Then we have

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} s_k - s_{k-1} = s^* - s^* = 0.$$

$\square$

Example (Harmonic Series). It is important to know that the converse of Lemma 9.2 is not true: $a_k \to 0$ does not imply that $\sum_{k=1}^{\infty} a_k$ is convergent. Here is a fundamental explicit counterexample: The *harmonic series* is given by $\sum_{k=1}^{\infty} a_k$ with $a_k = 1/k$. Here we obviously have $a_k = 1/k \to 0$, but $\sum_{k=1}^{\infty} a_k$ is divergent since

harmonic series

$$\begin{aligned}
s_1 &= 1, \\
s_2 &= s_1 + \frac{1}{2}, \\
s_4 &= s_2 + \frac{1}{3} + \frac{1}{4} \geq s_2 + 2\cdot\frac{1}{4} = s_2 + \frac{1}{2}, \\
s_8 &= s_4 + \frac{1}{5} + \cdots + \frac{1}{8} \geq s_4 + 4\cdot\frac{1}{8} = s_4 + \frac{1}{2}, \\
s_{16} &= s_8 + \frac{1}{9} + \cdots + \frac{1}{16} \geq s_8 + 8\cdot\frac{1}{16} = s_8 + \frac{1}{2},
\end{aligned}$$

and it can be proved by induction that $s_{2^n} \geq 1 + \frac{n}{2} \to \infty$.

We also have the following version of COLT for series. The proofs are straightforward reductions to COLT for sequences and will not be given here.

COLT for series

Theorem 9.3 (COLT). *Let $\sum_{k=1}^{\infty} a_k = a$, $\sum_{k=1}^{\infty} b_k = b$ and $c \in \mathbb{R}$. Then we have the following:*

(a) $\sum_{k=1}^{\infty} a_k + b_k$ *is convergent with limit* $a+b$.

(b) $\sum_{k=1}^{\infty} ca_k$ *is convergent with limit* ca.

(c) If $a_k \leq b_k$ *then* $a \leq b$.

9.2 Convergence tests

Here is our first convergence criterion for series.

Comparison Test

Theorem 9.4 (Comparison Test)**.** *Let* $0 \leq a_k \leq b_k$ *for all* $k \in \mathbb{N}$. *If* $\sum_{k=1}^{\infty} b_k$ *is convergent with limit* b *then* $\sum_{k=1}^{\infty} a_k$ *is also convergent with limit* $a \leq b$. *Equivalently, if* $\sum_{k=1}^{\infty} a_k$ *is divergent then so is* $\sum_{k=1}^{\infty} b_k$.

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$. By assumption, we have $0 \leq s_n \leq t_n \leq b$, since (t_n) is monotone increasing with limit b. But (s_n) is also monotone increasing and bounded above by b, so convergent to a number a by Theorem 6.2. Moreover, $s_n \leq t_n$ implies that

$$a = \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n = b.$$

The equivalent statement about divergence follows via contraposition.

□

Examples. We investigate $\sum_{k=1}^{\infty} a_k$ in each of the following cases.

(a) $a_k = \dfrac{\sqrt{k^2+1}}{k^2}$. Here we have $a_k \geq 1/k$ and $\sum_{k=1}^{\infty} a_k$ diverges by comparison with the harmonic series.

(b) $a_k = \dfrac{1}{k^2}$. Here we have $0 \leq a_{k+1} = \dfrac{1}{(k+1)^2} \leq \dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1} = b_k$ and

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{n+1} \to 1.$$

Therefore, $\sum_{k=1}^{\infty} a_{k+1}$ converges by comparison with $\sum_{k=1}^{\infty} b_k$ and, consequently, $\sum_{k=1}^{\infty} a_k = a_1 + \sum_{k=1}^{\infty} a_{k+1}$ converges as well. In fact, the limit is very beautiful: $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$.

(c) $a_k = \dfrac{(2k^2+2k+9)(k^2+4k+3)}{(k^3+6k+1)^2}$. Then we have $k^2 a_k \to 2$ as $k \to \infty$ and we can find $C \geq 0$ such that $k^2 a_k \leq C$ for all k.

Therefore, $0 \leq a_k \leq C/k^2$, and the series $\sum_{k=1}^{\infty} a_k$ converges by comparison with $\sum_{k=1}^{\infty} C/k^2$.

(d) **Claim:** Let $\alpha > 0$. Then $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges if and only if $\alpha > 1$.
This statement follows in the case $\alpha \geq 2$ by comparison with $\sum_{k=1}^{\infty} 1/k^2$ and in the case $\alpha \leq 1$ by comparison with the harmonic series. The proof in the case $1 < \alpha < 2$ follows from the *Integral Test* given later in Theorem 9.8.

(e) $a_k = \dfrac{2k+7}{\sqrt{3k^3-2}}$. Then $a_k \geq \frac{2}{\sqrt{3k}}$. Since $(2/\sqrt{3}) \sum_{k=1}^{\infty} 1/\sqrt{k}$ diverges, the series $\sum_{k=1}^{\infty} a_k$ diverges as well, by comparison.

(f) $a_k = \dfrac{7k^3 + 2\log(k)}{e^k}$. Then $k^2 a_k \to 0$ as $k \to \infty$ and we can find a constant $C > 0$ such that $0 \leq a_k \leq C/k^2$. Hence, $\sum_{k=1}^{\infty} a_k$ converges by comparison with $\sum_{k=1}^{\infty} C/k^2$.

The above Comparison Test requires both series to have non-negative summands. In the case that a series $\sum_{k=1}^{\infty} a_k$ has summands with positive and negative summands, it is often useful to consider the series $\sum_{k=1}^{\infty} |a_k|$, where all summands are the absolute values of the summands of the original series. We will see that convergence of $\sum_{k=1}^{\infty} |a_k|$ implies convergence of $\sum_{k=1}^{\infty} a_k$. To prepare this result, we first introduce the notion of *absolute convergence*.

Definition 9.5. *Let $\sum_{k=1}^{\infty} a_k$ be a series. If the series $\sum_{k=1}^{\infty} |a_k|$ converges, we say that the series $\sum_{k=1}^{\infty} a_k$* converges absolutely*. In the case that $\sum_{k=1}^{\infty} a_k$ is convergent but not $\sum_{k=1}^{\infty} |a_k|$, we say that the series $\sum_{k=1}^{\infty} a_k$* converges conditionally*.*

absolute convergence

conditional convergence

Theorem 9.6 (Absolute Convergence Theorem)**.** *Every absolutely convergent series $\sum_{k=1}^{\infty} a_k$ is also convergent.*

Absolute Convergence Theorem

Proof. Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Then $\sum_{k=1}^{\infty} 2|a_k|$ is convergent and so is $\sum_{k=1}^{\infty} |a_k| + a_k$, by comparison, since $0 \leq |a_k| + a_k \leq 2|a_k|$. Since $\sum_{k=1}^{\infty} |a_k| + a_k$ and $\sum_{k=1}^{\infty} |a_k|$ are both convergent, we conclude with COLT that also

$$\sum_{k=1}^{\infty} a_k = \left(\sum_{k=1}^{\infty} |a_k| + a_k \right) - \sum_{k=1}^{\infty} |a_k|$$

is convergent. □

Example. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+2k}$ is absolutely convergent since $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$ is convergent by comparison with $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Therefore $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+2k}$ is also convergent.

An example of a series which is only conditionally convergent is given by $\sum_{k=1}^{\infty}(-1)^k/k$. This series is not absolutely convergent since the harmonic series is divergent but it is convergent because of the following *Alternating Sign Test* due to Leibniz[7].

Alternating Sign Test

Theorem 9.7 (Alternating Sign Test)**.** *Let (a_k) be a monotone decreasing sequence of positive numbers with $a_k \to 0$. Then the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ is convergent.*

Proof. Let $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$. Then we have

$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n}$$

and

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \leq s_{2n-1}.$$

This implies that the subseqence (s_{2n}) is monotone increasing and that the subsequence (s_{2n-1}) is monotone decreasing. Moreover, since $s_{2n} = s_{2n-1} - a_{2n} \leq s_{2n-1}$, we have

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq s_{2n-1} \leq \cdots \leq s_5 \leq s_3 \leq s_1.$$

In particular, both subsequences (s_{2n}) and (s_{2n-1}) are monotone and bounded and, therefore, convergent. Since $a_{2n} = s_{2n-1} - s_{2n} \to 0$, their limits are equal, and we denote this limit by

$$s^* = \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1}.$$

Since both subsequences with even and with odd indices of (s_n) converge to the same limit s^*, it is easy to see that the full sequence (s_n) must also be convergent to s^*, finishing the proof. □

Integral Test

Theorem 9.8 (Integral Test)**.** *Let $f : [1,\infty) \to [0,\infty)$ be monotone decreasing. Let $a_k = f(k)$ and $F_k = \int_1^k f(x)dx$ for all $k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the sequence (F_k) is convergent.*

[7]Gottfried Wilhelm Leibniz (1646-1716) was a German universal scholar with particular important contributions in philosophy and mathematics. Along with Sir Isaac Newton, he can be considered as one the fathers of the differential and infinitesimal calculus, and many notions used today like the differential df/dx and the integral sign $\int$ (which stems from the letter S for sum) are his inventions. There was a priority controversy whether Leibniz had developed calculus independently of Newton.

Proof. Let $b_k = \int_k^{k+1} f(x)dx$. Then F_{k+1} are the partial sums of the series $\sum_{k=1}^{\infty} b_k$. Note that we have

$$0 \le a_{k+1} \le b_k \le a_k \tag{11}$$

by Monotonicity of the Integral (see the Calculus Script,p.60 (iii)) and the fact that f is monotone decreasing. Then applying the Comparison Test with both inequalities in (11) proves the theorem. $\square$

Examples. (a) Let $1 < \alpha < 2$ and $f(x) = \dfrac{1}{x^\alpha}$ for $x \ge 1$. Then f is monotone decreasing and ≥ 0. Moreover,

$$\begin{aligned} F_k = \int_1^k x^{-\alpha} dx &= \frac{1}{1-\alpha}(k^{1-\alpha} - 1) \\ &= -\frac{1}{\alpha - 1}\left(\frac{1}{k^{\alpha-1}} - 1\right) \to \frac{1}{\alpha-1} \qquad \text{as } k \to \infty. \end{aligned}$$

This shows that the series $\sum_{k=1}^{\infty} 1/k^\alpha$ is convergent, by the Integral Test.

(b) Let $f(x) = \dfrac{1}{x\log(x)}$ for $x \ge 2$. Then f is monotone decreasing and ≥ 0. Moreover,

$$F_k = \int_2^k \frac{1}{x \log x} dx = \log(\log k) - \log(\log 2)$$

is unbounded for $k \to \infty$. Therefore, the series $\sum_{k=2}^{\infty} 1/(k\log(k))$ is divergent, by the Integral Test.

Theorem 9.9 (Ratio Test)**.** *Let (a_k) be a sequence with $a_k \neq 0$ for all $k \in \mathbb{N}$.* *Ratio Test*

(a) If $|a_{k+1}|/|a_k| \to a < 1$ for $k \to \infty$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.

(b) If $|a_{k+1}|/|a_k| \ge 1$ for all but finitely many $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} a_k$ is divergent.

Proof. Ad (a): Choose $q \in (a, 1)$. Then there exists $K \in \mathbb{N}$ such that

$$\frac{|a_{k+1}|}{|a_k|} \le q \qquad \text{for all } k \ge K.$$

This implies for all $j \in \mathbb{N}$,

$$|a_{K+j}| \le q|a_{K+j-1}| \le \cdots \le q^j |a_K|.$$

Since $a < q < 1$, the geometric series $\sum_{j=0}^{\infty} q^j |a_K|$ converges and, therefore, also the series $\sum_{j=0}^{\infty} |a_{K+j}|$, by Comparison. Now we add the terms $|a_1|, \ldots, |a_{K-1}|$ to conclude that also $\sum_{k=0}^{\infty} |a_k|$ is convergent. This shows that $\sum_{k=0}^{\infty} a_k$ is absolutely convergent.
Ad (b): There exists $K \in \mathbb{N}$ such that we have $|a_{k+1}|/|a_k| \ge 1$ for all $k \ge K$. In other words, we have $|a_{k+1}| \ge |a_k| > 0$, but then (a_k) cannot converge to 0. Then $\sum_{k=1}^{\infty} a_k$ cannot be convergent, by Lemma 9.2. $\square$

Examples. We investigate $\sum_{k=1}^{\infty} a_k$ in the following cases.

(a) $a_k = c^k/k!$ with a constant $c \in \mathbb{R}$. Then

$$\frac{|a_{k+1}|}{|a_k|} = \frac{c^{k+1}k!}{c^k(k+1)!} = \frac{c}{k+1} \to 0 \quad \text{as } k \to \infty.$$

Therefore $\sum_{k=1}^{\infty} a_k$ converges absolutely, by the Ratio Test, for all values of $c \in \mathbb{R}$.

(b) $a_k = kc^k$ with a constant $c \in \mathbb{R}$. Then

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)c^{k+1}}{kc^k} = \frac{k+1}{k}c \to c \quad \text{as } k \to \infty.$$

Therefore $\sum_{k=1}^{\infty} a_k$ converges absolutely for $|c| < 1$ and diverges for $|c| > 1$, by the Ratio Test.

(c) $a_k = k!c^k$ with a constant $c \neq 0$. Then

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)!c^{k+1}}{k!c^k} = (k+1)c \quad \text{which is unbounded as } k \to \infty.$$

Therefore, $\sum_{k=1}^{\infty} a_k$ diverges by the Ratio Test.

(d) $a_k = c^k/(k^3 5^k)$ with a constant $c \neq 0$. Then

$$\frac{|a_{k+1}|}{|a_k|} = \frac{c^{k+1}k^3 5^k}{c^k(k+1)^3 5^{k+1}} = \frac{c}{5}\frac{1}{(1+1/k)^3} \to \frac{c}{5} \quad \text{as } k \to \infty.$$

Therefore, $\sum_{k=1}^{\infty} a_k$ converges absolutely for $|c| < 5$ and diverges for $|c| > 5$, by the Ratio Test.

Finally, we present another useful test without proof. We would only like to mention that the proof is, again, based on comparison with the geometric series.

n^{th} Root Test

Theorem 9.10 (n^{th} Root Test)**.** *Let (a_k) be a sequence with $|a_k|^{1/k} \to a$ for $k \to \infty$.*

(a) If $a < 1$ then $\sum_{k=1}^{\infty} a_k$ converges absolutely.

(b) If $a > 1$ then $\sum_{k=1}^{\infty} a_k$ is divergent.

Reader's Task. Give a proof of Theorem 9.10.

Example. Let $c \neq 0$ be a constant. We consider $\sum_{k=1}^{\infty}(2+(-1)^k)c^k$, i.e., $a_k = (2+(-1)^k)c^k$. Then

$$|a_k|^{1/k} = (2+(-1)^k)^{1/k}|c|.$$

We know that $1 \leq 2+(-1)^k \leq 3$ and $\lim_{k\to\infty} 3^{1/k} = 1$, so we conclude with the Squeezing Theorem or directly with its Corollary 3.10,

$$|a_k|^{1/k} \to |c| \quad \text{as } k \to \infty.$$

So the n^{th} Root Test tells us that $\sum_{k=1}^{\infty}(2+(-1)^k)c^k$ is absolutely convergent for $|c| < 1$ and divergent for $|c| > 1$.

9.3 Rearrangements of series

Given a *finite sum* $\sum_{k=1}^{n} a_k$, we can change the order of the summands without changing its value, by the Law of Commutativity, i.e., if $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is an arbitrary permutation, then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{\sigma(k)}.$$

It is natural to ask whether the same holds true for infinite sums. It turns out that, surprisingly, this is no longer true.

Example. We know that the sum $\sum_{k=1}^{\infty}(-1)^{k+1}/k$ is (conditionally) convergent, by the Alternating Sign Test. In fact, we have

$$\sum_{k=1}^{\infty}(-1)^{k+1}/k = \log(2) \approx 0.693147\ldots.$$

Now, we rearrange the summands in the sum that every positive term $1, 1/3, 1/5, 1/7, \dots$ is followed by two negative ones and obtain

$$
\begin{aligned}
&1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\frac{1}{16}+\frac{1}{9}-\frac{1}{18}-\frac{1}{20}+\dots = \\
&\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12} \\
&+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}+\left(\frac{1}{9}-\frac{1}{18}\right)-\frac{1}{20}+\dots = \\
&\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\frac{1}{18}-\frac{1}{20}+\dots = \\
&\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\dots\right)=\frac{1}{2}\log(2).
\end{aligned}
$$

This shows that this rearrangement leads to a series which converges to a limit which is only half the original limit.

CAUTION. The important message of the above example is that you have to be very careful when permuting the summands of an infinite series $\sum a_k$.

The above example is in agreement with the following hugely surprising fact, due to the famous 19th-century German mathematician Bernhard Riemann[8]. Even though the following results are of fundamental importance, we do not provide proofs for them for lack of time.

Riemann Rearrangement Theorem

Theorem 9.11 (Riemann Rearrangement Theorem)**.** *Let* $\sum_{k=1}^\infty a_k$ *a* conditionally *convergent real series and* $c \in \mathbb{R}$ *be an arbitrary real number. Then there exists a rearrangement*[9]

$$\sigma : \mathbb{N} \to \mathbb{N},$$

such that the rearranged sum $\sum_{k=1}^\infty a_{\sigma(k)}$ *converges to this number* c*. Moreover, the sum can also be rearranged that* $\sum_{k=1}^\infty a_{\sigma(k)}$ *is divergent.*

[8] Bernhard Riemann(1826-1866) was a famous German mathematician who contributed to many different mathematical areas like Real and Complex Analysis, Analytic Number Theory and Differential Geometry. He developed a geometric foundation of Complex Analysis and the fundamental notion of a *Riemann Surface* is named after him. He also introduced the basic notions for important spaces with a very general intrinsic geometry (manifolds with a *Riemannian Metric* and a *Riemannian Curvature Tensor*) which were also crucial in the later *Theory of General Relativity*, due to Albert Einstein(1879-1955). Riemann developed the *Riemann Integral* in his Habilitation thesis. The famous *Riemann Conjecture* about the zeroes of the *Riemann Zeta Function* is undoubtedly the most important open conjecture in mathematics with many important implications like the asymptotic distribution of the prime numbers.

[9] This means that σ is one-to-one (or bijective), which means that every natural number is in the image of σ (surjectivity) and that different natural numbers have different images (injectivity).

But there is also good news in stock: Such a rearrangement phenomenon can only occur for conditionally convergent series. If you know that a series $\sum_{k=1}^{\infty} a_k$ is *absolutely convergent*, every rearrangement of it is, again, convergent to the same limit.

Theorem 9.12. *Let $\sum_{k=1}^{\infty} a_k$ an absolutely convergent real series and $\sigma : \mathbb{N} \to \mathbb{N}$ a rearrangement. Then $\sum_{k=1}^{\infty} a_{\sigma(k)}$ is also absolutely convergent and we have*

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\sigma(k)}.$$

Another important rearrangement fact about the Cauchy product product of two series[10].

Cauchy Product

Theorem 9.13 (Cauchy Product Theorem). *Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be absolutely convergent series with limits $a, b \in \mathbb{R}$, respectively. For $n \geq 0$, let*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Then the series $\sum_{k=0}^{\infty} c_k$ is called the Cauchy Product *of $\sum a_k$ and $\sum b_k$. The series $\sum c_k$ is also absolutely convergent and we have*

$$\sum_{k=0}^{\infty} c_k = a \cdot b.$$

Remark. Note that the all terms in the product $(\sum a_k)(\sum b_k)$ can be listed in the following square array:

$$\begin{array}{cccccccc}
a_0b_0 & a_1b_0 & a_2b_0 & a_3b_0 & a_4b_0 & a_5b_0 & a_6b_0 & \ldots \\
a_0b_1 & a_1b_1 & a_2b_1 & a_3b_1 & a_4b_1 & a_5b_1 & a_6b_1 & \ldots \\
a_0b_2 & a_1b_2 & a_2b_2 & a_3b_2 & a_4b_2 & a_5b_2 & a_6b_2 & \ldots \\
a_0b_3 & a_1b_3 & a_2b_3 & a_3b_3 & a_4b_3 & a_5b_3 & a_6b_3 & \ldots \\
a_0b_4 & a_1b_4 & a_2b_4 & a_3b_4 & a_4b_4 & a_5b_4 & a_6b_4 & \ldots \\
a_0b_5 & a_1b_5 & a_2b_5 & a_3b_5 & a_4b_5 & a_5b_5 & a_6b_5 & \ldots \\
a_0b_6 & a_1b_6 & a_2b_6 & a_3b_6 & a_4b_6 & a_5b_6 & a_6b_6 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The terms $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ are then nothing but the finite sums in this square array along the diagonals starting at $a_0 b_n$ and ending with

[10] See page 41 in the Analysis 1 (Michaelmas Term) Lecture Notes for short biographical facts about Cauchy.

$a_n b_0$. So we see that, by summing up all c_n's, we are taking every term $a_k b_l$ into account once, namely when we consider the $(k+l)$-th diagonal in the above scheme.

Example. Here is an important application of the Cauchy product. Let $x, y \in \mathbb{R}$. We know from the Ratio Test that both series $\sum_{k=0}^{\infty} x^k/k!$ and $\sum_{k=0}^{\infty} y^k/k!$ are absolutely convergent. The terms c_n in the Cauchy product are then

$$c_n = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}.$$

Now we need an important fact which students in Discrete Mathematics have seen, the so called *Binomial Theorem*:

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

with the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. This implies that

Binomial Coefficient $\binom{n}{k}$

$$c_n = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} x^k y^{n-k} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \frac{1}{n!}(x+y)^n.$$

So the Cauchy product leads in this example to the important identity of absolutely convergent series

$$\left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left(\sum_{k=0}^{\infty} \frac{y^k}{k!} \right) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}.$$

Recalling the Taylor series of e^x from Calculus (see Section 6.1 in the Calculus Lecture Notes), this identity proves that

$$e^x e^y = e^{x+y}.$$

9.4 Complex series

Let us briefly discuss the question whether results for real series extend to complex series. Complex series play an important role in the 2H Course Complex Analysis, so we only provide a brief sketch. We introduced already earlier the notion of convergence of complex sequences and extensions of fundamental facts about real sequences like Bolzano-Weierstrass to complex sequences. We now state the crucial facts without proofs.

Proposition 9.14. *Let $z_k = x_k + iy_k \in \mathbb{C}$. Then $\sum_{k=1}^{\infty} z_k$ converges if and only if $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ converge. In this case, we have*

$$\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} x_k + i \sum_{k=1}^{\infty} y_k.$$

The following facts are still true for complex series:

(a) Convergence of $\sum_{k=1}^{\infty} z_k$ implies that $z_k \to 0$.

(b) COLT holds for complex series.

(c) The complex series $\sum z_k$ is said to *converge absolutely* if the real sequence $\sum |z_k|$ converges. Every absolutely convergent complex series is convergent.

(d) Ratio Test and n^{th} Root Test hold also for complex series.

(e) Rearrangements of absolutely convergent complex series lead to convergent series *with the same limit.*

(f) The Cauchy Product Theorem holds also for absolutely convergent complex series.

Examples. (a) Let $z = x + iy \in \mathbb{C}$ be arbitrary. Then we define e^z as

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}. \tag{12}$$

Note that this sum is absolutely convergent since

$$\sum_{k=0}^{\infty} \left| \frac{z^k}{k!} \right| = \sum_{k=0}^{\infty} \frac{|z|^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(\sqrt{x^2 + y^2}\right)^k}{k!},$$

and the last sum is convergent by the Ratio Test. Therefore e^z, given in (12), is a well-defined complex number. Moreover, the Cauchy Product Theorem for complex numbers yields for all $z_1, z_2 \in \mathbb{C}$:

$$e^{z_1+z_2} = e^{z_1} e^{z_1}.$$

(b) Let $z \in \mathbb{C}$. Recall from above and the Calculus Course that the Taylor series of e^z, $\sin z$ and $\cos z$ are

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \tag{13}$$

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \tag{14}$$

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}. \tag{15}$$

Using COLT, we obtain Euler's Identity

$$\cos z + i \sin z = \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!} z^{2k} + i \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k+1)!} z^{2k+1} =$$

$$\sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!} z^{2k} + \sum_{k=0}^{\infty} \frac{i^{2k+1}}{(2k+1)!} z^{2k+1} =$$

$$\sum_{k=0}^{\infty} \left(\frac{i^{2k}}{(2k)!} z^{2k} + \frac{i^{2k+1}}{(2k+1)!} z^{2k+1} \right) =$$

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} z^n = \sum_{n=0}^{\infty} \frac{1}{n!} (iz)^n = e^{iz}.$$

(c) Let $x, y \in \mathbb{R}$. Then we conclude from (a) and (b):

$$e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y).$$

Important concepts/typical problems in this chapter that you should try without looking anything up:

- Decide whether the following series are convergent and absolutely convergent:

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{\log(k)}, \quad \sum_{k=1}^{\infty} \frac{k^7 \sin(k)}{k!}, \quad \sum_{k=1}^{\infty} \frac{1}{1+ik}.$$

- Formulate the n^{th} Root Test and the Alternating Sign Test.
- How do you show that $\sum_{k=1}^{\infty} 1/k^{\alpha}$ is convergent for $1 < \alpha < 2$?
- For which values of $a \in \mathbb{R}$ is $\sum_{k=0}^{\infty} a^{k!}$ absolutely convergent?
- In which case can you rearrange a convergent complex series?

10 Integrals

You know the integral $\int_a^b f(x)dx$ from School and the Calculus Course. It describes the *signed area* between the graph of f and the x-axis on the interval $[a, b]$. In fact, the integral can be defined in different ways which all lead to the same result if f is "sufficiently nice", e.g., if f is continuous. The best known definitions of integrals are the *Riemann integral* and the *Lebesgue integral*. Whilst the Lebesgue integral is the more general concept, it needs a lot more time to be introduced properly, and we restrict ourselves here to the definition of the Riemann integral. The definition is based on the concepts of *lower and upper Riemann sums.*

10.1 The Riemann Integral and its properties

Definition 10.1. *A* partition $\mathcal{P}$ *of a compact interval* $[a, b]$ *is a finite set* $\{x_0, x_1, \ldots, x_n\}$ *satisfying*

partition of interval

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Let $f : [a, b] \to \mathbb{R}$ *be a bounded function and* $\mathcal{P}$ *be a partition of* $[a, b]$. *The* lower Riemann sum *of* f *relative to* $\mathcal{P}$ *is defined as*

Lower/Upper Riemann sum

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad \text{where } m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\},$$

and the upper Riemann sum *of* f *relative to* $\mathcal{P}$ *is defined as*

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}), \quad \text{where } M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}.$$

Examples. (a) We consider $f(x) = x$ on $[0, 1]$ with the partition

$$\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

Then we have

$$\begin{aligned} L(f, \mathcal{P}_n) &= \frac{1}{n}\sum_{i=0}^{n-1}\frac{i}{n} = \frac{1}{n}\frac{(n-1)n}{2n} = \frac{n-1}{2n}, \\ U(f, \mathcal{P}_n) &= \frac{1}{n}\sum_{i=1}^{n}\frac{i}{n} = \frac{1}{n}\frac{n(n+1)}{2n} = \frac{n+1}{2n}. \end{aligned}$$

(b) This time, let $f(x) = x^2$ on the same interval $[0,1]$ with the same partition as in (a). Then we have

$$\begin{aligned} L(f, \mathcal{P}_n) &= \frac{1}{n}\sum_{i=0}^{n-1} \frac{i^2}{n^2} = \frac{1}{n}\frac{(n-1)n(2n-1)}{6n^2} = \frac{(n-1)(2n-1)}{6n^2}, \\ U(f, \mathcal{P}_n) &= \frac{1}{n}\sum_{i=1}^{n} \frac{i^2}{n^2} = \frac{1}{n}\frac{n(n+1)(2n+1)}{6n^2} = \frac{(n+1)(2n+1)}{6n^2}. \end{aligned}$$

Lemma 10.2. *Let $f : [a,b] \to \mathbb{R}$ be bounded. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of $[a,b]$ with $\mathcal{P}_1 \subset \mathcal{P}_2$, then*

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_1).$$

Proof. The inequality $L(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_2)$ follows directly from

$$\inf\{f(x) \mid u \leq x \leq v\} \leq \sup\{f(x) \mid u \leq x \leq v\}.$$

The inequality $L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_2)$ follows from

$$\begin{aligned} &\inf\{f(x) \mid u \leq x \leq v\}((w-u) + (v-w)) \leq \\ &\qquad \inf\{f(x) \mid u \leq x \leq w\}(w-u) + \inf\{f(x) \mid w \leq x \leq v\}(v-w) \end{aligned}$$

for all $a \leq u \leq w \leq v \leq b$. The inequality $U(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_1)$ is treated similarly. □

Now we are in the position to introduce the Riemann integral.

Definition 10.3. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Let*

$$\begin{aligned} \mathcal{L}(f) &= \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \textit{ is a partition of } [a,b]\}, \\ \mathcal{U}(f) &= \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \textit{ is a partition of } [a,b]\}, \end{aligned}$$

where both sup *and* inf *are taken over all possible partitions. Then f is* Riemann integrable *on $[a,b]$ if $\mathcal{L}(f) = \mathcal{U}(f)$ and we define the* Riemann integral *of f as* *Riemann integrable* *Riemann integral*

$$\int_a^b f(x)dx = \mathcal{L}(f) = \mathcal{U}(f).$$

Remark. Note that we always have

$$\mathcal{L}(f) \leq \mathcal{U}(f). \tag{16}$$

We prove this indirectly: The assumption $\mathcal{L}(f) > \mathcal{U}(f)$ would lead to the existence of partitions $\mathcal{P}_L, \mathcal{P}_U$ with

$$L(f, \mathcal{P}_L) > U(f, \mathcal{P}_U).$$

On the other hand, Lemma 10.2 would give for the partition $\mathcal{P} = \mathcal{P}_L \cup \mathcal{P}_U$,

$$L(f, \mathcal{P}_L) \leq L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq U(f, \mathcal{P}_U),$$

in contradiction to the previous inequality.

Example. Let $f : [0, 1] \to \mathbb{R}$ be defined as follows

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \text{ rational}, \\ 0, & \text{if } x \in [0,1] \text{ irrational}. \end{cases}$$

Then we always have $L(f, \mathcal{P}) = 0$ and $U(f, \mathcal{P}) = 1$, and f is not Riemann integrable. It turns out that f is Lebesgue integrable and that the Lebesgue integral of f is 0. This is an example often used to show that the Lebesgue integral is the stronger concept.

The following integrability criterion is very useful.

Theorem 10.4. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if, for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that*

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon. \tag{17}$$

Proof. We first assume that f is Riemann integrable, i.e., $\mathcal{L}(f) = \mathcal{U}(f)$. Let $\epsilon > 0$ be given. Then we find partitions $\mathcal{P}_L, \mathcal{P}_U$ of $[a, b]$ such that

$$L(f, \mathcal{P}_L) \leq \mathcal{L}(f) < L(f, \mathcal{P}_L) + \frac{\epsilon}{2}$$

and

$$U(f, \mathcal{P}_U) - \frac{\epsilon}{2} < \mathcal{U}(f) \leq U(f, \mathcal{P}_U).$$

Now, consider the partition $\mathcal{P} = \mathcal{P}_L \cup \mathcal{P}_U$. By Lemma 10.2, we have

$$L(f, \mathcal{P}_L) \leq L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq U(f, \mathcal{P}_U),$$

which implies

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &\leq U(f, \mathcal{P}_U) - L(f, \mathcal{P}_L) \\ &< \mathcal{U}(f) + \frac{\epsilon}{2} - \left(\mathcal{L}(f) - \frac{\epsilon}{2}\right) = \epsilon, \end{aligned}$$

since $\mathcal{U}(f) = \mathcal{L}(f)$. This shows that we have (17).
Conversely, assume that we can find, for every $\epsilon > 0$, a partition $\mathcal{P}$ such that we have (17). This implies that we have, for every $\epsilon > 0$,

$$\mathcal{U}(f) - \mathcal{L}(f) \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

Since $\mathcal{U}(f) - \mathcal{L}(f) \geq 0$, by (16), and $\epsilon > 0$ was arbitrary, we have $\mathcal{U}(f) = \mathcal{L}(f)$ and, therefore, f is Riemann integrable. $\square$

Example. We can now prove that $f(x) = x^2$ is Riemann integrable on $[0, 1]$. Using the notation for partitions in the earlier example, we have

$$\begin{aligned} U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) &= \frac{(n+1)(2n+1)}{6n^2} - \frac{(n-1)(2n-1)}{6n^2} \\ &= \frac{2n^2 + 3n + 1 - (2n^2 - 3n + 1)}{6n^2} = \frac{1}{n}. \end{aligned}$$

Since $1/n \to 0$, we can find for every $\epsilon > 0$ a partition $\mathcal{P}$ such that (17) holds and f is Riemann integrable, by Theorem 10.4. The integral can be sandwiched between

$$\frac{(n-1)(2n-1)}{6n^2} \leq \int_0^1 x^2 dx \leq \frac{(n+1)(2n+1)}{6n^2},$$

for all $n \in \mathbb{N}$. Taking the limits on both sides, we find

$$\int_0^1 x^2 dx = \frac{2n^2}{6n^2} = \frac{1}{3}.$$

Theorem 10.5. *Every monotone increasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable. The same holds for monotone decreasing functions.*

Proof. We prove the statement for monotone increasing functions. We first introduce the equidistant partition $\mathcal{P}_n = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$, defined by

$$x_j = a + j\frac{b-a}{n}.$$

Since f is monotone increasing, we have $m_i = f(x_{i-1})$ and $M_i = f(x_i)$ and

$$L(f, \mathcal{P}_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1}),$$

$$U(f, \mathcal{P}_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i).$$

This implies that

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{b-a}{n}(f(b) - f(a)).$$

For every $\epsilon > 0$, we can therefore find $n \in \mathbb{N}$ with

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = \frac{b-a}{n}(f(b) - f(a)) < \epsilon,$$

which implies that f is Riemann integrable, by Theorem 10.4. □

The next result provides a large class of Riemann integrable functions.

Theorem 10.6. *Every continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Reader's Task. Theorem 10.6 can be proved with the help of the following fact: Every continuous function $f : [a,b] \to \mathbb{R}$ on a compact interval $[a,b]$ is *uniformly continuous*, i.e., for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in [a,b]$ with $|y - x| < \delta$ we have

uniform continuity

$$|f(y) - f(x)| < \epsilon.$$

Note the subtle difference between continuity and uniform continuity: In the latter case we can find, for a given $\epsilon > 0$, a $\delta > 0$ which works *simultaneously for every point* x in the domain $[a,b]$. A continuous function which is not uniformly continuous is, e.g., $f(x) = 1/x$ over $(0,1]$. Give a proof of Theorem 10.6, using the fact that f is uniformly continuous on $[a,b]$.

Theorem 10.7 (Properties of the Riemann Integral)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable functions. Then we have the following:*

(a) Linearity: For $c, d \in \mathbb{R}$, $cf + dg$ is also Riemann integrable and

$$\int_a^b cf(x) + dg(x)dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx. \qquad (18)$$

(b) Monotonicity: If $f(x) \geq g(x)$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

In particular, if $f(x) \geq 0$ for all $x \in [a,b]$, then also $\int_a^b f(x)dx \geq 0$.

(c) The product $fg : [a,b] \to \mathbb{R}$ is also Riemann integrable.

(d) For any $c \in (a,b)$, f is also Riemann integrable on $[a,c]$ and $[c,b]$ and we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{19}$$

(e) We have

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

Proof. **Ad (a):** We only prove the case $c = d = 1$. Let $\epsilon > 0$. By Theorem 10.4, we can find two partitions $\mathcal{P}$ and $\mathcal{P}'$ of $[a,b]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}), \quad U(g,\mathcal{P}') - L(g,\mathcal{P}') < \epsilon.$$

Taking the refinement $\mathcal{P}'' = \mathcal{P} \cup \mathcal{P}'$, we obtain

$$U(f+g,\mathcal{P}'') \leq U(f,\mathcal{P}'') + U(g,\mathcal{P}'') \leq U(f,\mathcal{P}) + U(g,\mathcal{P}') \tag{20}$$

and

$$L(f+g,\mathcal{P}'') \geq L(f,\mathcal{P}'') + L(g,\mathcal{P}'') \geq L(f,\mathcal{P}) + L(g,\mathcal{P}'), \tag{21}$$

i.e.,

$$\begin{aligned} &U(f+g,\mathcal{P}'') - L(f+g,\mathcal{P}'') \\ &\qquad \leq (U(f,\mathcal{P}) - L(f,\mathcal{P})) + (U(g,\mathcal{P}') - L(g,\mathcal{P}')) < 2\epsilon. \end{aligned}$$

This shows that $f+g$ is Riemann integrable. For the identity (18), note that

$$L(f,\mathcal{P}) + L(g,\mathcal{P}') \leq \int_a^b f(x)dx + \int_a^b g(x)dx \leq U(f,\mathcal{P}) + U(g,\mathcal{P}')$$

and

$$L(f+g,\mathcal{P}'') \leq \int_a^b f(x) + g(x)dx \leq U(f+g,\mathcal{P}''),$$

and therefore, using (20) and (21)

$$\begin{aligned} &\left|\int_a^b f(x) + g(x)dx - \left(\int_a^b f(x)dx + \int_a^b g(x)dx\right)\right| \\ &\qquad \leq U(f,\mathcal{P}) + U(g,\mathcal{P}') - (L(f,\mathcal{P}) + L(g,\mathcal{P})) < 2\epsilon. \end{aligned}$$

Ad (b): The inequality follows directly from

$$m_i(g) = \inf\{g(x) \mid x_{i-1} \le x \le x_i\} \le m_i(f) = \int\{f(x) \mid x_{i-1} \le x \le x_i\}$$

and, analogously, $M_i(f) \le M_i(g)$.

Ad (c): We first assume that we have $0 \le f, g \le C$. Let $\epsilon > 0$ be given. Then we can find partitions $\mathcal{P}_f$ and $\mathcal{P}_g$ such that

$$U(f, \mathcal{P}_f) - L(f, \mathcal{P}_f) < \epsilon \quad \text{and} \quad U(g, \mathcal{P}_g) - L(g, \mathcal{P}_g) < \epsilon.$$

Choosing the joint refinement $\mathcal{P} = \mathcal{P}_f \cup \mathcal{P}_g$, we have by Lemma 10.2 that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}), U(g, \mathcal{P}) - L(g, \mathcal{P}) < \epsilon.$$

Let $\mathcal{P} = \{a = x_0, x_1, \dots, x_n = b\}$. Using the notions $m_i(h) := \inf_{x \in [x_{i-1}, x_i]} h(x)$ and $M_i(h) := \inf_{x \in [x_{i-1}, x_i]} h(x)$ for a bounded function $h : [a, b] \to \mathbb{R}$ and the fact that $m_i(f), m_i(g), m_i(fg) \ge 0$, we have

$$\begin{aligned} M_i(fg) - m_i(fg) &\le M_i(f)M_i(g) - m_i(f)m_i(g) = \\ &M_i(f)(M_i(g) - m_i(g)) + m_i(g)(M_i(f) - m_i(f)) \le \\ &C(M_i(g) - m_i(g)) + C(M_i(f) - m_i(f)). \end{aligned}$$

This implies that

$$U(fg, \mathcal{P}) - L(fg, \mathcal{P}) = \sum_{i=1}^{n} (M_i(fg) - m_i(fg))(x_i - x_{i-1}) \le$$

$$C \sum_{i=1}^{n} (M_i(g) - m_i(g))(x_i - x_{i-1}) + C \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) < 2C\epsilon.$$

This shows that we can find partitions $\mathcal{P}$ of $[a, b]$ to make the difference $U(fg, \mathcal{P}) - L(fg, \mathcal{P})$ arbitrarily small. Therefore, fg is Riemann integrable by Theorem 10.4.

Now, we consider the general case of two arbitrary Riemann integrable functions f, g with $-C \le f, g \le C$. Since $f + C \ge 0$ and $g + C \ge 0$, the above considerations show that the function $(f + C)(g + C)$ is Riemann integrable. Since, by Linearity,

$$\int_a^b f(x)g(x)dx =$$

$$\int_a^b (f(x)+C)(g(x)+C)dx - C\int_a^b f(x)dx - C\int_a^b g(x)dx - C^2(b-a),$$

we conclude that fg is also Riemann integrable.

Ad (d): For $\epsilon > 0$, let $\mathcal{P}$ be a partition of $[a,b]$ with

$$L(f,\mathcal{P}) \leq \int_a^b f(x)dx \leq U(f,\mathcal{P}) \quad \text{and } U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

We can assume, without loss of generality, that $c \in \mathcal{P}$ (otherwise, we add c to the partition $\mathcal{P}$ and use Lemma 10.2). We assume that

$$\mathcal{P} = \{a = x_0, x_1, \ldots, c = x_k, x_{k+1}, \ldots, b = x_n\}$$

and introduce the partitions

$$\begin{aligned} \mathcal{P}_1 &= \{x_0, \ldots, x_k\}, \\ \mathcal{P}_2 &= \{x_k, \ldots, x_n\} \end{aligned}$$

of $[a,c]$ and $[c,b]$, respectively. Then we have

$$L(f,\mathcal{P}_1) + L(f,\mathcal{P}_2) \leq \int_a^c f(x)dx + \int_c^b f(x)dx \leq U(f,\mathcal{P}_1) + U(f,\mathcal{P}_2)$$

and

$$\underbrace{(U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1))}_{>0} + \underbrace{(U(f,\mathcal{P}_2) - L(f,\mathcal{P}_2))}_{>0} = U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon,$$

i.e., f is Riemann integrable on $[a,c]$ and $[c,b]$ and

$$\left| \int_a^b f(x)dx - \left(\int_a^c f(x)dx + \int_c^b f(x)dx \right) \right| < \epsilon.$$

This shows the identity (19).

Ad (e): Let $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \min\{f(x), 0\}$. Then we have $f = f^+ + f^-$ and $|f| = f^+ - f^-$. Assume we have shown that f^+ is Riemann integrable. Then $f^- = f - f^+$ is also Riemann integrable, by (a), and we conclude with the Triangle Inequality and (a) and (b) that

$$\begin{aligned} &\left| \int_a^b f(x)dx \right| = \left| \int_a^b f^+(x)dx + \int_a^b f^-(x)dx \right| \leq \\ &\left| \int_a^b f^+(x)dx \right| + \left| \int_a^b f^-(x)dx \right| = \left| \int_a^b f^+(x)dx \right| + \left| - \int_a^b f^-(x)dx \right| = \\ &\left| \int_a^b f^+(x)dx \right| + \left| \int_a^b (-f^-)(x)dx \right| = \int_a^b f^+(x)dx + \int_a^b (-f^-)(x)dx = \\ &\qquad \int_a^b f^+(x) - f^-(x)dx = \int_a^b |f(x)|dx. \end{aligned}$$

Now we show Riemann integrability of f^+. Let $\epsilon > 0$ be given. Then there exists a partition $\mathcal{P}$ of $[a,b]$ with

$$U(f,\mathcal{P}) - L(f,\mathcal{P})) = \sum_{i=1}^{n}(M_i(f) - m_i(f))(x_i - x_{i-1}) < \epsilon.$$

Note that

$$M_i(f^+) - m_i(f^+) = \begin{cases} M_i(f) - m_i(f) & \text{if } M_i(f) \geq m_i(f) \geq 0, \\ M_i(f) & \text{if } M_i(f) \geq 0 > m_i(f), \\ 0 & \text{if } 0 > M_i(f) \geq m_i(f). \end{cases}$$

This shows that we have always

$$0 \leq M_i(f^+) - m_i(f^+) \leq M_i(f) - m_i(f),$$

and, therefore,

$$U(f^+,\mathcal{P}) - L(f^+,\mathcal{P})) = \sum_{i=1}^{n}(M_i(f^+) - m_i(f^+))(x_i - x_{i-1}) \leq \sum_{i=1}^{n}(M_i(f) - m_i(f))(x_i - x_{i-1}) < \epsilon,$$

i.e., f^+ is Riemann integrable. □

Remark. Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. It is sometimes useful to also introduce the integral $\int_b^a f(x)dx$. We define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

You have already seen the the following fact of fundamental importance and a sketch of its proof in the Calculus Course, connecting differentiation and integration.

Fundamental Theorem of Calculus

Theorem 10.8 (Fundamental Theorem of Calculus)**.** *Let $f \in C([a,b])$. Then $F(c) = \int_a^c f(x)dx$ lies also in $C([a,b])$, is differentiable on $[a,b]$ (one-sided differentiable at a and b), and we have $F'(x) = f(x)$ for all $x \in [a,b]$.*

Remark. Another formulation of the Fundamental Theorem of Calculus reads as follows: If $g \in C([a,b])$ and $g = f'$ for some function $f : [a,b] \to \mathbb{R}$, then

$$f(c) = f(a) + \int_a^c g(x)dx. \tag{22}$$

Concrete rules for Riemann integrals like *Integration by Parts* or *Substitution* were discussed in the Calculus Course and will be omitted here.

Theorem 10.9 (Mean Value Theorem for integrals)**.** *Let $f \in C([a,b])$ and $g \geq 0$ be Riemann integrable on $[a,b]$. Then fg is also Riemann integrable on $[a,b]$ and there exists $c \in [a,b]$ such that* *Mean Value Theorem for integrals*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx. \tag{23}$$

Examples. Let us briefly discuss that both conditions "$g \geq 0$" and "f continuous" in the theorem are necessary. For this we present a counterexample for each of the cases that f is not continuous or g changes sign.

(a) Let $f, g : [0,1] \to \mathbb{R}$ with $g(x) = 1$ for all $x \in [0,1]$ and

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1/2], \\ 1 & \text{if } x \in (1/2,1]. \end{cases}$$

Note that g and fg are Riemann integrable, but f is not continuous. Then we have $\int_0^1 f(x)g(x)dx = 1/2$ and $\int_0^1 g(x)dx = 1$. Observe that f takes only the values 0 and 1. On the other hand, if the theorem were true, we could find $c \in [0,1]$ with

$$f(c) = f(c)\int_0^1 g(x)dx = \int_0^1 f(x)g(x)dx = \frac{1}{2},$$

which is a contradiction.

(b) Let $f, g : [-1,1] \to \mathbb{R}$ be defined by $f(x) = g(x) = x$ for all $x \in [-1,1]$. Then f and g are continuous and therefore, Riemann integrable, but we do not have $g \geq 0$. Moreover, note that $\int_{-1}^1 g(x)dx = 0$ and

$$\int_{-1}^1 f(x)g(x)dx = \int_{-1}^1 x^2 dx = \frac{2}{3}.$$

If the theorem were true, we could find $c \in [-1,1]$ such that

$$\frac{2}{3} = \int_{-1}^1 f(x)g(x)dx = f(c)\int_{-1}^1 g(x)dx = f(c)\cdot 0 = 0,$$

which is, again, a contradiction.

Proof. Since $f \in C([a,b])$, we know from Theorem 10.6 that f is Riemann integrable. Riemann integrability of fg follows then from Theorem 10.7(c).
Now we show the identity (23): By Theorem 7.12, there exist $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$. Since $g \geq 0$ and $mg \leq g \leq Mg$, we conclude from Theorem 10.7(a),(b) that

$$m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx.$$

If $\int_a^b g(x)dx = 0$, this implies that also $\int_a^b f(x)g(x)dx = 0$ and there is nothing to show. Therefore, we can assume that $\int_a^b g(x)dx > 0$ (the case $\int_a^b g(x)dx < 0$ is not possible since $g \geq 0$). So we can divide by $\int_a^b g(x)dx$ and obtain

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M.$$

Since f is continuous, by the Intermediate Value Theorem (Theorem 7.10), we can find $c \in [a,b]$ with

$$f(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx},$$

which shows (23). □

Definition 10.10. *Let us briefly introduce the Riemann Integral for a complex valued function $f : [a,b] \to \mathbb{C}$. Then we can write $f = f_1 + if_2$ with $f_1, f_2 : [a,b] \to \mathbb{R}$ and we say that f is Riemann integrable on $[a,b]$ if both f_1 and f_2 are Riemann integrable on $[a,b]$ and we define its Riemann Integral as*

Riemann Integral of complex functions

$$\int_a^b f(x)dx = \int_a^b f_1(x)dx + i\int_a^b f_2(x)dx.$$

Then Theorems 10.6 and 10.7(a,c,d) hold also for the Riemann Integral of a complex function. Integration of complex valued functions plays an important role in the 2H Course Complex Analysis.

10.2 Improper Integrals

The definition of the Riemann integral of f requires that f is defined on a compact interval and that f is bounded. A "nice" family of

Riemann integrable functions are continuous functions. We extend the definition of a Riemann integral of a continuous function to the case that either the interval is unbounded (f is defined on $(-\infty, a]$ or $[a, \infty)$) or that the interval is bounded but f may be unbounded, as we approach one of the finite boundary points of the interval. In this case we speak of an *improper integral* of f and give a definition via a limit procedure. Here is the precise definition.

Definition 10.11. *Let $I = [a, b)$ with $a < b$ (bounded domain) of $b = \infty$ (unbounded domain). Let $f : I \to \mathbb{R}$ be continuous. For $c \in I$, let*

$$F(c) = \int_a^c f(x)dx.$$

If $F(c) \to L$ if $c \to b$, we say that the integral $\int_a^b f(x)dx$ converges *to L. If $F(c)$ has no limit, we say that $\int_a^b f(x)dx$* diverges. *The expression $\int_a^b f(x)dx$ is called an* improper integral *of f.*

The improper integrals of a continuous function f over the bounded intervals $(a, b]$ and (a, b) with $a < b$ and the unbounded intervals $(-\infty, b]$ and $\mathbb{R}$ are defined similarly.

convergence and divergence of an improper integral

Remark. Let $f \in C([a, b])$. Then we have both a ordinary Riemann integral of f on $[a, b]$ and the improper integral on $[a, b)$. Then we also have $F \in C([a, b])$ (see Theorem 10.8) and the improper integral agrees with the ordinary Riemann integral:

$$\lim_{c \to b-} \int_a^c f(x)dx = \int_a^b f(x)dx.$$

Examples. (a) Let $f(x) = x^{-\alpha}$ on $[1, \infty)$. If $\alpha = 1$, we have $F(c) = \log(c)$ and the improper integral $\int_1^\infty x^{-1}dx$ diverges. If $\alpha \in (0, \infty)$ and $\alpha \neq 1$, then $F(c) = (c^{1-\alpha} - 1)/(1 - \alpha)$ and the improper integral $\int_1^\infty x^{-\alpha}dx$ converges if and only if $\alpha \in (1, \infty)$, in which case we have

$$\int_1^\infty \frac{dx}{x^\alpha} = \frac{1}{\alpha - 1}.$$

(b) Let $f(x) = x^{-\alpha}$ on $(0, 1]$. If $\alpha = 1$, we have $F(c) = \log(c)$ and the improper integral $\int_0^1 x^{-1}dx$ diverges. If $\alpha \in (0, \infty)$ and $\alpha \neq 1$, then $F(c) = (1 - c^{1-\alpha})/(1 - \alpha)$ and the improper integral $\int_0^1 x^{-\alpha}dx$ converges if and only if $\alpha \in (0, 1)$, in which case we have

$$\int_0^1 \frac{dx}{x^\alpha} = \frac{1}{1 - \alpha}.$$

(c) We conclude from (a) and (b) that the improper integral $\int_0^\infty x^{-\alpha} dx$ diverges for any choice of $\alpha \in (0, \infty)$.

(d) Let $f(x) = \cos x$ on $[0, \infty)$. Then $F(c) = \sin x$ and the improper integral $\int_0^\infty \cos(x) dx$ diverges.

(e) Let $f(x) = e^{-\alpha t}$ with $\alpha \in \mathbb{R} \backslash \{0\}$ on $[0, \infty)$. Then $F(c) = \frac{1-e^{-\alpha c}}{\alpha}$ and the improper integral converges to $1/\alpha$ if and only if $\alpha > 0$.

(f) Let $f(x) = \log x$ on $(0, 1]$. Then

$$F(c) = \int_c^1 \log(x) dx = [x \log(x) - x]_c^1 = c - 1 - c \log(c).$$

Since (with $c = 1/t$)

$$\lim_{c \to 0+} c \log(c) = \lim_{t \to \infty} \frac{\log(1/t)}{t} = - \lim_{t \to \infty} \frac{\log(t)}{t} = 0,$$

we have

$$\lim_{c \to 0+} F(c) = -1,$$

and the improper integral $\int_0^1 \log(x) dx$ converges to -1.

(g) Let $f(x) = (3 - x)^{-\alpha}$ on $[1, 3)$ with $\alpha \geq 0$. Then we have

$$F(c) = \int_1^c \frac{dx}{(3-x)^\alpha} = \int_{3-c}^2 \frac{dt}{t^\alpha} = \frac{1}{1-\alpha} \left(2^{1-\alpha} - (3-c)^{1-\alpha} \right),$$

and the improper integral $\int_1^3 \frac{dx}{(3-x)^\alpha}$ converges if and only if $0 \leq \alpha < 1$. In this case we have

$$\int_1^3 \frac{dx}{(3-x)^\alpha} = \lim_{c \to 3} F(c) = \frac{2^{1-\alpha}}{1-\alpha}.$$

For example, if $\alpha = 1/2$, we have

$$\int_1^3 \frac{dx}{\sqrt{3-x}} = 2\sqrt{2}.$$

(h) Consider $\displaystyle\int_0^\infty \frac{dx}{1+e^x}$. Direct calculation of the integral leads to

$$\int_0^c \frac{dx}{1+e^x} = \left[-\log(1+e^{-x}) \right]_0^c = \log\left(\frac{2}{1+e^{-c}}\right) \to \log(2) \quad \text{as } c \to \infty.$$

So we have

$$\int_0^\infty \frac{dx}{1+e^x} = \log 2.$$

Linearity (see Theorem 10.7(a)) holds also for improper integrals. This follows via a limit process directly from Theorem 10.7(a). Like in the case of infinite series, we have a *Comparison Test* and an *Absolute Convergence Theorem* for improper integrals.

Comparison Test

Theorem 10.12 (Comparison Test)**.** *Let $0 \leq f \leq g$ be continuous functions on $[a, b)$ with $b > a$ or $b = \infty$. If the improper integral $\int_a^b g(x)dx$ converges with limit L then $\int_a^b f(x)dx$ is also convergent with limit $K \leq L$. Equivalently, if the improper integral $\int_a^b f(x)dx$ is divergent then so is $\int_a^b g(x)dx$.*

The proof of this fact is straightforward and will be omitted.

absolute convergence

conditional convergence

Definition 10.13. *Let f be a continuous function on $[a, b)$ with $b > a$ or $b = \infty$. If the improper integral $\int_a^b |f(x)|dx$ converges, we say that the improper integral $\int_a^b f(x)dx$* converges absolutely. *In the case that $\int_a^b f(x)dx$ is convergent but not $\int_a^b |f(x)|dx$, we say that the improper integral $\int_a^b f(x)dx$* converges conditionally.

Absolute Convergence Theorem

Theorem 10.14 (Absolute Convergence Theorem)**.** *Let f be a continuous function on $[a, b)$ with $b > a$ or $b = \infty$. If $\int_a^b f(x)dx$ converges absolutely, then it converges.*

The proof follows very much the same lines as the proof of Theorem 9.6:

Proof. Let $\int_a^b f(x)dx$ be absolutely convergent. Then $\int_a^b 2|f(x)|dx$ is convergent and so is $\int_a^b(|f(x)| + f(x))dx$, by comparison, since $0 \leq |f| + f \leq 2|f|$. Since $\int_a^b(|f(x)| + f(x))dx$ and $\int_a^b |f(x)|dx$ are both convergent, we conclude with Linearity that also

$$\int_a^b f(x)dx = \int_a^b (|f(x)| + f(x))dx - \int_a^b |f(x)|dx$$

is convergent. □

Examples. (a) Consider $\displaystyle\int_0^\infty \exp(-x^2)dx$. Then $x^2 \exp(-x^2) \to 0$ as $x \to \infty$ and there exists a constant $C > 0$ such that $0 \leq x^2 \exp(-x^2) \leq C$ for all $0 \leq x \leq \infty$. This shows that we have $0 \leq \exp(-x^2) \leq Cx^{-2}$ on $[1, \infty)$ and, since $\int_1^\infty x^{-2}dx$ converges, the improper integral $\int_1^\infty \exp(-x^2)dx$ is also convergent, by Comparison. Note that $\exp(-x^2)$ is continuous and, therefore, Riemann integrable on $[0,1]$. This shows convergence of the improper integral

$$\int_0^\infty \exp(-x^2)dx = \int_0^1 \exp(-x^2)dx + \int_1^\infty \exp(-x^2)dx.$$

In fact, while $F(c) = \int_0^c \exp(-x^2)dx$ cannot be expressed explicitly, one can prove that

$$\int_0^\infty \exp(-x^2)dx = \lim_{c\to\infty} F(c) = \frac{\sqrt{\pi}}{2}.$$

You may know from Statistics that the function $\frac{2}{\pi}\int_0^c \exp(-x^2)dx$ is called the *(Gauss) error function.* *Gauss error function*

(b) Consider $\displaystyle\int_\pi^\infty \frac{\cos x}{x^2}dx$. Since this improper integral is absolutely convergent ($\int_\pi^\infty x^{-2}dx$ converges), it must also be convergent.

(c) The improper integral $\displaystyle\int_\pi^\infty \frac{\sin x}{x}dx$ is conditionally convergent. We first show convergence: Integration by Parts leads to

$$F(c) = \int_\pi^c \frac{\sin x}{x}dx = -\left[\frac{\cos x}{x}\right]_\pi^c - \int_\pi^c \frac{\cos x}{x^2}dx.$$

We have

$$\left[\frac{\cos x}{x}\right]_\pi^c = \frac{\cos c}{c} + \frac{1}{\pi} \to \frac{1}{\pi} \quad \text{as } c \to \infty,$$

and $\int_\pi^c \cos(x)/x^2 dx$ is convergent, as seen in (b). Therefore, the improper integral is convergent.

Next we show that $\displaystyle\int_\pi^\infty \left|\frac{\sin x}{x}\right| dx$ is divergent. Let

$$I_n = \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx = \int_0^\pi \frac{\sin t}{t+n\pi}dt \geq \frac{1}{(n+1)\pi}\int_0^\pi \sin t dt = \frac{2}{(n+1)\pi}.$$

Since

$$F((n+1)\pi) = \int_\pi^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{i=1}^n \frac{2}{(i+1)\pi},$$

the improper integral is not absolutely convergent since the harmonic series $\sum_{i=1}^\infty 1/(i+1)$ diverges.

Sometimes, it is useful in the proof of convergence of an improper integral $\int_a^b f(x)dx$ to split up the interval, i.e., to choose $c \in (a,b)$ and to prove separately that $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent.

Example. Consider $\int_0^\infty f(x)dx$ with $f(x) = \log(x)/(1+x^3)$.
On $(0,1]$, we have $0 \leq |f(x)| \leq -\log(x)$ and the integral $\int_0^1 f(x)dx$ is absolutely convergent, by Comparison with the convergent improper integral $\int_0^1 \log(x)dx$.
Next, we look at $\int_1^\infty f(x)dx$. Since $\log(x)/x \to 0$ as $x \to \infty$, we can find $C > 0$ such that $0 \leq \log x/x \leq C$ for all $x \in [1,\infty)$, i.e., $0 \leq \log x \leq Cx$. This implies that

$$|f(x)| \leq \frac{Cx}{1+x^3} \leq \frac{C}{x^2}$$

for all $x \in [1,\infty)$. Then the integral $\int_1^\infty f(x)dx$ is also absolutely convergent, by Comparison with the convergent improper integral $\int_1^\infty dx/x^2$.
Combining both facts shows that the improper integral $\displaystyle\int_0^\infty \frac{\log(x)}{1+x^3}dx$ is convergent.

Important concepts/typical problems in this chapter that you should try:

- Let $f(x) = x$ if $x \in [0,1] \cap \mathbb{Q}$ and $f(x) = 0$ for $x \in [0,1] \backslash \mathbb{Q}$. Compute $L(f,\mathcal{P})$ for all partitions $\mathcal{P}$ of $[0,1]$ and $\mathcal{U}(f)$.
- Let $f \in C([a,b])$ and $g : [\alpha,\beta] \to [a,b]$ be differentiable on (α,β). For $\gamma \in (\alpha,\beta)$, let $F(\gamma) = \int_a^{g(\gamma)} f(x)dx$. Calculate $F'(\gamma)$. Show that the integral $\int_0^T \frac{dx}{1+x^2} + \int_0^{1/T} \frac{dx}{1+x^2}$ is independent of $T > 0$.
- Formulate the Mean Value Theorem for integrals.
- Show that the improper integral $\int_0^\infty dx/(x^{1/3}+x^2)$ is convergent.

11 Sequences of functions and uniform convergence

In this chapter, we consider sequences (f_n) of functions and will discuss different notions of convergence for these sequences.

11.1 Pointwise convergence of functions

Let $I \subset \mathbb{R}$ be an interval and (f_n) be a sequence in $C(I)$. If, for every $x \in I$, the sequence of real numbers $f_n(x) \in \mathbb{R}$ converges, it is natural to define the "limit function" $f : I \to \mathbb{R}$ of such a sequence f_n as

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Example. Let $f_n : [0,1] \to \mathbb{R}$ be defined as $f_n(x) = x^n$. Then $f_n \in C([0,1])$. For $x \in [0,1)$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0.$$

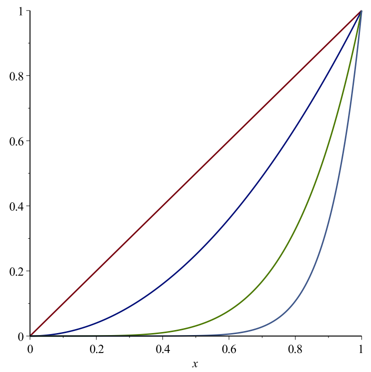

Figure 1: The functions f_1, f_2, f_5 and f_{10} defined by $f_n(x) = x^n$.

Moreover,

$$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1.$$

This behaviour is illustrated in Figure 1. So the *pointwise limit* of (f_n) is the function $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & \text{if } x \in [0,1), \\ 1, & \text{if } x = 1. \end{cases}$$

Note that the *limit function* $f : [a,b] \to \mathbb{R}$ is discontinuous at $x = 1$, so $f \notin C([0,1])$.

Let us formalise the notion of a pointwise limit.

Definition 11.1. *Let (f_n) be a sequence of functions in $C(I)$. We say that this sequence has a* pointwise limit *if, for all $x \in I$, $\lim_{n\to\infty} f_n(x)$ exists, and the* limit function *$f : I \to \mathbb{R}$ is then defined as*

pointwise limit

limit function

$$f(x) = \lim_{n\to\infty} f_n(x).$$

11.2 Uniform convergence of functions

There is a stronger notion of convergence of continuous functions f_n, the so called *uniform convergence.* We say that f_n converges uniformly to f in $I \subset \mathbb{R}$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that the graph of f_n stays within the graphs of $f - \epsilon$ and $f + \epsilon$, for all $n \geq N$ or, in other words, $|f - f_n| < \epsilon$. In Figure 2, the graphs of $f - \epsilon$ and $f + \epsilon$ are blue and while f_7 stays within both graphs, the function f_2 clearly does not.

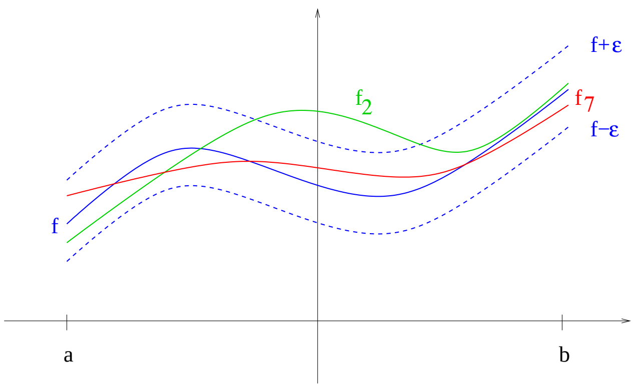

Figure 2: The function f_7 satisfies $|f - f_7| < \epsilon$ while the function f_2 does not.

Definition 11.2. *Let (f_n) be a sequence of functions in $C(I)$. We say that f_n* converges uniformly *to $f : I \to \mathbb{R}$ if, for every $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in I$, we have*

uniform convergence

$$|f(x) - f_n(x)| < \epsilon.$$

If f_n converges uniformly to f, we write "$f_n \to f$ uniformly".

Example. Let $f_n(x) = x^2 + \frac{sin x}{n}$ on $I = [-\pi, \pi]$. Then $f_n \to f$ with $f(x) = x^2$ uniformly since, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ with $1/N < \epsilon$ and we have for all $n \geq N$ and $x \in I$:

$$|f_n(x) - f(x)| = \frac{|\sin x|}{n} \leq \frac{1}{N} < \epsilon.$$

Uniform convergence is a stronger property than pointwise convergence:

Proposition 11.3. *Let $f_n \in C(I)$ be converging uniformly to $f : I \to \mathbb{R}$. Then f_n converges also pointwise to f.*

Proof. For every fixed $x \in I$, uniform convergence implies directly that $\lim_{n\to\infty} f_n(x) = f(x)$. $\square$

In contrast to pointwise convergence, uniform convergence has the important property that it *preserves continuity*, which is formulated in the following theorem.

Theorem 11.4. *Let $f_n \in C(I)$. If $f_n \to f$ uniformly, then the limit function f is also continuous, i.e., $f \in C(I)$.*

Proof. We show that f is continuous, using the (ϵ, δ)-formalism. Let $c \in I$ and $\epsilon > 0$ be given. Since $f_n \to f$ uniformly, we can find $N \in \mathbb{N}$ such that for all $n \geq N$:

$$|f(x) - f_n(x)| < \frac{\epsilon}{3} \qquad \forall\, x \in I. \tag{24}$$

Now we use continuity of the function f_N: Since f_N is continuous at $c \in I$, we find $\delta > 0$ such that for all $x \in I$ with $|x - c| < \delta$:

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}. \tag{25}$$

Combining (24) and (25) via the triangle inequality implies that we have for all $x \in I$ with $|x - c| < \delta$:

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

which shows that f is continuous at c. Since $c \in I$ was arbitrary, f is continuous. $\square$

Remark. Theorem 11.4 implies that the pointwise convergence $f_n \to f$ in the example in Section 11.1 cannot be a uniform convergence, since the limit function is discontinuous at $x = 1$.

Finally, we mention further important properties of uniform convergence.

Theorem 11.5. *Let $I = [a,b]$ and (f_n) be a sequence in $C(I)$. If $f_n \to f$ uniformly, then we have for all $c \in [a,b]$*

$$\int_a^c f_n(x)dx \to \int_a^c f(x)dx.$$

Reader's Task. Give a proof of Theorem 11.5.

Theorem 11.6. *Let $I = [a,b]$ and (f_n) be a sequence in $C^1(I)$. Assume that*

- *the sequence f_n converges pointwise to a limit function f.*
- *the sequence f_n' converges uniformly in I.*

Then $f \in C^1(I)$, i.e., the limit function f has also a continuous derivative, and we have at every $c \in I$:

$$f'(c) = \lim_{n\to\infty} f_n'(c).$$

Proof. Let $f_n \in C^1(I)$ be as in the theorem. Let $f_n' \to g$ uniformly. Then we have $g \in C(I)$, by Theorem 11.4. Since $f_n' \in C(I)$, (22) tells us that

$$f_n(c) = f_n(a) + \int_a^c f_n'(x)dx. \tag{26}$$

We know from Theorem 11.5 that $\int_a^c f_n'(x)dx \to \int_a^c g(x)dx$, and taking limits in (26), as $n \to \infty$, leads to

$$f(c) = f(a) + \int_a^c g(x)dx.$$

Differentiating both sides w.r.t. c and using the Fundamental Theorem of Calculus yields

$$f'(c) = g(c) = \lim_{n\to\infty} f_n'(c).$$

Note that $f \in C^1(I)$, since $f' = g \in C(I)$. □

Example. Let $f_n \in C([0,2])$ be defined by (see Figure 3)

$$f_n(x) = \begin{cases} 4^{n-2}x, & \text{if } 0 \le x \le 1/2^{n-1}, \\ 2^{n-3} - 4^{n-2}(x - 1/2^{n-1}), & \text{if } 1/2^{n-1} < x \le 1/2^{n-2}, \\ 0, & \text{if } 1/2^{n-2} < x \le 2. \end{cases}$$

Then we have

$$\int_0^2 f_n(x)dx = 1/4$$

for all $n \in \mathbb{N}$, but f_n converges pointwise to the zero function $f \in C([0,2])$, $f(x) = 0$, since for $x \in (0,2]$ we can find $N \in \mathbb{N}$ such that $1/2^{N-2} < x$ and therefore $f_n(x) = 0$ for all $n \geq N$ and for $x = 0$ we have $f_n(0) = 0$ for all $n \in \mathbb{N}$. The convergence $f_n \to f$ cannot be uniform, since in this case Theorem 11.5 would imply that

$$\int_0^2 f(x)dx = \lim_{n\to\infty} \int_0^2 f_n(x)dx = 1/4,$$

which is a contradiction to $\int_0^2 f(x)dx = 0$.

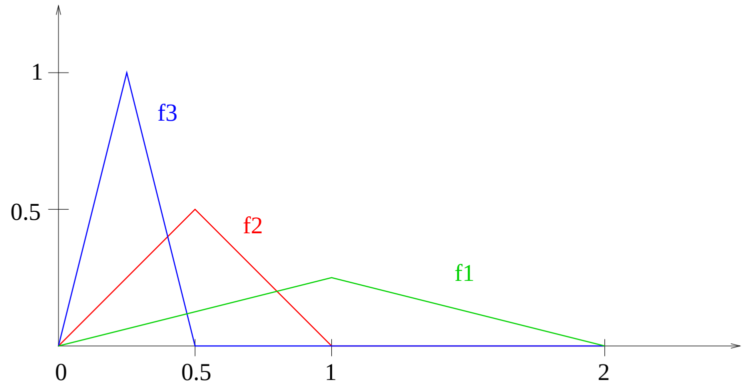

Figure 3: The functions $f_1, f_2, f_3, \ldots$ converging pointwise to zero.

Remark. The Definitions 11.1 and 11.2 of pointwise and uniform convergence extend canonically to complex-valued functions $f_n : I \to \mathbb{C}$. Then Proposition 11.3 and Theorems 11.4 and 11.5 hold also without change for complex-valued functions.

Important concepts/typical problems in this chapter that you should try:

- Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \sin(nx)/n$. Show that f_n converges uniformly to the zero function but that f_n' is not even pointwise convergent.
- Calculate the pointwise limit of the sequence $f_n(x) = \frac{e^x}{x^n}$ on the interval $(1, \infty)$ and decide whether the convergence is uniform.
- Let $f_n, f : [a, b] \to \mathbb{R}$. Pointwise convergence $f_n \to f$ can be expressed as follows with the help of quantifiers and the (ϵ, δ)-formalism:

  $$\forall\, x \in [a, b]\, \forall\, \epsilon > 0\, \exists\, N \in \mathbb{N}\, \forall\, n \geq N : \quad |f_n(x) - f(x)| < \epsilon. \tag{27}$$

  Give a corresponding formulation for "$f_n \to f$ uniformly".

12 Power series and Taylor series

A very useful method to describe functions is via *power series*. A power series is an expression of the form $\sum_{k=0}^{\infty} a_k x^k$. The coefficients a_k can be real or complex and, choosing an explicit real or complex value for x, we obtain an infinite series in the sense of Chapter 9 which might converge or not. To understand the convergence domain of a power series it is better to consider it as a complex power series $\sum_{k=0}^{\infty} a_k z^k$ with $z \in \mathbb{C}$. It turns out that there is a special number $R \in [0,\infty]$, the so called *radius of convergence* of the power series, such that the power series converges for all complex values $z \in \mathbb{C}$ with $|z| < R$ and diverges for all complex values $|z| > R$. Note that the set $\{z \in \mathbb{C} \mid |z| < R\}$ describes the interior of a circle of radius R around the origin (see Figure 4). This is the reason for the name radius of convergence.

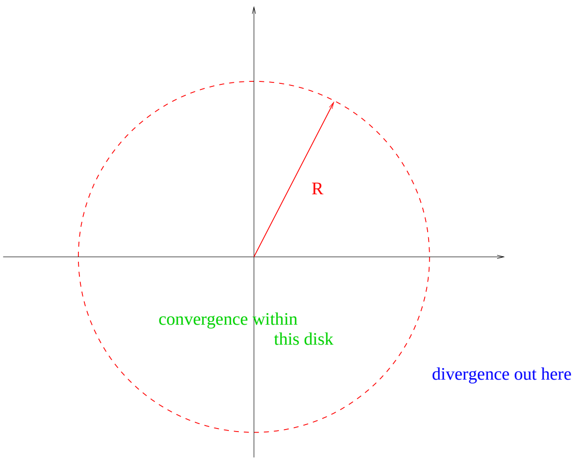

Figure 4: Radius of convergence of a complex power series $\sum_{k=0}^{\infty} a_k z^k$.

Here are some examples.

Examples. (a) Every polynomial $\sum_{k=0}^{n} a_k z^k$ is a power series with $a_k = 0$ for $k \geq n+1$. In this case the radius of convergence is $R = \infty$, since the sum is a finite value for every choice of $z \in \mathbb{C}$.

(b) Recall that the *geometric series* is $\sum_{k=0}^{\infty} z^k$. Here we have $a_k = 1$ for all $k \in \mathbb{N} \cup \{0\}$. Note that we have for $|z| < 1$ the identity

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z},$$

and divergence for $|z| > 1$. So the radius of convergence here is $R = 1$. The power series represents the function $f(z) = 1/(1-z)$ on $\{z \in \mathbb{C} \mid |z| < 1\}$, but note that, while the power series diverges for $|z| > 1$, the function f is well defined on the whole complex plane except for $z = 1$.

(c) Recall from the Calculus Course that the *exponential function* can be represented via the power series (Taylor series)

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

In this case we have $a_k = 1/k!$ and the power series converges for all $z \in \mathbb{C}$, so the radius of convergence is here, again, $R = \infty$.

(d) Consider the power series $\sum_{k=0}^{\infty} k^{k+1} z^k$. In this case we have $a_k = k^{k+1}$, and it turns out that the power series diverges for any complex number $z \neq 0$. So the radius of convergence is $R = 0$ in this case.

Remark. In many cases we have $R \in (0, \infty)$ and the domain $\{z \in \mathbb{C} \mid |z| < R\}$ is called the *disk of convergence*. While we know what happens for complex numbers $z \in \mathbb{C}$ with $|z| < R$ (convergence) and with $|z| > R$ (divergence), convergence/divergence behaviour of the power series for points *on the circle* $|z| = R$ is a subtle question.

12.1 Radius of convergence and absolute convergence

Definition 12.1. *A* complex power series *is an expression of the form* $\sum_{k=0}^{\infty} a_k z^k$ *with* $a_k \in \mathbb{C}$. *Given such a power series and a number* $R \geq 0$ *or* $R = \infty$ *such that the series converges for* $|z| < R$ *and diverges for* $|z| > R$*, then* R *is called the* radius of convergence *of the power series.*

power series

radius of convergence

The Ratio Test and the n^{th} Root Test provide tools to calculate the radius of convergence for particular power series $\sum_{k=0}^{\infty} a_k z^k$. For $z \neq 0$, we have $|a_{k+1}/a_k| \to L$ as $k \to \infty$ if and only if

$$\left| \frac{a_{k+1} z^{k+1}}{a_k z^k} \right| \to L|z|,$$

and the Ratio Test tells us that we have convergence for $|z| < 1/L$ and divergence for $|z| > 1/L$. Similarly, for $z \neq 0$, we have $|a_k|^{1/k} \to L$ as

$k \to \infty$ if and only if

$$\left|a_k z^k\right|^{1/k} \to L|z|,$$

and the n^{th} Root Test tells us that we have convergence for $|z| < 1/L$ and divergence for $|z| > 1/L$. In both cases, the radius of convergence is, therefore $R = 1/L$.

Examples. We calculate the radius of convergence for the following power series $\sum_{k=0}^{\infty} a_k z^k$.

(a) $a_k = 2^k/k$. Then we have

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2k}{k+1} \to 2,$$

so the radius of convergence is $R = 1/2$.

(b) $a_0 = 1$ and $a_k = k^k/k!$ for $k \geq 1$. Then we have for $k \geq 1$:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^{k+1}}{(k+1)k^k} = \left(\frac{k+1}{k}\right)^k = \left(1 + \frac{1}{k}\right)^k \to e,$$

so the radius of convergence is $R = 1/e$.

Lemma 12.2. *Let* $\sum_{k=0}^{\infty} a_k c^k$ *be convergent for some* $c \in \mathbb{C}\backslash\{0\}$. *Then* $\sum_{k=0}^{\infty} a_k z^k$ *converges absolutely for all* $|z| < |c|$.

Proof. Since $\sum_{k=0}^{\infty} a_k c^k$ converges, we have $a_k c^k \to 0$ as $k \to \infty$. Therefore, there exists $M > 0$ such that $|a_k c^k| \leq M$. This implies that

$$|a_k z^k| = |a_k c^k| \cdot (|z|/|c|)^k \leq M|z/c|^k.$$

Since $|z/c| < 1$, The geometric series $\sum_{k=0}^{\infty} M|z/c|^k$ is convergent and, therefore, $\sum_{k=0}^{\infty} A a_k z^k|$ is also convergent, by Comparison. □

A similar proof yields the following fact for the term by term derivative of a power series.

Lemma 12.3. *Let* $\sum_{k=0}^{\infty} a_k c^k$ *be convergent for some* $c \in \mathbb{C}\backslash\{0\}$. *Then* $\sum_{k=0}^{\infty}(k+1)a_{k+1} z^k$ *converges absolutely for all* $|z| < |c|$.

Reader's Task. Modify the proof of Lemma 12.2 in such a way that you obtain a proof of Lemma 12.3.

Theorem 12.4. *Consider* $\sum_{k=0}^{\infty} a_k z^k$. *Then we must have one of the following cases:*

(a) $\sum a_k z^k$ *converges only for* $z = 0$*, i.e., the radius of convergence is* $R = 0$*.*

(b) There exists a number $R \in (0, \infty)$ *(radius of convergence) such that* $\sum a_k z^k$ *converges absolutely for* $|z| < R$ *and diverges for* $|z| > R$*.*

(c) $\sum a_k z^k$ *converges absolutely for all* $z \in \mathbb{C}$*, i.e., the radius of convergence is* $R = \infty$*.*

Proof. Let

$$X = \{r \in [0, \infty) \mid \sum a_k z^k \text{ converges for some } z \in \mathbb{C} \text{ with } |z| = r\}.$$

Note that $0 \in X$, so X is not empty. Then X is either unbounded from above or it has a supremum, by the Completeness Axiom of $\mathbb{R}$. In the first case, we must have case (c), by Lemma 12.2. In the case $X = \{0\}$, we must have case (a). It remains to consider the case that X is bounded and $R = \sup(X) \in (0, \infty)$. Then Lemma 12.2 tells us that $\sum a_k z^k$ converges absolutely for all $|z| < R$, since then we can find $c \in X$ with $|z| < c$. Now let $|z| > R$. If $\sum a_k z^k$ were convergent, then $R < |z| \in X$ and we would have a contradiction to the fact $R = \sup(X)$. So we must be in case (b). □

Of course, real power series $\sum a_k x^k$ with $a_k \in \mathbb{R}$ can be considered as restrictions of complex power series $\sum a_k z^k$. In this case, if $R > 0$ is the radius of convergence, then $\sum a_k x^k$ is absolutely convergent for all $x \in (-R, R)$ and divergent for all real x with $|x| > R$. In the next section, we return to real power series and consider another type of convergence, namely, uniform convergence.

12.2 Uniform convergence and the Weierstrass M-Test

After considerations about *absolute convergence* we will focus in this section on *uniform convergence*. The precise facts about the different types of convergence (conditional/absolute convergence and pointwise/uniform convergence) might be a bit confusing when seeing them for the first time, but they are important and will play, again, an important role in the 2H Course Complex Analysis. To make things a bit easier, we restrict ourselves to uniform convergence of real power series, even though the concepts can be also extended naturally to the complex case.

Note that the partial sums of a real power series $\sum_{k=0}^{\infty} a_k x^k$ are the polynomials $f_n(x) = \sum_{k=0}^{n} a_k x^k$ and that the polynomials $f_n(x)$ can be written as the finite sums $f_n(x) = \sum_{k=0}^{n} g_k(x)$ with monomials $g_k(x) = a_k x^k$. In this way, we can consider the power series as a limit of the functions f_n in the sense of Chapter 11 and we may ask whether the sequence (f_n) converges uniformly in a suitable domain $D \subset \mathbb{R}$. In this case we can apply results like Proposition 11.3 or Theorems 11.4, 11.6 and 11.5 to make statements about the functions represented by power series, since they can then be viewed as the uniform limits of the partial sums of these power series. A useful tool to prove uniform convergence in a domain $D \subset \mathbb{R}$ is the so-called *Weierstrass M-Test*:

Weierstrass M-Test

Theorem 12.5 (Weierstrass M-Test)**.** *Let $D \subset \mathbb{R}$ and (g_k) be a sequence of functions $g_k : D \to \mathbb{R}$ satisfying*

$$|g_k(x)| \leq M_k \quad \textit{for all } x \in D.$$

Let $f_n : D \to \mathbb{R}$ be defined as $f_n = \sum_{k=0}^{n} g_k$. Assume that $\sum_{k=0}^{\infty} M_k$ is convergent. Then there exists a function $f : D \to \mathbb{R}$ such that "$f_n \to f$ uniformly in D".

Proof. For each $x \in D$, the series $\sum g_k(x)$ converges absolutely, by Comparison with the series $\sum M_k$. Therefore $\sum g_k(x)$ is also convergent and we define $f(x) = \sum g_k(x) \in \mathbb{R}$. Now we need to show uniform convergence $f_n \to f$ on D. Let $\epsilon > 0$. Since $\sum M_k$ is convergent, i.e.,

$$\sum_{k=0}^{n} M_k \to L \quad \text{as } n \to \infty,$$

we can find $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} M_k = |L - \sum_{k=0}^{N} M_k| < \epsilon.$$

But this implies for all $x \in D$ and all $n \geq N$ that

$$|f(x) - f_n(x)| = |\sum_{k=n+1}^{\infty} g_k(x)| \leq \sum_{k=n+1}^{\infty} |g_k(x)| \leq \sum_{k=N+1}^{\infty} M_k < \epsilon.$$

This shows that f_n converges uniformly to f in D. $\square$

Using the Weierstrass M-Test, we can now prove the following important fact.

Theorem 12.6. *Let $\sum a_k x^k$ be a real power series and $R \in (0, \infty]$ be its radius of convergence. Let $0 < r < R$. Then the partial sums converge uniformly to $\sum a_k x^k$ in $[-r, r]$.*

Proof. Let $g_k(x) = a_k x^k$, $M_k = |a_k r^k|$ and $D = [-r, r]$. Since $0 < r < R$, we know from Theorem 12.4 that $\sum a_k r^k$ converges absolutely. Therefore, $\sum M_k$ is convergent. Moreover, we have for all $x \in D$ (i.e., $-r \leq x \leq r$) that

$$|g_k(x)| = |a_k x^k| = |a_k| \cdot |x|^k \leq |a_k| \cdot r^k = |a_k r^k| = M_k.$$

This shows that the requirements of the Weierstrass M-Test are satisfied and we have uniform convergence of

$$f_n(x) = \sum_{k=0}^{n} g_k(x) = \sum_{k=0}^{n} a_k x^k$$

to $f(x) = \sum a_k x^k$ in D. □

The next result shows that we can differentiate real power series term by term.

Corollary 12.7. *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a real power series and $R \in (0, \infty]$ be its radius of convergence. Then f is differentiable at all points $x \in (-R, R)$ and we have*

$$f'(x) = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k.$$

Proof. Let $f_n(x) = \sum_{k=0}^{n} a_k x^k$ be a sequence of real functions defined on the interval $(-R, R)$. Then their derivatives are given by $g_n(x) = \sum_{k=0}^{n-1} (k+1) a_{k+1} x^k$ and the radius of convergence of the power series $g(x) = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$ is at least R, by Lemma 12.3. Let $x \in (-R, R)$ and $r = |x|$. Then both sequences f_n and g_n converge uniformly in $[-r, r]$ to f and g, respectively, by Theorem 12.6. Note that $f_n \in C^1([-r, r])$. Then we can apply Theorem 11.6 and find that f is differentiable at x and we have

$$f'(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} (k+1) a_{k+1} x^k = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k.$$

□

Repeated application of the above corollary shows that a power series is an infinitely many times differentiable function within its radius of convergence.

Corollary 12.8. *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a real power series and $R \in (0, \infty]$ be its radius of convergence. Then f is infinitely many times differentiable at x at all points $x \in (-R, R)$ and we have for all $n \in \mathbb{N} \cup \{0\}$:*

$$f^{(n)}(0) = n! \cdot a_n. \tag{28}$$

Proof. Applying Corollary 12.7 repeatedly, we see that f is infinitely many times differentiable at all points $x \in (-R, R)$ and that we have

$$f^{(n)}(x) = \sum_{k=0}^{\infty} (k+1)(k+2) \cdots (k+n) a_{k+n} x^k.$$

Evaluation at $x = 0$ yields

$$f^{(n)}(0) = n! \cdot a_n.$$

□

(28) can be used to show that power series representing a function within its radius of convergence are unique.

Identity Theorem for Power Series

Theorem 12.9 (Identity Theorem for Power Series)**.** *Let $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ be two real power series with positive radii of convergence $R_a, R_b > 0$, respectively. If both series agree as functions on an interval $(-r, r)$ with $0 < r < \min\{R_a, R_b\}$, then we have $a_k = b_k$ for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Since $f = g$ on $(-r, r)$, we have

$$n! \cdot a_n = f^{(n)}(0) = g^{(n)}(0) = n! \cdot b_n,$$

i.e., $a_n = b_n$ after division by $n! \in \mathbb{N}$. □

Remark. In fact, an even stronger identity result holds for complex power series $f(z) = \sum a_k z^k$ with positive radius of convergence. If there exists a sequence $z_n \in \mathbb{C}$ within the circl of convergence with $z_n \to 0$ and $f(z_n) = 0$ for all n, then we have $a_k = 0$ for all $k \in \mathbb{N} \cup \{0\}$, i.e., the power series is trivial.

12.3 Taylor Series

Recall from the Calculus Course that, for a given function $f : \mathbb{R} \to \mathbb{R}$ with arbitrarily high derivatives at $x = 0$, we can define its *Taylor series*[11] as

Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Given such a Taylor series, we can use the Ratio Test or the n^{th} Root Test to calculate its radius of convergence R. But even though we know for $|x| < R$ that the power series is convergent, its value may not agree with the original function f. Here is such an example.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

In this case, it can be checked that f is infinitely many times differentiable and that there are polynomials p_k such that

$$f^{(k)}(x) = \begin{cases} p_k(1/x)e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Since $f^{(k)}(0) = 0$ for all $k \geq 0$, the Taylor series of f is $\sum_{k=0}^{\infty} 0 \cdot x^k$, i.e., the zero function and, therefore, has radius of convergence $R = \infty$. But the Taylor series of f agrees with f only at the origin $x = 0$.

In order to check whether the Taylor series represents the function f at a point x, we need to consider the remainder term, given in the Calculus Course, and need to show that this remainder term, at a given point $x \in \mathbb{R}$, converges to 0.
We will not delve further into this subtle topic. In many cases, we have agreement of a function and its Taylor series, and we can manipulate these power series in a straightforward manner. Here are examples.

Example. (a) A polynomial $f(x) = a_n x^n + \dots a_1 x + a_0$ is its own Taylor series since $f^{(k)}(0) = k! \cdot a_k$ for all k.

[11] This kind of series was formally introduced by the English mathematician Brook Taylor (1685-1731) in 1715. Taylor studied at Cambridge University and became a fellow of the Royal Society in 1712. He was also a member of the committee dealing the priority dispute between Sir Isaac Newton (1685-1731) and Gottfried Leibniz (1646-1716) on the invention of calculus.

(b) Let $f(x) = \sin x$. Then we have $f^{(4l)}(x) = \sin x$, $f^{(4l+1)}(x) = \cos x$, $f^{(4l+2)}(x) = -\sin x$ and $f^{(4l+3)}(x) = -\cos x$, which leads to

$$f^{(4l)}(0) = f^{(4l+2)}(0) = 0$$

and

$$f^{(4l+1)}(0) = -f^{(4l+3)}(0) = 1.$$

This shows that only the higher derivatives of f of odd orders at $x = 0$ are non-vanishing and the Taylor series of $\sin x$ is

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

Here, the radius of convergence is $R = \infty$ and the Taylor series agrees with $\sin x$ on all of $\mathbb{R}$.

(c) Similarly as in (b), we could also calculate the Taylor series of $\cos x$ directly. Another way to obtain a power series to represent $\cos x$ is to use Corollary 12.7 and to differentiate the power series of (b) term by term. Doing so leads to

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(2k+1)x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

By Lemma 12.3 and the Identity Theorem 12.9 we conclude that this power series represents $\cos x$ on all of $\mathbb{R}$ and agrees with the Taylor series of $\cos x$.

(d) Two other important functions are $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$, represented by their Taylor series on all of $\mathbb{R}$:

$$\sinh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} \quad \text{and} \quad \cosh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}.$$

(e) The Taylor series of $\log(1+x)$ is given by

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k.$$

Its radius of convergence is $R = 1$ and the power series represents $\log(1+x)$ on $-1 < x \leq 1$. Choosing, in particular, $x = 1$, leads to the identity

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log(2).$$

Finally, let us mention the general Binomial Expansion, derived via the Taylor series.

Binomial Expansion

Example (Binomial Expansion). For $|x| < 1$ and $c \in \mathbb{R}$, we have

$$(1+x)^c = 1 + \sum_{k=1}^{\infty} \binom{c}{k} x^k,$$

where

$$\binom{c}{k} = \frac{c(c-1)(c-2)\cdots(c-k+1)}{k!}.$$

Here are some special cases:

(a) If $c \in \mathbb{N}$, then $\binom{c}{k} = 0$ for $k > c$ and the power series is the polynomial

$$(1+x)^c = 1 + \sum_{k=1}^{\infty} \binom{c}{k} x^k.$$

(b) Since $\binom{-1}{k} = (-1)^k$, we recover the geometric series

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k.$$

(c) We have for $c = 1/2$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{16}x^3 - \cdots \quad \text{for } |x| < 1.$$

Important concepts/typical problems in this chapter that you should try without looking anything up:

- Calculate the radius of convergence of $\sum_{k=1}^{\infty} x^k/(k3^k)$.
- Using the Weierstrass M-Test, show that $\sum_{k=1}^{\infty} 1/(k^2+x^2)$ converges uniformly in the whole real line $\mathbb{R}$.
- Let $\sum a_k x^k$ be the Taylor series of a function f. How are a_k and derivatives of f related? Let the radius of convergence of the Taylor series be $R = \infty$. Does this imply that the Taylor series represents the function on the whole real line?
- Calculate the Taylor series of $f(x) = e^x$. Derive from it a power series representing the function $g(x) = e^{-x^2}$.