Analysis 1 (Michaelmas Term 2014)

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Lectures (all in CLC013): Mon 14, Tues 16

Contents

Some Books

a) *Mathematical Analysis, a straightforward approach*, K. G. Binmore.

b) *Calculus*, Michael Spivak.

c) *Limits, Limits Everywhere, The Tools of Mathematical Analysis*, David Applebaum.

d) *Calculus*, Schaum's Outlines, F. Ayres and E. Mendelson.

e) *Advanced Calculus*, Schaum's Outlines, R. Wrede and M. Spiegel.

f) *How to Think Like a Mathematician*, Kevin Houston.

g) *How to Read and Do Proofs*, Daniel Solow.

The books a)-c) are good introductions into material of Analysis. However, c) falls short on the concepts of continuity, differentiation and integration. d) and e) contain a lot of solved problems and is a good exercise source. Finally, the books f) and g) cover the logic as well as proof techniques which are important in the study of Analysis.

This lecture notes are not meant to be complete, but they are a useful additional source of information!

The symbol at the margin of the page informs you about pitfalls to be avoided.

At the end of each chapter you find a box with important points which are useful for you to check whether you understood the crucial concepts and can apply the methods introduced in this chapter.

1 Basic logic and sets

1.1 Logic

In mathematics, we formulate mathematical statements and prove them.

Examples. (a) There are infinitely many prime numbers.

(b) $\sqrt{2}$ is irrational.

In mathematics, we introduce the objects we work with by *definitions*. Here is the definition of a mathematical statement.[1] *definition*

statement

Definition 1.1. *A* statement *is a sentence which is either true or false – but not both.*

Remarks. (a) and (b) above are examples of true statements.[2] While "$3 < 2$" is a false statement, "$x > 1$" is not a statement! It can be true or false, depending on the number we choose for x. "$x > 1$" is an example of a *conditional statement*. A conditional statement contains variables which can be specified to obtain a statement which is then true or false. *conditional statement*

Statements can be manipulated and combined via connectives to obtain new statements.

and/or/not

Definition 1.2. *Let A and B be two statements.*

(a) "$A\,\mathrm{and}\,B$" is a statement which is true only if both A and B are true.

(b) "$A\,\mathrm{or}\,B$" is a statement which is false only if both A and B are false.

(c) "$\mathrm{not}\,A$" is a statement which is true (false) if A is false (true). "$\mathrm{not}\,A$" is called the negation *of A.*

truth tables

The truth values of combined statements like "$A\,\mathrm{or}\,(\mathrm{not}\,B)$" in terms of A and B can be illustrated via *truth tables*.

[1] Note that definitions and other mathematical structures carry numbers so that we can refer to them at a later stage via their numbers.

[2] There are statements like "There are infinitely many pairs of numbers $n, n+2$ which are both prime", for which we do not know at present whether they are true or false. We call them *conjectures*. This one is called the *Twin Prime Conjecture*.

A	B	A and B	A or B	not A
false	false	false	false	true
false	true	false	true	true
true	false	false	true	false
true	true	true	true	false

We say, two expressions built up by unspecified statements A, B, C are *equivalent* if the truth tables made from their inputs and outputs are the same. We use the symbol "$\Leftrightarrow$". Here is an example due to De Morgan.[3]

equivalent statements
"$\Leftrightarrow$"
De Morgan's Law

Example (De Morgan's Law). "not (A or B)" is equivalent to "(not A) and (not B)". We check this via truth tables:

A	B	not (A or B)	not A	not B	(not A) and (not B)
false	false	true	true	true	true
false	true	false	true	false	false
true	false	false	false	true	false
true	true	false	false	false	true

We write, in short:

$$\text{not}\,(A \,\text{or}\, B) \quad \Leftrightarrow \quad (\text{not}\, A)\,\text{and}\,(\text{not}\, B).$$

There is a second De Morgan's Law:

$$\text{not}\,(A \,\text{and}\, B) \quad \Leftrightarrow \quad (\text{not}\, A)\,\text{or}\,(\text{not}\, B).$$

Other important logical rules which are proved via truth tables are the following:

(a) **Law of Commutativity:**

$$\begin{aligned} A \,\text{and}\, B &\Leftrightarrow B \,\text{and}\, A, \\ A \,\text{or}\, B &\Leftrightarrow B \,\text{or}\, A. \end{aligned}$$

(b) **Law of Associativity:**

$$\begin{aligned} A \,\text{and}\,(B \,\text{and}\, C) &\Leftrightarrow (A \,\text{and}\, B)\,\text{and}\, C, \\ A \,\text{or}\,(B \,\text{or}\, C) &\Leftrightarrow (A \,\text{or}\, B)\,\text{or}\, C. \end{aligned}$$

(c) **Law of Distributivity:**

$$\begin{aligned} A \,\text{and}\,(B \,\text{or}\, C) &\Leftrightarrow (A \,\text{and}\, B)\,\text{or}\,(A \,\text{and}\, C), \\ A \,\text{or}\,(B \,\text{and}\, C) &\Leftrightarrow (A \,\text{or}\, B)\,\text{and}\,(A \,\text{or}\, C). \end{aligned}$$

[3] Augustus De Morgan (1806-1871) was a British mathematician and logician who also introduced the term "induction", which is an important mathematical proof technique.

1.2 Sets

We think of a *set* as a well defined unordered collection of elements where each element is contained only once. For example, the set $\{a, 4, z, 4, a, 22, a\}$ has in fact only four elements and coincides with $\{a, z, 4, 22\}$. We use the following well-known notation in connection with two sets X, Y: $X \cup Y$ is the *union*[4], $X \cap Y$ is the *intersection*[5], $X \backslash Y$ is the set of all elements in X and not in Y, $x \in X$ ($x \notin X$) means that x is (not) an element in X, $X \subset Y$ means that X is a subset of Y[6]. The above logical rules have counterparts in set operations, where "and" and "or" are replaced by "$\cap$" and "$\cup$". We collect these rules in a mathematical structure, called a *proposition*. Propositions contain interesting mathematical facts and *need to be proved*. *Proposition*

Proposition 1.3. *Let X, Y, Z be sets. Then the following hold:*

(a) **Law of Commutativity:**

$$\begin{aligned} X \cup Y &= Y \cup X, \\ X \cap Y &= Y \cap X. \end{aligned}$$

(b) **Law of Associativity:**

$$\begin{aligned} X \cup (Y \cup Z) &= (X \cup Y) \cup Z, \\ X \cap (Y \cap Z) &= (X \cap Y) \cap Z. \end{aligned}$$

(c) **Law of Distributivity:**

$$\begin{aligned} X \cup (Y \cap Z) &= (X \cup Y) \cap (X \cup Z), \\ X \cap (Y \cup Z) &= (X \cap Y) \cup (X \cap Z). \end{aligned}$$

(d) **De Morgan's Law:** *Let $X, Y \subset Z$. Then we have:*

$$\begin{aligned} Z \backslash (X \cup Y) &= (Z \backslash X) \cap (Z \backslash Y),^{7} \qquad (1) \\ Z \backslash (X \cap Y) &= (Z \backslash X) \cup (Z \backslash Y). \end{aligned}$$

equality of sets

Note that this proposition presents a collection of set equalities. The *general method* to prove that two sets U and V are equal is the following: We prove that every element of U is also an element of V and

[4] $X \cup Y$ is the set of all elements which are contained in at least one of the two sets X, Y.

[5] $X \cap Y$ the set of all elements which are contained in both sets X, Y.

[6] X is a subset of Y if every element of X is also an element of Y.

[7] This formula carries a number, namely (1). We use this marking to refer to this identity later in the text.

conversely. Often, both directions can be carried out in one go via a sequence of equivalences:

$$x \in U \quad \Leftrightarrow \quad \cdots \quad \Leftrightarrow \quad x \in V.$$

Sometimes, it is better to prove each direction separately, i.e., first[8]

$$x \in U \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad x \in V,$$

and then

$$x \in V \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad x \in U.$$

This separation is necessary if each direction needs a different sequence of arguments.

Proof

A proof is another mathematical structure. Since proofs may be long (some of them may go over several pages), we mark the end of a proof by a special symbol, the square "$\square$" . Here is the first example of such a proof. We do not prove all statements of the proposition and restrict ourselves to one of the two De Morgan's Laws. The proofs of the other identities are carried out similarly. Generally, all equations of the proposition are proved by reduction to the corresponding logical rules.

end of proof symbol

Proof of (1). We have

$$\begin{aligned}
& x \in Z \backslash (X \cup Y) \\
\Leftrightarrow \quad & x \in Z \text{ and } x \notin X \cup Y \\
\Leftrightarrow \quad & x \in Z \text{ and not } (x \in X \text{ or } x \in Y) \\
\overset{(*)}{\Leftrightarrow} \quad & x \in Z \text{ and } (\text{not } x \in X) \text{ and } (\text{not } x \in Y) \\
\Leftrightarrow \quad & x \in Z \text{ and } x \notin X \text{ and } x \notin Y \\
\overset{(**)}{\Leftrightarrow} \quad & (x \in Z \text{ and } x \notin X) \text{ and } (x \in Z \text{ and } x \notin Y) \\
\Leftrightarrow \quad & x \in Z \backslash X \text{ and } x \in Z \backslash Y \\
\Leftrightarrow \quad & x \in (Z \backslash X) \cap (Z \backslash Y).
\end{aligned}$$

In $(*)$ we used De Morgan's Law for statements, and in $(**)$ we used the equivalence "$A \Leftrightarrow A \text{ and } A$", which can be easily checked via a truth table. We also made frequent use of commutativity and associativity (bracketing) without futher mentioning. $\square$

[8] At present, we have not yet properly introduced the symbol "$\Rightarrow$', but we assume that you have an intuitive understanding that "$A \Rightarrow B$" means that "if A is true then also B is true", but not necessarily conversely. A prime example is "$x = y \Rightarrow x^2 = y^2$" (since we have $(-3)^2 = 3^2$ but not $-3 = 3$). This symbol is called *implication*. Its precise logical meaning will be introduced later in Chapter 4.

CAUTION. Do not mix up connectives with set operations! The connectives " and", " or", "not" are only used to create new statements from given ones and must be understood as set operations. For example, if X, Y are sets, "X and not (Y)" **makes no sense!** The set operations "$\cup$", "$\cap$", "$\setminus$" are only used to manipulate sets and not statements. For example, if A, B are statements, the expression "$A \cup B$" **is nonsense!**

Venn diagram

Reader's Task. You might know from school how Venn Diagrams can be used to illustrate set relations. Check the De Morgan's Laws with the help of Venn diagrams. But be alert that Venn diagrams do not replace a proper proof!!

Here is another concrete example for an equality of sets.

Proposition 1.4. *Let S be set of all differences of two square numbers. Let O be the set of all odd integers and Z be the set of all integers divisible by 4. Then we have*

$$S = O \cup Z.$$

It turns out that both inclusions need different arguments and have to be proved separately.

Proof. We first show that $S \subset O \cup Z$: Let $x = a^2 - b^2 \in S$. Then we have $x = (a-b)(a+b)$ and we can have two situations. If $a-b$ is even, then so is $a+b$ (since $a+b = (a-b)+2b$), and x is therefore divisible by 4 and $x \in Z$. If $a-b$ is odd, then so is $a+b$, and x is therefore also odd and $x \in O$.

Finally, we show that $O \cup Z \subset S$: If $x \in O \cup Z$ is odd, then $x = 2k+1$ for some integer k and $x = (k+1)^2 - k^2$, i.e., $x \in S$. If $x \in O \cup Z$ is divisible by 4, then $x = 4k$ for some integer k and $x = (k+1)^2 - (k-1)^2$, i.e., $x \in S$. $\square$

Important concepts/typical problems in this chapter that you should try without looking anything up:

- State De Morgan's Law.
- Prove "$A \,\text{and}\, (B \,\text{or}\, C) \Leftrightarrow (A \,\text{and}\, B) \,\text{or}\, (A \,\text{and}\, C)$" via truth tables.
- The symmetric difference of two sets X, Y is defined as $X \Delta Y = (X \backslash Y) \cup (Y \backslash X)$. Draw a Venn Diagram to illustrate $X \Delta Y$. Give a formal proof that $X \Delta Z \subset (X \Delta Y) \cup (Y \Delta Z)$.

2 Numbers and Inequalities

number sets

Fundamentally important sets in mathematics are *number sets*. There is a hierarchy of number sets: the natural numbers, the integers and the rational numbers "appear in Nature" in the sense that measurements yield answers that belong to $\mathbb{Q}$; the others are "invented"[9].

- The *Natural Numbers* or positive integers $\mathbb{N} = \{1, 2, 3, \dots\}$. In some textbooks the number 0 is added to the natural numbers.
- The *Integers* $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.
- The *Rational Numbers* $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$.
- The *Real Numbers* $\mathbb{R}$ will not be defined here. A crucial property of the real numbers is that $\mathbb{R}$ is complete, i.e., there are no gaps on the number line. We come back to this later in more detail. The rational numbers lie dense on the number line, but there are certain positions on the number line which cannot be represented by a rational number (e.g., $\sqrt{2}$, the diagonal in the unit square).
- The *Complex Numbers* $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i^2 = -1\}$. Complex Numbers can be represented geometrically as points in the Argand plane[10] and they are also complete.

We have the proper inclusions

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

On the number line, the real numbers $\mathbb{R}$ are *ordered*, and the order relations $<, >, \leq, \geq$ satisfy the following properties:

(a) If $x < y$ and $y < z$, then $x < z$ (transitivity).

(b) If $x < y$ and $a \in \mathbb{R}$, then $x + a < y + a$.

(c) If $x < y$ and $c > 0$, then $cx < cy$.

(d) If $x < y$ and $c < 0$, then $cx > cy$.

(e) If $0 < x < y$, then $0 < 1/y < 1/x$.

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[9] Leopold Kronecker (1823-1891) was even more restrictive. His is quoted as having said "God made natural numbers; all else is the work of man."

[10] Jean-Robert Argand (1768-1822) was a French amateur mathematician who published the idea to represent complex numbers geometrically in the plane. The same geometric interpretation of complex numbers was also considered by Carl Friedrich Gauss (1777-1855) and Caspar Wessel (1745-1818).

CAUTION. Recall that the complex numbers $\mathbb{C}$ are points in the Argand plane. Complex numbers $\mathbb{C}$ are **NOT** ordered, so we **cannot use the concept of inequality**.

In mathematics, we often use known facts to derive new facts from them. The mathematical structure containing a relatively straightforward consequence of other facts is called a *corollary*. *Corollary*

Corollary 2.1. *Rules (a)-(e) imply the following fact: If $x < y$ and $a < b$, then $x + a < y + b$.*

Proof. Since $x < y$ and $a \in \mathbb{R}$, we conclude from (b) that

$$x + a < y + a. \tag{2}$$

Since $a < b$ and $y \in \mathbb{R}$, we conclude from (b) that[11]

$$y + a = a + y < b + y = y + b. \tag{3}$$

Using (2) and (3), we conclude from (a) that

$$x + a < y + b.$$

□

Here are examples to find the real solutions of inequalities.

Examples. (a) Find all $x \in \mathbb{R}$ such that $-3(4 - x) \leq 12$. We have

$$\begin{aligned} & -3(4-x) \leq 12 \\ \Leftrightarrow\ & 4 - x \geq -4 \quad \text{(Division by } -3\text{)} \\ \Leftrightarrow\ & 8 \geq x \quad \text{(Adding } 4 + x \text{ to both sides)} \end{aligned}$$

(b) Solve $\dfrac{x+2}{3} < \dfrac{5-2x}{4}$.[12] We have

$$\begin{aligned} & \frac{x+2}{3} < \frac{5-2x}{4} \\ \Leftrightarrow\ & 4x + 8 < 15 - 6x \quad \text{(Multiplication by 12)} \\ \Leftrightarrow\ & 10x < 7 \quad \text{(Adding } 6x - 8 \text{ to both sides)} \\ \Leftrightarrow\ & x < \frac{7}{10} \quad \text{(Division by 10)} \end{aligned}$$

[11] In fact, we also use here without proof the commutativity of the addition, in other words $y + a = a + y$.

[12] Here we ask, without mentioning it explicitly, for the set of all possible *real* solutions of this inequality.

(c) Solve $x^2 - 4x + 3 > 0$. We have

$$\begin{aligned}
& x^2 - 4x + 3 > 0 \\
\Leftrightarrow\ & (x-3)(x-1) > 0 \quad \text{(Factorising the polynomial)} \\
\Leftrightarrow\ & (x-3 > 0 \text{ and } x-1 > 0) \text{ or } (x-3 < 0 \text{ and } x_3 < 0) \\
& \qquad\qquad\qquad\qquad \text{(Both factors must have the same sign)} \\
\Leftrightarrow\ & (x-3 > 0) \,\text{or}\, (x-1 < 0) \\
\Leftrightarrow\ & x > 3 \,\text{or}\, x < 1
\end{aligned}$$

(d) Solve $\dfrac{3}{x-2} \le x$.[13] We have

$$\begin{aligned}
& \frac{3}{x-2} \le x \\
\Leftrightarrow\ & \frac{3}{x-2} - x \le 0 \quad \text{(Subtracting } x \text{ from both sides)} \\
\Leftrightarrow\ & \frac{3 - x(x-2)}{x-2} \le 0 \quad \text{(Bringing to a common denominator)} \\
\Leftrightarrow\ & \frac{x^2 - 2x - 3}{x-2} \ge 0 \quad \text{(Multiplication by } -1) \\
\Leftrightarrow\ & \frac{(x-3)(x+1)}{x-2} \ge 0 \quad \text{(Factorising the numerator).}
\end{aligned}$$

The last inequality is equivalent to

$$(x-2 > 0 \,\text{and}\, (x-3)(x+1) \ge 0) \,\text{or}\, (x-2 < 0 \,\text{and}\, (x-3)(x+1) \le 0).$$

This simplifies to

$$(x \ge 3) \,\text{or}\, (-1 \le x < 2).$$

Definition 2.2. *The* absolute value *of a complex number $z = x + iy$ is a non-negative real number, given by*

absolute value in $\mathbb{C}$

$$|z| = \sqrt{x^2 + y^2} \ge 0.$$

$|z|$ measures the Euclidean distance between the point z and the origin in the Argand plane.

Remark. Since $\mathbb{R} \subset \mathbb{C}$, the absolute value is also defined for real numbers $x \in \mathbb{R}$, and we have $|x| = \sqrt{x^2} \ge 0$.

[13] Observe here that we need to have $x \neq 2$. Otherwise the expression $3/(x-2)$ is not defined.

It makes sense to consider inequalities involving *absolute values* of real and complex numbers. The following rules are important:

(a) *Triangle Inequality:* $|z_1 + z_2| \leq |z_1| + |z_2|$ for all $z_1, z_2 \in \mathbb{C}$. *Triangle inequality*

(b) Variant of (a): $|\,|z_1| - |z_2|\,| \leq |z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$.

(c) $|z| = |(-z)|$ for all $z \in \mathbb{C}$.

(d) $x \in \mathbb{R}$ and $|x| \leq c$ is equivalent to $-c \leq x \leq c$. Here $-c \leq x \leq c$ means "$-c \leq x$ and $x \leq c$". (There is an analogous rule for $|x| < c$.)

The triangle inequality is one of the most fundamental inequalities in mathematics.

Reader's Task. (i) It is a useful exercise to derive the variant (b) from the Triangle Inequality (a). You may also use the rules (c) and (d).

(ii) Find an equivalent statement to "$x \in \mathbb{R}$ and $|x| > c$" for some positive $c \in \mathbb{R}$.

Here are examples how to find real solutions of inequalities with absolute values.

Examples. (a) Solve $|3x - 4| \leq 2$. This is equivalent to

$$\begin{aligned} & -2 \leq 3x - 4 \leq 2 \\ \Leftrightarrow\ & 2 \leq 3x \leq 6 \\ \Leftrightarrow\ & \frac{2}{3} \leq x \leq 2 \end{aligned}$$

(b) Solve $|2x + 3| > 5$. This is equivalent to

$$\begin{aligned} & 2x + 3 < -5 \quad \text{or} \quad 5 < 2x + 3 \\ \Leftrightarrow\ & 2x < -8 \quad \text{or} \quad 2 < 2x \\ \Leftrightarrow\ & x < -4 \quad \text{or} \quad 1 < x \end{aligned}$$

(c) Solve $|x + 2| \leq |2x - 1|$. This is equivalent to

$$\begin{aligned} & (x + 2)^2 \leq (2x - 1)^2 \\ \Leftrightarrow\ & x^2 + 4x + 4 \leq 4x^2 - 4x + 1 \\ \Leftrightarrow\ & 0 \leq 3x^2 - 8x - 3 \\ \Leftrightarrow\ & 0 \leq (3x + 1)(x - 3) \\ \Leftrightarrow\ & x \geq 3 \quad \text{or} \quad 3x \leq -1 \\ \Leftrightarrow\ & x \geq 3 \quad \text{or} \quad x \leq -\frac{1}{3} \end{aligned}$$

Important concepts/typical problems in this chapter that you should try without looking anything up:

- Find all $x \in \mathbb{R}$ such that $\left|\frac{x}{x-2}\right| \leq 5$.
- Write down the Triangle Inequality.
- Complex numbers are multiplied as follows:

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

  Prove for complex numbers $z_1, z_2 \in \mathbb{C}$ that $|z_1z_2| = |z_1| \cdot |z_2|$.

3 Basics about sequences and limits

real sequence

Definition 3.1. *A* real sequence *is a function from $\mathbb{N}$ to $\mathbb{R}$, i.e., it assigns to every natural number $n \in \mathbb{N}$ a real number, say, $x_n \in \mathbb{R}$. We denote such a sequence by $(x_n)_{n\in\mathbb{N}}$. We sometimes write the elements of the sequence consecutively as*

$$x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \ldots$$

Examples. 1. If $x_n = 6$ for all $n \in \mathbb{N}$, then we have the constant sequence

$$x_1 = 6, \quad x_2 = 6, \quad x_3 = 6, \quad x_4 = 6, \quad x_5 = 6, \quad \ldots$$

2. If $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then we have the sequence

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{4}, \quad a_5 = \frac{1}{5}, \quad \ldots$$

   Here, we chose a different name for the sequence. This sequence is denoted concisely by $(a_n)_{n\in\mathbb{N}}$.

3. If $b_n = (-1)^n$ for all $n \in \mathbb{N}$, then we have the sequence

$$b_1 = -1, \quad b_2 = 1, \quad b_3 = -1, \quad b_4 = 1, \quad b_5 = -1, \quad \ldots$$

4. Finally, let $y_n = n^2$, i.e.,

$$y_1 = 1, \quad y_2 = 4, \quad y_3 = 9, \quad y_4 = 16, \quad \ldots$$

index

Remark. We call the parameter $n \in \mathbb{N}$ of a element x_n the *index* of this element. In Example 2 above, the element with the small index $n = 2$ is $a_2 = \frac{1}{2}$, and the element with large index $n = 1000$ is $a_{1000} = \frac{1}{1000}$. We see that the elements a_n of the sequence (a_n) become smaller and smaller as their index n increases, and they approach the limit value 0 as n goes to infinity.

Now we introduce the crucial notion of *limit* of a real sequence. We call a sequence $(x_n)_{n\in\mathbb{N}}$ convergent, if its elements come closer and closer to a real number $x^* \in \mathbb{R}$ as their index increases. We characterise convergence by measuring the distance $|x_n - x|$ between x_n and x, which should become arbitrarily small as the index $n \in \mathbb{N}$ increases. Now, we make this concept precise. Small positive real numbers are often denoted by the Greek letters ϵ and δ, pronounced "epsilon" and "delta".

Definition 3.2. *A real sequence* $(x_n)_{n\in\mathbb{N}}$ *has the* limit $x^* \in \mathbb{R}$ *if, for every* $\epsilon > 0$*, there exists an index* $N \in \mathbb{N}$ *such that* *limit of sequence*

$$|x_n - x^*| < \epsilon \quad \textit{for all } n \geq N.$$

We write "$\lim_{n\to\infty} x_n = x^*$*" or also "*$x_n \to x^*$ *as* $n \to \infty$*". A sequence which has a limit is called a* convergent *sequence. If a sequence is not convergent, it is called* divergent*.* *convergent/divergent sequence*

Remark. All $y \in \mathbb{R}$ satisfying $|y - x^*| < \epsilon$ define precisely the small real open interval

$$B_\epsilon(x^*) := (x^* - \epsilon, x^* + \epsilon) \subset \mathbb{R}.$$

We call this interval *the* ϵ*-interval around* x^*. Convergence of x_n to a limit x^* means therefore that, given $\epsilon > 0$, from some start index N onwards, all elements of the sequence lie in the interval $B_\epsilon(x^*)$. Of course, as we choose $\epsilon > 0$ smaller, the start index N may need to be larger (we say that the start index N depends on the choice of ϵ). But, whatever number $\epsilon > 0$ is chosen, only finitely many elements x_n of the sequence can lie outside the ϵ-interval $B_\epsilon(x^*)$ around x^*.

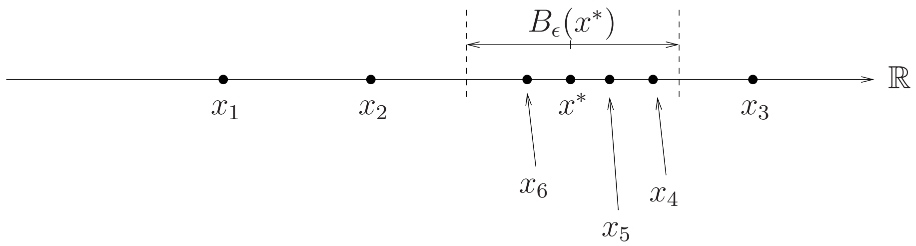

Figure 1: A sequence (x_n) entering $B_\epsilon(x^*)$ from $N = 4$ onwards

Examples. Coming back to our earlier examples of sequences, we have the following limiting behaviour.

1. Here we have $x_n \to 6$ since, for every $\epsilon > 0$ and every $n \in \mathbb{N}$, we have $|x_n - 6| < \epsilon$. In other words, we can choose $N = 1$ for all choices of $\epsilon > 0$. Therefore, the sequence (x_n) is convergent with limit 6.

2. Here we have $a_n \to 0$. This can be seen as follows: For a given $\epsilon > 0$ we choose a natural number $N > \frac{1}{\epsilon}$. This implies that $\frac{1}{N} < \epsilon$ and, therefore,

$$|a_n - 0| = |a_n| = \frac{1}{n} \leq \frac{1}{N} < \epsilon \quad \text{for all } n \geq N.$$

Therefore, the sequence (a_n) is convergent with limit 0.

3. Here, the elements b_n with even indices $n = 2k$ are 1 and the elements with odd indices $n = 2k + 1$ are -1. If we had $b_n \to b^*$ for some real number $b^* \in \mathbb{R}$ then, for any choice of $\epsilon > 0$, only finitely many b_n could lie outside the ϵ-interval $I = (b^* - \epsilon, b^* + \epsilon)$ around b^*. But if we choose $\epsilon < 1$, I cannot contain both numbers -1 and 1. Since b_n assumes both values -1 and 1 infinitely many times, there are always infinitely many elements of the sequence outside the ϵ-interval around b^*. This shows that (b_n) is not convergent.

4. Note here that the elements of the sequence (y_n) become arbitrarily large with increasing indices. If we had $y_n \to y^* \in \mathbb{R}$, then all but finitely many elements of y_n would have to lie in the bounded open interval (y^*-1, y^*+1) (by taking $\epsilon = 1$), which would violate the fact that the elements y_n become arbitrarily large. Therefore, (y_n) is not convergent. We could say that "$y_n \to \infty$ as $n \to \infty$", but ∞ is not a real number.

CAUTION. If $x_n \to x^*$, x^* has to be a fixed real number, **not dependent** on n. In other words, a statement of the form "$\frac{1}{n+1} \to \frac{1}{n}$ as $n \to \infty$" does not make sense!!

While propositions contain interesting mathematical facts which can be useful in certain instances, *theorems* are mathematical structures stating facts of fundamental importance. Of course, theorems need to be proved as well. The following theorem establishes *uniqueness of the limit* for any convergent sequence. *Theorem*

Theorem 3.3 (Uniqueness of the limit)**.** *Every convergent sequence $(z_n)_{n\in\mathbb{N}}$ has precisely one limit.*

The idea for the proof of uniqueness of the limit is already contained in the arguments showing that (b_n) in Example 3 above is not convergent.

Proof. If a sequence (z_n) had two different limits $z^* \neq z'$, then we could choose $\epsilon > 0$ small enough such that the ϵ-intervals around z^* and z' do not intersect (choose, for example, $\epsilon < \frac{|z^*-z'|}{2}$). Then we would have infinitely many elements z_n lying in $B_\epsilon(z^*) = (z^* - \epsilon, z^* + \epsilon)$ since z^* is a limit of (z_n). These elements would necessarily lie outside $B_\epsilon(z') = (z' - \epsilon, z' + \epsilon)$. But this contradicts the assumption that z' is also a limit of (z_n). $\square$

Theorem 3.4. *A sequence* (x_n) *is called a* bounded sequence*, if there exists* $C > 0$ *such that* $|x_n| \leq C$ *for all* $n \in \mathbb{N}$*. A sequence which is not bounded is called an* unbounded sequence*. Then we have the following fact: Every convergent sequence* (x_n) *is a bounded sequence.*

(un)bounded sequence

Proof. Exercise. □

Note that the first three sequences in our example above are bounded sequences, while $y_n = n^2$ in Example 4 is unbounded. Theorem 3.4 tells us that (y_n) cannot be convergent.
Here is another fundamental result, called the Squeezing Theorem. It allows to derive for many sequences that they are convergent with limit equal to zero.

Theorem 3.5 (Squeezing Theorem)**.** *If* $|x_n| \leq y_n$ *for all* $n \in \mathbb{N}$ *and* $y_n \to 0$ *as* $n \to \infty$*, then also* $x_n \to 0$ *as* $n \to \infty$*.*

Squeezing Theorem

Proof. Let $\epsilon > 0$ be given. Since $y_n \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $y_n = |y_n - 0| < \epsilon$ for all $n \geq N$. This implies that

$$|x_n - 0| = |x_n| \leq y_n < \epsilon \quad \text{for all } n \geq N.$$

This shows that $x_n \to 0$ as $n \to \infty$, as well. □

Theorem 3.6. *Let* $x_n \to 0$ *as* $n \to \infty$*. Let* (y_n) *be a bounded sequence. Then we have* $x_n y_n \to 0$ *as* $n \to \infty$*.*

Reader's Task. It is a good exercise for you to prove Theorem 3.6 by a slight modification of the proof of the Squeezing Theorem.

The next theorem is called the "Calculus of Limits Theorem", abbreviated as COLT. This theorem is crucial in the explicit calculation of limits by deducing the limits of more complicated sequences via limits of simpler ones.

Theorem 3.7 (COLT)**.** *Let* $x^* = \lim_{n\to\infty} x_n$ *and* $y^* = \lim_{n\to\infty} y_n$*. Let* $a, b \in \mathbb{R}$ *be constants. Then we have*

COLT

(i) $ax_n + by_n \to ax^* + by^*$ *as* $n \to \infty$*.*

(ii) $x_n y_n \to x^* y^*$ *as* $n \to \infty$*.*

(iii) $\frac{x_n}{y_n} \to \frac{x^*}{y^*}$ *as* $n \to \infty$*, provided* $y^* \neq 0$ *and all elements of* (y_n) *are non-zero.*

Proof. We provide a sample proof for statement (ii), the other proofs are left to the reader. A crucial step is to estimate the difference $x_n y_n - x^* y^*$ via the triangle inequality:

$$\begin{aligned} |x_n y_n - x^* y^*| &\leq |x_n y_n - x_n y^*| + |x_n y^* - x^* y^*| \\ &\leq |x_n| \cdot |y_n - y^*| + |y^*| \cdot |x_n - x^*|. \end{aligned}$$

Since (x_n) is convergent, we have $C > 0$ such that $|x_n| \leq C$ for all $n \in \mathbb{N}$, by Theorem 3.4. By increasing $C > 0$, if necessary, we can also assume that $|y^*| \leq C$ and, therefore,

$$|x_n y_n - x^* y^*| \leq C(|y_n - y^*| + |x_n - x^*|).$$

Recall that $C > 0$ is a fixed positive real number. Now, choose N such that $|x_n - x^*| < \frac{\epsilon}{2C}$ for all $n \geq N$. Analogously, choose N' such that $|y_n - y^*| < \frac{\epsilon}{2C}$ for all $n \geq N'$. Consequently, we have for all $n \geq \max\{N, N'\}$:

$$|x_n y_n - x^* y^*| \leq C(|y_n - y^*| + |x_n - x^*|) < C(\frac{\epsilon}{2C} + \frac{\epsilon}{2C}) = \epsilon.$$

This shows that we have $x_n y_n \to x^* y^*$ as $n \to \infty$. $\square$

Using COLT and the fact $\frac{1}{n} \to 0$ as $n \to \infty$, we can conclude for every $k \in \mathbb{N}$ that $\frac{1}{n^k} \to 0$ as $n \to \infty$. Using the Squeezing Theorem, we can also conclude that $\frac{1}{n^\alpha} \to 0$ as $n \to \infty$, for all $\alpha \geq 1$. In fact, this even holds for all $\alpha > 0$.
The Euler number e is named after Leonhard Euler[14] and is defined as the limit of a sequence. At present, we will not give a proof that this sequence has a limit. We return to the convergence proof later. The sequence can be motivated as a limit of growth processes with shorter and shorter increment times.

Definition 3.8. *The* Euler number

$$e = 2.718281828459045\ldots$$

is defined as the limit

Euler number

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

[14]The famous Swiss Mathematician LEONHARD EULER(1707-1783) is one of the most productive and greatest mathematician to have ever lived. His picture is on an earlier, out of date version of the Swiss 10-francs banknote.

Facts. In determining limits, you can use the following very important and useful facts without proof:

i) **Exponentials beat powers:** For any $c > 0$ and $k \in \mathbb{N}$,

$$\frac{n^k}{e^{cn}} \to 0 \quad \text{as } n \to \infty.$$

(ii) **Powers beat logarithms:** For any $c > 0$ and $\alpha > 0$,

$$\frac{\log(cn)}{n^\alpha} \to 0 \quad \text{as } n \to \infty.$$

Here log means logarithm to the base e (in other words, log means the same as ln).

In determining limits, we also often use the following fact about continuous functions:

Theorem 3.9. *If $x_n \to x^*$ as $n \to \infty$ and if $f(x)$ is continuous at x^*, then we have $f(x_n) \to f(x^*)$ as $n \to \infty$.*

We do not prove this theorem here but mention that this theorem could also be used as a *definition of continuity.* We will discuss continuity in detail in Chapter 7. Here we only list some functions which are known to be continuous: polynomials $p(x)$; ratios $p(x)/q(x)$ of polynomials away from the zeros of $q(x)$; $\sin(x), \cos(x), \exp(x) = e^x$ and $\sqrt{x}$ for $x \geq 0$ and $\log(x)$ for $x > 0$. So, for example, if we know that $x_n \to x^*$ as $n \to \infty$, we can deduce that $\exp(x_n) \to \exp(x^*)$ as $n \to \infty$.

Now we consider explicit examples of convergent sequences and derive their limits.

Examples. 1. Compute $\lim_{n\to\infty} \frac{n\sqrt{3n^2-2}}{\sqrt{1+8n^4}}$. We have, using COLT and continuity of the square root function,

$$\frac{n\sqrt{3n^2-2}}{\sqrt{1+8n^4}} = \frac{\sqrt{3-2/n^2}}{\sqrt{1/n^4+8}} \to \frac{\sqrt{3-0}}{\sqrt{0+8}} = \sqrt{\frac{3}{8}} \quad \text{as } n \to \infty.$$

Be aware that the limit appears on the right hand side of the "$\to$" symbol and must not depend any more on the index n.

2. Compute $\lim_{n\to\infty} \dfrac{n+\sin(n)}{\sqrt{4n^2+1}}$. We obtain, using COLT, continuity of the square root function, and Theorem 3.6 (note that the sequence $\sin(n)$ is bounded),

$$\frac{n+\sin(n)}{\sqrt{4n^2+1}} = \frac{1+n^{-1}\sin(n)}{\sqrt{4+1/n}} \to \frac{1+0}{\sqrt{4+0}} = \frac{1}{2} \quad \text{as } n\to\infty.$$

3. Compute $\lim_{n\to\infty} \dfrac{n^2+n^3e^{-n}}{\left(\log(2^n)+\log(n^8)\right)^2}$. Here we use COLT and the facts that "exponentials beat powers" and that "powers beat logarithms" to obtain

$$\frac{n^2+n^3e^{-n}}{\left(\log(2^n)+\log(n^8)\right)^2} = \frac{1+ne^{-n}}{\frac{1}{n^2}\left(n\log(2)+8\log(n)\right)^2} =$$
$$\frac{1+ne^{-n}}{\left(\log(2)+\frac{8\log(n)}{n}\right)^2} \to \frac{1+0}{(\log(2)+0)^2} = (\log(2))^{-2} \quad \text{as } n\to\infty.$$

4. Compute $\lim_{n\to\infty} \dfrac{n^2 n!}{(n+2)!}$. Here we obtain

$$\frac{n^2 n!}{(n+2)!} = \frac{n^2}{(n+1)(n+2)} =$$
$$\frac{1}{(1+1/n)(1+2/n)} \to \frac{1}{1\cdot 1} = 1 \quad \text{as } n\to\infty.$$

5. Compute $\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n})$. While $\sqrt{n}$ becomes larger and larger as $n\to\infty$, the difference $\sqrt{n+1}-\sqrt{n}$ goes to 0 as $n\to\infty$ (the latter is not obvious). So it is not clear how the product behaves as $n\to\infty$. We need a trick to find the limit. The trick is to write

$$(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n}) = (n+1)-n = 1,$$

which leads to

$$\sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$
$$= \frac{1}{\sqrt{1+1/n}+1} \to \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \quad \text{as } n\to\infty.$$

6. Compute $\lim_{n\to\infty} x_n$ with $x_n = n^{5/n}$. We obviously have $x_n > 0$ and we can consider the sequence $y_n = \log(x_n) = \frac{5\log(n)}{n}$ instead. Since "powers beat logarithms", we have $\lim_{n\to\infty} y_n = 0$ and, therefore,
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \exp(y_n) = \exp(0) = 1.$$

7. Compute $\lim\limits_{n\to\infty} \dfrac{\log(3^n + n^3)}{n}$. We have
$$\frac{\log(3^n + n^3)}{n} = \frac{\log(3^n(1 + 3^{-n}n^3))}{n} =$$
$$\frac{n\log(3) + \log(1 + 3^{-n}n^3)}{n} = \log(3) + \frac{1}{n}\log(1 + 3^{-n}n^3) \to$$
$$\log(3) + 0 \cdot \log(1 + 0) = \log(3) \quad \text{as } n \to \infty.$$

Corollary 3.10. *Let $(a_n), (b_n), (c_n)$ be three real sequences with $a_n \leq b_n \leq c_n$ and let (a_n) and (c_n) be convergent with the same limit, in other words, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = x^*$. Then (b_n) is also convergent and we have $\lim_{n\to\infty} b_n = x^*$.*

Proof. From the assumptions we conclude that
$$|b_n - a_n| \leq c_n - a_n$$
with $c_n - a_n \to 0$ as $n \to \infty$. The Squeezing Theorem then implies that also $b_n - a_n \to 0$ as $n \to \infty$. Therefore, by COLT,
$$b_n = (b_n - a_n) + a_n \to 0 + x^* = x^* \quad \text{as } n \to \infty.$$
□

The next fact is very important and useful.

Theorem 3.11. *Let $|c| < 1$. Then the sequence $(c^n)_{n\in\mathbb{N}}$ is convergent and we have*
$$\lim_{n\to\infty} c^n = 0.$$

Proof. We first consider the special case $0 \leq c < 1$. Let $\epsilon > 0$. We need to find $N \in \mathbb{N}$ such that
$$|c^n - 0| = c^n < \epsilon \quad \text{for all } n \geq N. \tag{4}$$

$c^n < \epsilon$ is equivalent to $n\log(c) < \log(\epsilon)$ and, since $\log(c) < 0$, this is equivalent to

$$n > \frac{\log(\epsilon)}{\log(c)}.$$

Now, choose $N \in \mathbb{N}$ such that $N > \frac{\log(\epsilon)}{\log(c)}$. Then (4) is satisfied and we have $c^n \to 0$ as $n \to \infty$.

Now we consider the general case, in other words, we have a constant $c \in \mathbb{R}$ with $|c| < 1$ and $x_n = c^n$. Our first part of the proof showed that $|x_n| = |c|^n \to 0$ as $n \to \infty$. Finally, we use the fact that $|x_n| \to 0$ implies $x_n \to 0$, which is a very easy exercise. $\square$

Remark. It can also be shown that for $|c| > 1$ the sequence $(c^n)_{n\in\mathbb{N}}$ is divergent. There are two remaining cases: If $c = 1$, then (c^n) is constant $= 1$ and, therefore, convergent with limit 1. If $c = -1$, we have a divergent sequence with values alternating between -1 and 1.

CAUTION. Observe that if (x_n) is a convergent sequence of positive real numbers, the limit **does not need to be positive**. The easiest example is $x_n = 1/n$. But the limit is certainly non-negative. This is the statement of the next proposition.

Proposition 3.12. *Let (x_n) be a convergent real sequence with $x_n \geq 0$. Then we have*

$$\lim_{n\to\infty} x_n \geq 0.$$

Proof. If $x_n \to x^* < 0$ then, choosing $\epsilon = \frac{|x^*|}{2}$, we must have $x_n \in B_\epsilon(x^*) = (x^* - \epsilon, x^* + \epsilon)$ from some index $N \in \mathbb{N}$ onwards. Since $x^* + \epsilon = x^*/2 < 0$, we conclude that these elements satisfy $x_n < x^*/2 < 0$, in other words, these elements of the sequence must be negative. $\square$

Example. We now prove: If $c \in \mathbb{R}$ is a fixed number, then

$$\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c.$$

This shows, in the case $c = 1$, that the sequence $(1 + 1/n)^n$ is convergent, which was not proved in the definition of e (see Definition 3.8). We use the following well known formula for the logarithm

$$\log(t) = \int_1^t \frac{dx}{x}.$$

We start with the following fact: If $x \in [1, 1 + c/n]$ then

$$\frac{n}{c+n} \leq \frac{1}{x} \leq 1.$$

Integrating from 1 to $t = 1 + c/n$, we obtain

$$\frac{c}{n}\frac{n}{c+n} = \int_1^{1+c/n} \frac{n}{c+n} dx \leq \int_1^{1+c/n} \frac{dx}{x} \leq \int_1^{1+c/n} dx = \frac{c}{n}.$$

Multiplying by n we conclude

$$\frac{cn}{c+n} \leq \log\left(\left(1 + \frac{c}{n}\right)^n\right) \leq c.$$

Since $cn/(c+n) \to c$ as $n \to \infty$, we obtain by applying Corollary 3.10

$$\lim_{n\to\infty} \log\left(\left(1 + \frac{c}{n}\right)^n\right) = c.$$

Now using the continuity of the exponential function, we end up with

$$\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c,$$

finishing the proof. □

CAUTION. Note that the index of a sequence is not always denoted by n. For example, $(u_j = j/(j^2+1))_{j\in\mathbb{N}}$ is a perfectly fine sequence and the index here is j. In this example, it makes **no sense** to write $\lim_{n\to\infty} u_j = 0$, since n does not appear as index of the sequence. Here the correct notation is "$\lim_{j\to\infty} u_j = 0$" or "$u_j \to 0$ as $j \to \infty$".

The following result states that the convergence and the limit of a sequence is not affected if we remove some elements at its beginning by an index shift:

Proposition 3.13. *Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence and $K \in \mathbb{N}$. Let $(y_n)_{n\in\mathbb{N}}$ be the sequence defined by $y_n = x_{n+K}$, in other words,*

$$y_1 = x_{K+1}, y_2 = x_{K+2}, y_3 = x_{K+3}, y_4 = x_{K+4}, \ldots$$

Then (y_n) is also convergent and we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n.$$

We leave the straightforward proof of this result to the reader.

Remark. Proposition 3.13 is a manifestation of the general fact that the limit of a sequence is independent of the concrete values of finitely many elements at the beginning. In fact, we may also add some elements at the beginning. For example, if $(x_n)_{n\in\mathbb{N}}$ is convergent then so is the sequence $(y_n)_{n\in\mathbb{N}}$ with the same limit

$$y_n = \begin{cases} 2 & \text{if } n = 1 \\ -1100 & \text{if } n = 2 \\ x_{n-2} & \text{if } n \geq 3. \end{cases}$$

So far, we only discussed real sequences and their limits. It turns out that the same concepts can be introduced for complex sequences. In the case of a complex sequence $(z_n)_{n\in\mathbb{N}}$, we have $z_n \in \mathbb{C}$. *complex sequence*

Remark. Also the definition of limit is not restricted to real sequences. We say that $\lim_{n\to\infty} z_n = z^* \in \mathbb{C}$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ with *limit of complex sequence*

$$|z_n - z^*| < \epsilon \quad \text{for all } n \geq N. \tag{5}$$

The set of complex numbers $z \in C$ satisfying $|z - z^*| < \epsilon$ is an open Euclidean ball of radius $\epsilon > 0$ around z^*, which we denote by $B_\epsilon(z^*)$. We call this set the *open ϵ-ball around $z^* \in \mathbb{C}$*. Therefore, property (5) can be reformulated equivalently, that z_n lies in $B_\epsilon(z^*)$ for all indices $n \geq N$ (see Figure 2 for illustration). *open ϵ-ball*

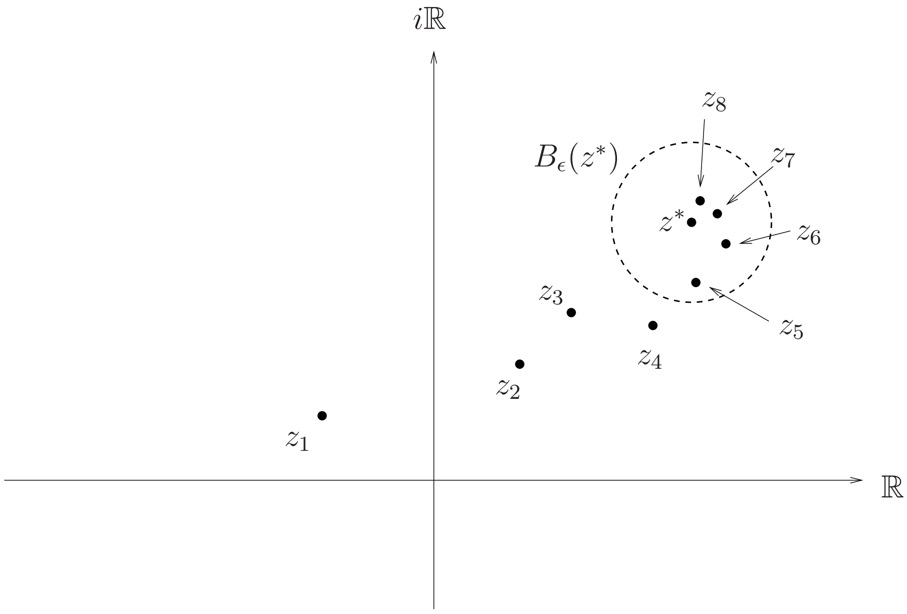

Figure 2: A sequence (z_n) entering $B_\epsilon(z^*)$ from $N = 5$ onwards

Important concepts/typical problems in this chapter that you should try without looking anything up:

- Give the definition that a sequence $(x_n)_{n\in\mathbb{N}}$ has the limit x^*.
- Give an example of a sequence which is bounded but not convergent.
- What is the convergence/divergence behaviour of $(c^n)_{n\in\mathbb{N}}$ for $c \in \mathbb{R}$?
- Formulate COLT and the Squeezing Theorem.
- Calculate $\lim\limits_{k\to\infty} \dfrac{(k^3+\log(2k))(\cos(5k)-3k^2)}{(2k+1)^5}$.
- Calculate the limit of $\sqrt{n+\log(5n^3)}(\sqrt{n+2}-\sqrt{n})$ as $n \to \infty$.

4 More logic: Quantifiers, negation and proof techniques

4.1 Quantifiers

Recall that an expression $A(x)$ like "$x > 1$" is a *conditional statement*. By choosing real values for the variable x, we obtain true or false statements. So $A(2)$ is a true statement while $A(-5)$ is false. Conditional statements can be combined with *quantifiers* to obtain new statements. The two quantifiers which are used are the symbols "$\forall$" and "$\exists$":

$\forall$, $\exists$ *quantifiers*

- The symbol "$\forall$" abbreviates the phrase "For all".
- The symbol "$\exists$" abbreviates the phrase "There exists".

Examples. (i) "$\forall x \in \mathbb{R}$: $x^2 \geq 0$" This is an abbreviation for "for all $x \in \mathbb{R}$ we have $x^2 \geq 0$", which is a true statement.

(ii) "$\forall k \in \mathbb{N}$: $2k+1$ is odd" This is an abbreviation for "for all $k \in \mathbb{N}$ we have that $2k+1$ is odd", which is again a true statement.

(iii) "$\exists x \in \mathbb{R}$: $x^2 = -1$" This is an abbreviation for "there exists $x \in \mathbb{R}$ such that $x^2 = -1$", which is a false statement.

(iv) "$\forall \epsilon > 0 \; \exists n \in \mathbb{N} : \epsilon > 1/n$" This is an abbreviation for "for all $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\epsilon > 1/n$", which is a true statement.

(v) "$\forall x \in \mathbb{Z}$: if x^2 even then x even" This is an abbreviation for "for all $x \in \mathbb{Z}$ we have: if x^2 is even then x is even", which is a true statement.

Using quantifiers is a very efficient way to make mathematical statements. In fact, the fundamental statement from Chapter 3 that "the sequence (x_n) is convergent with limit x^*" can be written in symbols as follows:

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : |x_n - x^*| < \epsilon. \tag{6}$$

Reader's Task. Translate the following statements into symbols.

(a) The sequence $(x_n)_{n \in \mathbb{N}}$ is bounded.

(b) The sequence $(a_n)_{n \in \mathbb{N}}$ is unbounded.

(c) The rational numbers $\mathbb{Q}$ lie dense on the number line [15].

CAUTION. The meaning of a statement can change dramatically if the **order of the quantifiers is permuted**. While

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : y > x$$

means that "for every real number x there exists another real number y which is larger than x", which is a true statement, the sequence

$$\exists y \in \mathbb{R}\ \forall x \in \mathbb{R} : y > x$$

means that "there exists a real number $y \in R$ which is larger than every other real number $x \in \mathbb{R}$", which is obviously a false statement. So be always careful with the order in which you place your quantifiers.

However, sometimes, we may place $\forall$-quantifiers after conditional statements instead of before them without changing the meaning. So the following symbol expression is equivalent to (6):

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} : |x_n - x^*| < \epsilon \quad \forall n \geq N.$$

This means, in words: "For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that we have $|x_n - x^*| < \epsilon$ for all $n \geq N$."

4.2 Negation

The use of quantifiers allows us also to negate certain statements correctly, even if they are involved. Let us start with a simple example and consider the statement A_1, given by

$$\forall x \in X : B(x),$$

where $B(x)$ is a conditional statement. This statement is not true if and only if "there exists $x \in X$ such that not $B(x)$". In other words, A_1 is false if and only if

$$\exists x \in X : \text{not}\, B(x) \tag{7}$$

is true. Therefore, (7) represents the negation "not A_1". Similarly, if A_2 is the statement

$$\exists x \in X : C(x),$$

[15] In other words, for every real number $x \in \mathbb{R}$ we find rational numbers which are arbitrarily close to x.

where $C(x)$ is another conditional statement, then not A_2 means that there is no $x \in X$ with $C(x)$ or, in other words, we have not $C(x)$ for all $x \in X$. Therefore, the negation of A_2 is

$$\forall x \in X : \text{not}\, C(x).$$

Rule for negation. *Generally, we have the following formal rule to negate a logic statement, consisting of a list of quantifiers with a concluding conditional statement of the form* *rule for negation*

$$Q_1 x_1 Q_2 x_2 \dots Q_n x_n : P(x_1, \dots, x_n),$$

where $Q_i \in \{\forall, \exists\}$.

- *Change every* $\forall$*-quantifier to an* $\exists$*-quantifier and every* $\exists$*-quantifier to a* $\forall$*-quantifier.*
- *Replace "*$P(x_1, \dots, x_n)$*" by "*$\text{not}\, P(x_1, \dots, x_n)$*".*

Examples.

1. The statement

$$\forall n \in \mathbb{N} : n \text{ is a prime number}$$

is obviously false. Its negation is

$$\exists n \in \mathbb{N} : n \text{ is not a prime number},$$

which is obviously true (choose, for example $n = 6$).

2. Let us recall the statement, denoted by A, that "x^* is the limit of the sequence (x_n)". This was formulated in (6). Applying formally the negation rule, we obtain

$$\exists \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N : |x_n - x^*| \geq \epsilon,$$

to express that "x^* is not the limit of the sequence (x_n)".

In other words, "not A" means that there exists a $\epsilon > 0$ such that for every start index N there is an element x_n with an index $n \geq N$ outside the open ϵ-interval $B_\epsilon(x^*)$.

A shorter formulation of "not A" is that "there exists $\epsilon > 0$ such that there are elements x_n with arbitrarily high indices $n \in \mathbb{N}$ outside $B_\epsilon(x^*)$".

An even shorter formulation of "not A" is that "there exists $B_\epsilon(x^*)$ with infinitely many elements x_n outside $B_\epsilon(x^*)$".

We see in this example that is often useful to combine strictly formal logical rules with the concrete understanding of the meaning of a statement to end up with a short and efficient formulation.

4.3 Proof techniques

An important connective between statements besides ”and” and ”or” is ”if ... then”.

Definition 4.1. *Let A and B be two statements. ”If A then B” is a statement which is false only if A is true and B is false. The concise mathematical symbol for ”If A then B” is ”$A \Rightarrow B$”, and we call this connective also an* implication.

if...then

”$\Rightarrow$”

implication

Let us quickly justify this definition. Since a false statement cannot follow from a true one, we need to have that ”If *true* then *false*” is false. Since true statements can follow from true statements and also false statements can follow from false statements, ”If *true* then *true*” is true as well as ”If *false* then *false*”. What about ”If *false* then *true*”? In fact, a true statement can follow from a false one; here is an example: The equation ”$0 = 1$” is obviously false, but multiplying both sides with 0 yields the true statement ”$0 = 0$”. So a true statement can follow from a false one and we therefore define ”If *false* then *true*” as true.
The following is an important fact which is often used in connection with a certain proof technique, called ”indirect proof”.

negation of ”if...then”

Theorem 4.2. *The negation of ”If A then B” is ”A and (not B)”.*

Proof. The proof is given via truth tables:

A	B	if A then B	A	not B	A and (not B)
false	false	true	false	true	false
false	true	true	false	false	false
true	false	false	true	true	true
true	true	true	true	false	false

□

CAUTION. Here it is important to avoid a dangerous misconception. The negation of ”If A then B” is **NOT** ”If A then (not B)”.

Reader’s Task. It is a good exercise to write down the truth tables for both ”If A then B” and ”If A then (not B)” to convince yourself that they are not negations of each other.

Example. Let us give an example for an **Indirect Proof**: Assume we want to prove for integers n that

indirect proof

$$\text{"If } n^2 \text{ is even then } n \text{ is even."} \tag{8}$$

In an indirect proof, we start with the negation of the statement and derive from it a logical contradiction. This implies that the negation must be a false statement and, therefore, the original statement must be true.[16] Let us carry out this method in the above example (8). Recall for integers that beging "not even" means being "odd". The negation of (8) is

$$\text{"}n^2 \text{ is even and } n \text{ is odd."}$$

But if n is odd, we can write $n = 2k + 1$ for some $k \in \mathbb{Z}$ and derive

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1,$$

in contradiction to the assumption that n^2 is even. This finishes the proof.

Remark. In fact, we already used the indirect proof technique in Chapter 3. Theorem 3.3 and Proposition 3.12 were proved indirectly. You may go back and convince yourself about this fact.

Finally, we mention another important fact which is often used in connection with another certain proof technique, called "contrapositive proof".

Theorem 4.3. *Let A, B be two statements. Then the following two statements are equivalent: "If A then B" and "If* (not B) *then* (not A)*". The second statement is called the* contrapositive *of the first one.*

contrapositive statement

We omit the proof which is again given by a truth table. Be aware that the statements A and B **must be** interchanged when switching to the contrapositive statement.

Example. Let us give an example of an **Contrapositive Proof**: We return to our statement (8) and want to prove for integers n that

contrapositive proof technique

$$\text{"If } n^2 \text{ is even then } n \text{ is even".}$$

[16]The method is also called *reductio ad absurdum.* The British mathematician G. H. Hardy (1877-1947) described the indirect proof as "one of a mathematician's finest weapons", saying "It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game." (see Wikipedia)

Instead we prove the equivalent contrapositive statement

$$\text{"If } n \text{ is odd then } n^2 \text{ is odd",}$$

which is much easier to prove: We set $n = 2k + 1$ for some $k \in \mathbb{Z}$ and derive

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1,$$

in other words n^2 is odd.

Remark. Above, you learnt two important proof techniques which we will use over and over again in the following chapters. Revisiting both examples, we observe that the central argument of both proofs is "if n is odd then so is n^2". But the logic of both proofs is very different:

- In the *indirect proof* we start with the negation of the original statement (which has the opposite truth value as the original statement) and derive a contradiction.
- In the *contrapositive proof* we start with the contrapositive statement (which has the same truth value as the original statement) and prove it directly.

It is important that you fully understand the logical difference between the two proof techniques.

Important concepts/typical problems in this chapter that you should try without looking anything up:

- Write formally that the sequence $(x_n)_{n\in\mathbb{N}}$ has limit x^*.
- Write in symbols that a function $f : \mathbb{R} \to \mathbb{R}$ is surjective.
- The fact that a function $f : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing can be written in symbols by

  $$\forall x, y \in \mathbb{R} : \text{if } x < y \text{ then } f(x) < f(y)$$

  Derive the formal negation of this statement.
- Find two concrete mathematical statements A, B such that "If A then B" is true but "If B then A" is false. In other words, "If...then" is not commutative.
- Show that if $x \in \mathbb{R}\backslash\mathbb{Q}$ then $\sqrt{x} \in \mathbb{R}\backslash\mathbb{Q}$, giving first an indirect proof and then a contrapositive proof.

5 The Completeness Axiom for $\mathbb{R}$

Recall that, while the rational numbers $\mathbb{Q}$ are "dense" on the real number line, there are points which do not represent rational numbers (for example $\sqrt{2}$). Obviously, we can find sequences of rational numbers with elements approximating $\sqrt{2}$ better and better, but their limit is no longer rational. In some sense, the rational numbers $\mathbb{Q}$ are *incomplete*. Before we give a precise formulation of the *Completeness Axiom* for $\mathbb{R}$, we need some preparations. The completeness of $\mathbb{R}$ is expressed in this chapter with the help of sets rather than sequences. But we will return to sequences in the next chapter and then discuss implications of the Completeness Axiom for sequences.

Definition 5.1. *Let $X \subset \mathbb{R}$ be a set of real numbers. A number $M \in X$ is called the* maximum *of X, if $x \leq M$ for all $x \in X$. In other words, the maximum is the largest number in the set X. Similarly, we define $m \in X$ to be the* minimum *of X, if $m \leq x$ for all $x \in X$.* *maximum/ minimum*

Examples. 1. $X = \{2, 3, 6, 8, 9\}$. Then we have $\min(X) = 2$ and $\max(X) = 9$.

2. $X = \{1/n \mid n \in \mathbb{N}\}$. Then we have $\max(X) = 1$, but $\min X$ does not exist.

3. $X = [0, \infty)$. Then we have $\min(X) = 0$, but $\max(X)$ does not exist.

Definition 5.2. *A set $X \subset \mathbb{R}$ is* bounded above *if there exists a number $C \in \mathbb{R}$ such that $x \leq C$ for all $x \in X$. C is called an* upper bound *of X. Similarly, X is* bounded below *if there exists $c \in \mathbb{R}$ such that $c \leq x$ for all $x \in X$. c is called a* lower bound *of X. A set X is called* bounded*, if it is bounded above and below.* *bounded above/below* *upper/ lower bound*

Examples. 1. $X = \{1/n \mid n \in \mathbb{N}\}$ is bounded, since it is bounded above (upper bounds are, for example, $1, e, 17$) and bounded below (lower bounds are, for example, $-5, -3/2, 0$).

2. $X = \{x \in \mathbb{Q} : x^2 \leq 2\}$ is bounded below by $-\sqrt{2}$ and above by $\sqrt{2}$, but these two numbers are not minimum and maximum of X, since they are not elements of X.

If a set $X \subset \mathbb{R}$ is bounded above, it has many upper bounds. But there might be an optimal upper bound, namely the smallest one. A priori, it is not clear that such a smallest upper bound exists. If it exists, we call it the supremum of the set X.

Definition 5.3. *Let $X \subset \mathbb{R}$ be bounded above. A number $C \in \mathbb{R}$ is called* least upper bound *or* supremum *of X if*

supremum/ infimum

(i) C is an upper bound of X and

(ii) no number less than C is an upper bound of X.

We write $C = \sup(X)$. Similarly, we can define the greatest lower bound *or* infimum *of X, denoted by* $\inf(X)$.

In contrast to the maximum of X, the supremum of X need not be element of X. But if $\sup(X) \notin X$, then there exists a sequence $x_n \in X$ with $x_n \to \sup(X)$ as $n \to \infty$. Therefore, (ii) is equivalent to
(ii') $C \in X$ or there exists $x_n \in X$ such that $x_n \to C$ as $n \to \infty$.
In the following examples, we use the defining properties (i) and (ii') to derive supremum and infimum.

Examples.

1. $X = \{1/n \mid n \in \mathbb{N}\}$. Then we have $\inf(X) = 0$ and $\sup(X) = 1$.
2. $X = (0, \infty)$. Then we have $\inf(X) = 0$ but $\sup(X)$ does not exist.
3. $X = \{\frac{n}{1+n^2} \mid n \in \mathbb{N}\}$. We have

$$X = \left\{ \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \ldots \right\},$$

and we guess that $\sup(X) = 1/2$ and $\inf(X) = 0$.
Proof of $\sup(X) = 1/2$: $n/(1+n^2) \leq 1/2$ is equivalent to $(n-1)^2 \geq 0$, which is true. Since $1/2 \in X$, we have $\sup(X) = 1/2$ by (i) and (ii').
Proof of $\inf(X) = 0$: We have $n/(1+n^2) \geq 0$ for all $n \in \mathbb{N}$. Moreover, $n/(1+n^2) \to 0$ as $n \to \infty$. Therefore, (i) and (ii') are satisfied and we have $\inf(X) = 0$.

4. $X = \{x_{n,m} = \dfrac{nm}{1+n^2+m^2} \mid n, m \in \mathbb{N}\}$. Looking at some values of X we guess that $\sup(X) = 1/2$ (by choosing $n = m$ large) and $\inf(X) = 0$ (by choosing $n = 1$ and m large).
Proof of $\sup(X) = 1/2$: $x_{n,m} \leq 1/2$ is equivalent to

$0 \leq 1 + (n-m)^2$, which is true. Therefore, $1/2$ is an upper bound. Moreover, we have

$$\lim_{n\to\infty} x_{n,n} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} = \frac{1}{2}.$$

Therefore, $\sup(X) = 1/2$ by (i) and (ii').
Proof of $\inf(X) = 0$: Since $x_{n,m} \geq 0$, 0 is a lower bound of X. Moreover,

$$\lim_{n\to\infty} x_{n,1} = \lim_{n\to\infty} \frac{n}{2+n^2} = 0.$$

Therefore, $\inf(X) = 0$ by (i) and(ii').

5. $X = \{x_n = \dfrac{n^2-4n+4}{1+2n^2} \mid n \in \mathbb{N}\}$. We have

$$X = \left\{\frac{1}{3}, 0, \frac{1}{19}, \frac{4}{33}, \frac{9}{51}, \ldots\right\},$$

so we guess that $\inf(X) = 0$ and $\sup(X) = 1/2$ (by looking at $\lim_{n\to\infty} x_n$).
Proof of $\sup(X) = 1/2$: $x_n \leq 1/2$ is equivalent to
$0 \leq 8n - 7$, which is true. We then conclude $\sup(X) = 1/2$ from

$$\lim_{n\to\infty} \frac{n^2-4n+4}{1+2n^2} = \frac{1}{2}.$$

Proof of $\inf(X) = 0$: We have $x_n \geq 0$ since $n^2-4n+4 = (n-2)^2 \geq 0$. Moreover, $x_1 = 0$, which implies $\inf(X) = 0$.

Completeness Axiom for $\mathbb{R}$. *Every non-empty set of real numbers which is bounded above has a supremum.*

completeness of $\mathbb{R}$

Remark. When reading the completeness axiom, we may be tempted to think that this is a theorem which needs to be proved. The problem is that we *never defined what real numbers are.* It is possible, but time-consuming, to give a proper definition of the real numbers and then to prove the completeness axiom as a theorem. Another approach is to introduce the real numbers axiomatically. Then the completeness axiom is one of their defining features, distinguishing the real numbers from the rational numbers. This means that completeness should not hold if we consider it within the set $\mathbb{Q}$ of rational numbers. Here the set $X = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$ is non-empty and bounded above, but there is no supremum of X within the rationals, since we cannot choose the irrational $\sqrt{2}$ and, for every other upper bound $C \in \mathbb{Q}$ of X there exists a smaller one, ruling out that there is an optimal rational upper bound of X.

Definition 5.4. *Let X be a set and $f : X \to \mathbb{R}$ be a real-valued function. The* image *of f, denoted by $f(X)$ is the set*

image set of a function

$$f(X) = \{f(x) \mid x \in X\}.$$

We say that f is bounded above *if $f(X) \subset \mathbb{R}$ is bounded above. Similarly, we define* bounded below *and* bounded. *For $\sup(f(X))$ we also use the notation $\sup(f)$ and $\sup_{x\in X} f(x)$.*

supremum/ infimum of a function

Examples. 1. $f(x) = x^2$ on $X = \mathbb{R}$ has $\inf(f) = 0$ and $\sup(f)$ does not exist.

2. $f(x) = \dfrac{x^2 \cos(x)}{1+x^2}$ for $x > 0$. Since $|\cos(x)| \leq 1$, we have

$$-1 \leq -\frac{x^2}{1+x^2} \leq f(x) \leq \frac{x^2}{1+x^2} \leq 1$$

for all $x > 0$. So -1 and 1 are lower and upper bound of $f(x)$. The question is whether we can find sequences $x_n, y_n > 0$ such that $f(x_n) \to -1$ and $f(y_n) \to 1$ as $n \to \infty$. Choosing $x_n = 2n\pi > 0$ and $y_n = (2n-1)\pi > 0$ for $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \frac{(2n\pi)^2}{1+(2n\pi)^2} = 1, \\ \lim_{n\to\infty} f(y_n) &= \lim_{n\to\infty} -\frac{(2n-1)^2\pi^2}{1+(2n-1)^2\pi^2} = -1. \end{aligned}$$

3. $f(x) = -\dfrac{1}{x^2} + \dfrac{1}{x} - 1$ for $x > 1$. We conclude from $x > 1$ that $x^2 > x$ and, therefore, $1/x > 1/x^2$. This shows that

$$f(x) = -\frac{1}{x^2} + \frac{1}{x} - 1 > -1.$$

On the other hand, we have both[17]

$$\lim_{x\to 1} \left(-\frac{1}{x^2} + \frac{1}{x} - 1 \right) = -1 + 1 - 1 = -1$$

[17] We have not yet defined the following symbols $\lim_{x\to 1}$ and $\lim_{x\to\infty}$ rigorously, so we ask you to use your intuition about their meaning. We will introduce this notation later in Chapter 7 in full rigour.

and
$$\lim_{x\to\infty}\left(-\frac{1}{x^2}+\frac{1}{x}-1\right) = -0+0-1 = -1.$$
This shows that $\inf(f) = -1$. For the supremum we need some school maths and look out for local maxima of $f(x)$. Since f is differentiable, a necessary condition for such a maximum is $f'(x) = 0$. This leads to
$$f'(x) = \frac{2}{x^3} - \frac{1}{x^2} = \frac{1}{x^3}(2-x) = 0,$$
with the only solution $x = 2$. Considering
$$f''(x) = \frac{2}{x^2}\left(1-\frac{3}{x}\right),$$
we obtain $f''(2) < 0$, which shows that $x = 2$ is a local maximum with
$$f(2) = -\frac{1}{4}+\frac{1}{2}-1 = -\frac{3}{4}.$$
This is our guess: $\sup(f) = 3/4$, which we need to prove.
Proof of $\sup(f) = 3/4$: $f(x) \leq -3/4$ for $x > 1$ is equivalent to $(x-2)^2 \geq 0$, which is true. So $-3/4$ is an upper bound of f. Finally, $f(2) = -3/4$ shows that $\sup(f) = -3/4$.

Reader's Task. Let $f : X \to \mathbb{R}$. Check that we have
$$\sup(-f) = -\inf(f).$$

Proposition 5.5. *Let $f, g : X \to \mathbb{R}$. Then we have*
$$\sup(f) + \inf(g) \leq \sup(f+g) \leq \sup(f) + \sup(g).$$

Proof. Note that $f(x) \leq \sup(f)$ for all $x \in X$, since $\sup(f)$ is an upper bound of $f(X)$. This implies that, for all $x \in X$,
$$f(x) + g(x) \leq \sup(f) + \sup(g),$$
in other words, $\sup(f) + \sup(g)$ is an upper bound for
$$(f+g)(X) = \{f(x) + g(x) \mid x \in X\}.$$
Since $\sup(f+g)$ is the smallest upper bound for $(f+g)(X)$, we have
$$\sup(f+g) \leq \sup(f) + \sup(g).$$

For the other inequality, we start with

$$f(x) + g(x) \leq \sup(f + g) \quad \forall\, x \in X,$$

which implies

$$f(x) \leq \sup(f + g) - g(x) \leq \sup(f + g) - \inf(g) \quad \forall\, x \in X.$$

In other words, $\sup(f+g) - \inf(g)$ is an upper bound for f. Therefore,

$$\sup(f) \leq \sup(f + g) - \inf(g),$$

which finishes the proof. $\square$

Important concepts/typical problems in this chapter that you should try without looking anything up:

- Formulate the Completeness Axiom for $\mathbb{R}$.
- Give an example of a set $X \subset \mathbb{R}$ which is bounded above but does not have a maximum.
- Show rigorously the following fact: If $X \subset \mathbb{R}$ has a maximum, then X has also a supremum and $\sup(X) = \max(X)$.
- Find infimum and supremum of the following set:

  $$X = \left\{ x + \frac{1}{x} \mid \frac{1}{2} < x < 2 \right\}.$$

  Does X have a minimum or a maximum?

6 More on limits of sequences

6.1 Roadway to Bolzano-Weierstrass

Definition 6.1. *A sequence* $(x_n)_{n\in\mathbb{N}}$ *is called* monotone increasing *if* $x_n \le x_{n+1}$ *for all* $n \in \mathbb{N}$. *Analogously,* (x_n) *is called* monotone decreasing *if* $x_n \ge x_{n+1}$ *for all* $n \in \mathbb{N}$.

monotone increasing/ decreasing sequences

Recall that we already introduced the notion of boundedness of sequences in Theorem 3.4. Now we state a crucial consequence of the Completeness Axiom for bounded increasing sequences, namely that they must have a limit in the set of real numbers. In fact, this limit is nothing but the least upper bound of the sequence considered as a set.

Theorem 6.2. *Let* (x_n) *be a* monotone increasing real sequence. *If* (x_n) *is* bounded, *then* (x_n) *is convergent and we have*

$$\lim_{n\to\infty} x_n = \sup(X),$$

where $X = \{x_n \mid n \in \mathbb{N}\}$. *There is an analogous statement for bounded, monotone decreasing sequences, namely,*

$$\lim_{n\to\infty} x_n = \inf(X).$$

A simple example here is the sequence $x_n = 1/n$. This sequence is bounded and monotone decreasing and we have $\lim_{n\to\infty} x_n = \inf(X) = 0$, in accordance with the theorem.

Remarks. (a) Note that we **do not have** monotone increasing **complex** sequences, since $x_n \le x_{n+1}$ does not make sense generally if $x_n \in \mathbb{C}$. Therefore, Theorem 6.2 does not have a counterpart in the context of complex sequences.

(b) Note also in the theorem above that X and (x_n) are two very different mathematical objects. While (x_n) is a sequence, in other words, a map $\mathbb{N} \to \mathbb{R}$ (namely, $n \mapsto x_n$), the object X is a set, consisting of all the elements of the sequence. For example, if $x_n = (-1)^n$, then we have, on the one hand,

$$x_1 = (-1), x_2 = 1, x_3 = (-1), x_4 = 1, x_5 = (-1), \dots,$$

and $X = \{-1, 1\}$ on the other hand. So X is here a finite set with only two elements. Note that the set X does not change if we change the sequence $x_n = (-1)^n$ to the very different sequence

$$x_n = \begin{cases} 1, & \text{if } n = 1, \\ -1, & \text{if } n \geq 2. \end{cases}$$

Proof. Since X is bounded above, there exists $\sup(X)$ by the Completeness Axiom for $\mathbb{R}$. Note that $x_n \leq \sup(X)$ for all $n \in \mathbb{N}$. For every $\epsilon > 0$, there must be $x_N \in X$ with $x_N > \sup(X) - \epsilon$ by the definition of the supremum. By the monotonicity of (x_n), we have for all $n \geq N$,

$$\sup(X) - \epsilon < x_N \leq x_n \leq \sup(X),$$

in other words

$$|x_n - \sup(X)| < \epsilon \quad \forall\, n \geq N.$$

This shows that $x_n \to \sup(X)$ as $n \to \infty$. □

Next we introduce the important notion of a subsequence.

Definition 6.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence. A* subsequence *of (x_n) is a sequence $(x_{n_j})_{j\in\mathbb{N}}$ with $n_1 < n_2 < n_3 < \dots$.* *subsequence*

Example. Let $x_n = (-1)^n(1 - 1/n)$. Then we have

$$x_1 = 0,\ x_2 = \frac{1}{2},\ x_3 = -\frac{2}{3},\ x_4 = \frac{3}{4},\ x_5 = -\frac{4}{5},\ x_6 = \frac{5}{6},\ x_7 = -\frac{6}{7},\ \dots$$

It is easy to check that this sequence does not have a limit. But the subsequence $(x_{2j})_{j\in\mathbb{N}}$, given by $x_{2j} = 1 - 1/(2j)$, is convergent and we have

$$\lim_{j\to\infty} x_{2j} = 1.$$

Note, however, that the sequence $y_n = x_{1+2|3-n|}$ is **not** a subsequence of (x_n), since we have

$$y_1 = x_{\mathbf{5}}, y_2 = x_{\mathbf{3}}, y_3 = x_{\mathbf{1}}, y_4 = x_{\mathbf{3}}, y_5 = x_{\mathbf{5}}, y_6 = x_{\mathbf{7}}, y_7 = x_{\mathbf{9}}, \dots,$$

and the sequence of x-indices $5, 3, 1, 3, 5, 7, 9, \dots$ is not strictly monotone increasing. But this condition is required for a subsequence.

Proposition 6.4. *Let (x_n) be convergent with limit $\lim_{n\to\infty} x_n = x^*$ and (x_{n_j}) be a subsequence. Then (x_{n_j}) is also convergent and*

$$\lim_{j\to\infty} x_{n_j} = x^*.$$

Reader's Task. It is a very useful exercise for the reader to proof Proposition 6.4 and to check that our earlier Proposition 3.13 is a special case of this fact (by choosing $n_j = j + K$).

Here we state our first lemma. Lemmas are mathematical structures containing smaller facts that may become useful at a later stage. *Lemma*

Lemma 6.5. *Every real sequence (x_n) contains a subsequence which is either increasing or decreasing.*

Here is the neat proof of this fact:

Proof. Given a sequence $(x_n)_{n\in\mathbb{N}}$. We call an element x_{n_0} a *peak element* of the sequence if we have $x_{n_0} \geq x_n$ for all $n > n_0$. The corresponding index n_0 is called a *peak index.* Now one of the following two cases must be true.

Case 1: There are infinitely many peak indices $n_1 < n_2 < n_3 < \dots$. This means that $(x_{n_j})_{j\in\mathbb{N}}$ is a monotone decreasing subsequence.

Case 2: There are only finitely many peak indices. Then there is $n_1 \in \mathbb{N}$ such that all peak indices are $< n_1$. Since n_1 is not a peak index, there must be $n_2 > n_1$ with $x_{n_1} < x_{n_2}$. Since n_2 is not a peak index, there must be $n_3 > n_2$ with $x_{n_2} < x_{n_3}$. Continuing this way, we obtain a subsequence $(x_{n_j})_{j\in\mathbb{N}}$, which is monotone increasing (even strictly).

$\square$

Now we formulate the main result of this chapter. The theorem is named after the mathematicians Bolzano[18] and Weierstrass[19]. We like to mention as a rule of thumb, that theorems which have names attached are usually **particularly important facts**.

Bolzano-Weierstrass

Theorem 6.6 (Bolzano-Weierstrass)**.** *Let (x_n) be a bounded real sequence. Then (x_n) has a subsequence which is convergent.*

Proof. Let (x_n) be a bounded real sequence. Lemma 6.5 tells us that there exists a monotone subsequence (x_{n_j}). This monotone subsequence is also a bounded sequence and, by Theorem 6.2, convergent. $\square$

[18] Bernard Bolzano (1781-1848) was born and lived in Prague. Besides being a mathematician and a philosopher, he was also a Catholic priest.

[19] Karl Weierstrass (1815-1897) was a German mathematician. Before he became professor at the Technical University of Berlin, he worked as a highschool teacher. He is considered to be the "father of modern analysis".

Remarks. (a) If we reconsider the proof sequence leading to the Bolzano-Weierstrass Theorem, we see that a crucial ingredient is the Completeness Axiom for $\mathbb{R}$.

(b) While Theorem 6.2 has no counterpart for complex sequences, the Bolzano-Weierstrass Theorem holds also for bounded complex sequences (since $\mathbb{C}$ is also a complete space in an appropriate sense). But the proof of the Bolzano-Weierstrass Theorem in the case of a complex sequence cannot use Theorem 6.2.

6.2 Cauchy sequences

Often, we have to show that a sequence (x_n) is convergent without knowing its limit x^*. In this case, we cannot give a direct proof that $|x_n - x^*| \to 0$ as $n \to \infty$. Theorem 6.2 may help in special situations, but it only treats the very restricted case of bounded monotone real sequences. A very useful concept to prove convergence without knowing the limit is the concept of a Cauchy sequence[20].

Definition 6.7. *A sequence $(x_n)_{n\in\mathbb{N}}$ is called a* Cauchy sequence *if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that* *Cauchy sequence*

$$|x_m - x_n| < \epsilon \quad \forall\, n, m \geq N.$$

Intuitively, the elements x_n of a Cauchy sequence become closer and closer to each other as their indices increase.

Reader's Task. The following two facts are important and their proofs are useful exercises for the reader. The proof of the first theorem is very similar to the proof of Theorem 3.4. For the proof of the second theorem, you need to use the triangle inequality.

Theorem 6.8. *Let (x_n) be a Cauchy sequence. Then (x_n) is bounded.*

Theorem 6.9. *Let (x_n) be a convergent sequence. Then (x_n) is also a Cauchy sequence.*

Of particular importance is the converse of Theorem 6.9, which holds in $\mathbb{R}$ but not in $\mathbb{Q}$, since a crucial ingredient is the Completeness Axiom for $\mathbb{R}$.

[20]Baron Augustin-Louis Cauchy (1789-1857) was a French mathematician who became a full professor at École Polytechnique in 1816. He was the first mathematician to introduce techniques and notions (the ϵ, δ-terminology) to prove certain fundamental theorems in calculus with full rigour.

Theorem 6.10. *Let (x_n) be a real Cauchy sequence. Then (x_n) is convergent with real limit $x^* \in \mathbb{R}$.*

Proof. Let (x_n) be a real Cauchy sequence. Then (x_n) is bounded, by Theorem 6.8. Therefore, there exists a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$, by the Bolzano-Weierstrass Theorem. Let

$$x^* = \lim_{j\to\infty} x_{n_j} \in \mathbb{R}.$$

It remains to prove that x^* is also the limit of (x_n). Let $\epsilon > 0$ be given. Then there exists $J \in \mathbb{N}$ such that

$$|x_{n_j} - x^*| < \frac{\epsilon}{2} \quad \forall\, j \geq J. \tag{9}$$

Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ with

$$|x_n - x_m| < \frac{\epsilon}{2} \quad \forall\, n, m \geq N. \tag{10}$$

Choose $j^* \geq J$ such that $n_{j^*} \geq N$. Then we have for all $n \geq N$, using (9), (10) and the triangle inequality,

$$|x_n - x^*| \leq |x_n - x_{n_{j^*}}| + |x_{n_{j^*}} - x^*| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Remark. Here again, Theorem 6.10 has an analogue for complex sequences. Once, the Bolzano-Weierstrass Theorem for complex sequences is established, the same proof as the one above can be used to derive Theorem 6.10 for complex sequences.

Example. Let $a, c > 0$ and $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by $u_1 = c$ and

$$u_{n+1} = \frac{1}{2}\left(u_n + \frac{a}{u_n}\right) \quad \forall\, n \geq 1. \tag{11}$$

Our goal is to prove that (u_n) is convergent. To do so we first derive for all $n, m \geq N \geq 2$ that

$$|u_n - u_m| \leq \frac{1}{2^{N-3}}|u_3 - u_2|, \tag{12}$$

which implies that (u_n) is Cauchy. The convergence of (u_n) follows then via Theorem 6.10.

So we only need to prove (12). Let $n \geq 2$. From the recursion relation (11) and $a, c > 0$ we conclude that $u_n > 0$ for all $n \in \mathbb{N}$. Moreover, we obtain

$$u_{n+2} - u_{n+1} = \left(\frac{1}{2} - \frac{a}{2u_{n+1}u_n}\right)(u_{n+1} - u_n). \tag{13}$$

If we show that $0 < a/(2u_{n+1}u_n) < 1$, we derive from (13) the useful relation

$$|u_{n+2} - u_{n+1}| < \frac{1}{2}|u_{n+1} - u_n| \quad \forall\, n \geq 2. \tag{14}$$

But $0 < a/(2u_{n+1}u_n) < 1$ follows directly from

$$u_{n+1}u_n = \frac{u_n}{2}\left(u_n + \frac{a}{u_n}\right) = \frac{a}{2} + \frac{u_n^2}{2} > \frac{a}{2} > 0.$$

Applying (14) repeatedly, we obtain

$$|u_{n+1} - u_n| \leq \frac{1}{2^{n-2}}|u_3 - u_2| \quad \forall\, n \geq 2.$$

Assuming $n \geq m \geq N \geq 2$, we derive via the geometric series and the triangle inequality

$$\begin{aligned}
|u_n - u_m| &\leq |u_n - u_{n-1}| + \cdots + |u_{m+1} - u_m| \\
&\leq \left(\frac{1}{2^{n-3}} + \cdots + \frac{1}{2^{m-2}}\right)|u_3 - u_2| \\
&\leq \frac{1}{2^{m-2}}\left(1 + \frac{1}{2} + \ldots \frac{1}{2^{n-m-1}}\right)|u_3 - u_2| \\
&\leq \frac{1}{2^{m-2}}\underbrace{\frac{1-(1/2)^{n-m}}{1-(1/2)}}_{<2}|u_3 - u_2| \leq \frac{1}{2^{N-3}}|u_3 - u_2|,
\end{aligned}$$

finishing the proof of (12). □

Interestingly, knowing that (u_n) satisfying (11) is convergent, we are now even able to derive the limit u^*. We know from Proposition 3.13 that

$$u^* = \lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1}.$$

Multiplying (11) by $2u_n > 0$, we obtain

$$2u_{n+1}u_n = a + u_n^2, \tag{15}$$

and we conclude from (11) and COLT that

$$2(u^*)^2 = 2\lim_{n\to\infty} u_{n+1}u_n = a + \lim_{n\to\infty} u_n^2 = a + (u^*)^2,$$

which yields $u^* = \pm\sqrt{a}$. Since $u_n > 0$, we must have $u^* = \sqrt{a}$. So we derived the explicit value of the limit u^*, using the fact that (u_n) is convergent.

Note that this result has a practical application: We can use the sequence (u_n) to calculate arbitrarily good rational approximations of $\sqrt{a}$ for any $a \in \mathbb{N}$. Moreover, letting $m \to \infty$, the inequality (12) leads to

$$|u_n - u^*| \leq \frac{1}{2^{n-3}}|u_3 - u_2| \quad \forall\, n \geq 2,$$

which can be used as a very crude estimate of the *error* of the approximation. In most cases, the method is much more efficient and has an extremely good convergence rate (namely, *quadratic convergence*: with every iteration the number of correct decimals roughly doubles).

Moreover, the sequence (11) is derived from a general numerical method to find the zeros of a differentiable function f, the so-called *Newton method*[21]. The Newton method recursion takes the form

Newton method

$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)} \quad \forall\, n \in \mathbb{N}, \tag{16}$$

where f' denotes the derivative of f, and we obtain (11) by choosing $f(x) = x^2 - a$.

[21] The method is named after the English mathematician and physicist Sir Isaac Newton (1642-1726), who was Lucasian Professor of Mathematics at the University of Cambridge. We like to mention, amongst his many fundamental contributions, the development of Calculus and notations used today. He shares this achievement with the German mathematician and philosopher Gottfried Wilhelm Leibniz (1646-1716), who developed similar concepts at the same time and independently.

Important concepts/typical problems in this chapter that you should try without looking anything up:

- State the Theorem of Bolzano-Weierstrass.
- Give an example of a Cauchy sequence (x_n) in $\mathbb{Q}$ which is not convergent in $\mathbb{Q}$.
- Give an example of a bounded sequence which has two subsequences with different limits.
- Let $a, b \in \mathbb{R}$. Show that the sequence (a_n) given by $a_1 = a$, $a_2 = b$ and $a_{n+2} = (a_{n+1} + a_n)/2$ for $n \geq 1$ is convergent.
- Give an indirect proof for the following fact: The real sequence (w_n), given by $w_{n+1} = w_n^2 + 1$ for all $n \geq 1$, does not have a limit for any real initial value $w_1 \in \mathbb{R}$.

7 Functions, Limits and continuity

7.1 Preimage of a function

Recall that a function $f : X \to Y$ is a map between sets. In accordance with Definition 5.4, we define the *image* of a set $X_0 \subset X$ under the function by

image of a set under a function

$$f(X_0) = \{f(x) \mid x \in X_0\} \subset Y.$$

In other words, $f(X_0)$ is the set of all $y \in Y$ which are obtained as values $f(x)$ for $x \in X_0$. Another useful concept is the notion of the *preimage* of a set under the function f. We will see below the surprising fact that the concept of preimage is more consistent with set theoretical operations than the concept of an image set.

Definition 7.1. *Let $f : X \to Y$ be a function and $Y_0 \subset Y$. The* preimage *of the set Y_0 under f is a subset of X and given by*

preimage

$$f^{-1}(Y_0) = \{x \in X \mid f(x) \in Y_0\}.$$

In other words, the preimage $f^{-1}(Y_0)$ consists of all elements in X which are mapped into Y_0 under the function f (see the Venn Diagram illustation in Figure 3).

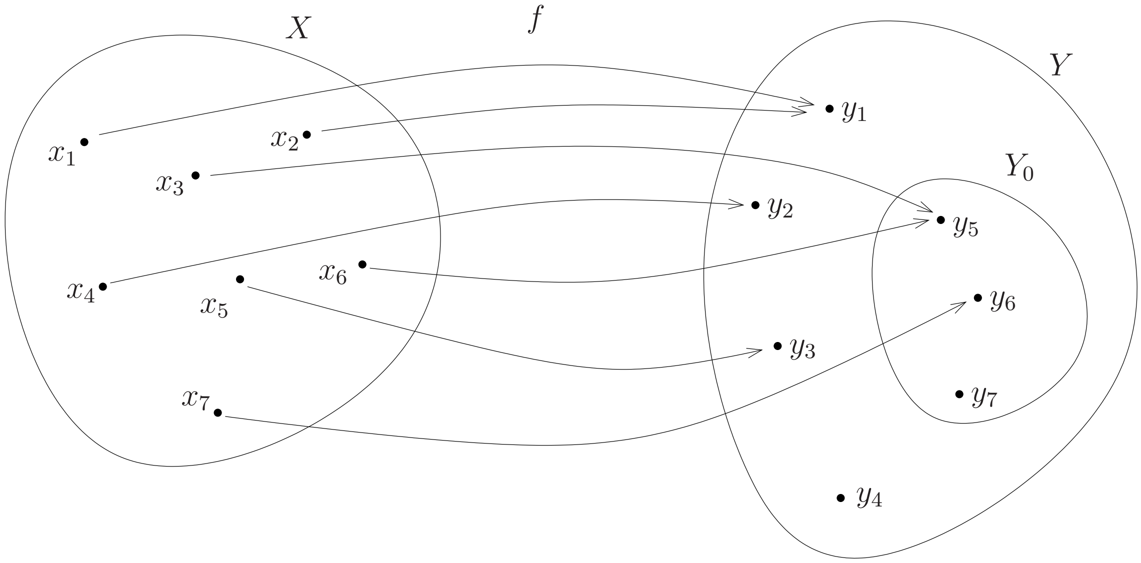

Figure 3: Only the elements x_3, x_6, x_7 are mapped into $Y_0 = \{y_5, y_6, y_7\} \subset Y$ under f, so we have here $f^{-1}(Y_0) = \{x_3, x_6, x_7\} \subset X$.

CAUTION. Note that the preimage of a set is defined for any function f. In contrast to the definition of the *inverse function*, which has the same notation f^{-1}, we do not need bijectivity[22] of f to consider preimages. However, if f is bijective with inverse f^{-1} and $Y_0 \subset Y$, then the preimage $f^{-1}(Y_0)$ agrees with the image set of Y_0 under the function f^{-1}. Be aware that the preimage $f^{-1}(\{y\})$ is a subset of X (namely the set of all elements $x \in X$ which satisfy $f(x) = y$), whereas in the case of bijectivity with inverse function f^{-1}, the expression $f^{-1}(y)$ is an element of X.

Examples. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Since $f(-1) = f(1) = 1$, the function f is not injective. But we have

$$\begin{aligned}
f^{-1}(\{3\}) &= \{-\sqrt{3}, \sqrt{3}\}, \\
f^{-1}(\{0\}) &= \{0\}, \\
f^{-1}(\{-1\}) &= \{\}, \\
f^{-1}((1, 10]) &= [-\sqrt{10}, -1) \cup (1, \sqrt{10}], \\
f^{-1}((-1, 4)) &= (-2, 2).
\end{aligned}$$

(b) Let $g : [0, 2\pi] \to \mathbb{R}$ be given by $g(x) = \sin(x)$. Then we have

$$\begin{aligned}
g^{-1}(\{0\}) &= \{0, \pi, 2\pi\}, \\
g^{-1}([-1, 1]) &= [0, 2\pi], \\
g^{-1}([0, 1)) &= [0, \pi/2) \cup (\pi/2, \pi] \cup \{2\pi\}.
\end{aligned}$$

(c) Let $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^2 - 4x + 2$. Then the preimage $h^{-1}((-\infty, 2])$ is

$$h^{-1}((-\infty, 2]) = \{x \in \mathbb{R} \mid h(x) \leq 2\}.$$

$h(x) \leq 2$ is equivalent to $x(x-4) \leq 0$, which implies

$$\{x \in \mathbb{R} \mid h(x) \leq 2\} = [0, 4].$$

Figure 4 illustrates the preimage $h^{-1}((-\infty, 2])$ as the set of points of the real axis whose images are ≤ 2.

Proposition 7.2. *Let $f : X \to Y$ be a function and $Y_0, Y_1 \subset Y$. Then we have the following facts.*

a) $f^{-1}(Y) = X$

b) $f^{-1}(Y_0 \cup Y_1) = f^{-1}(Y_0) \cup f^{-1}(Y_1)$

[22]Recall from the Calculus 1 course that *bijectivity* of f is defined as follows: if $x \neq y$ then $f(x) \neq f(y)$.

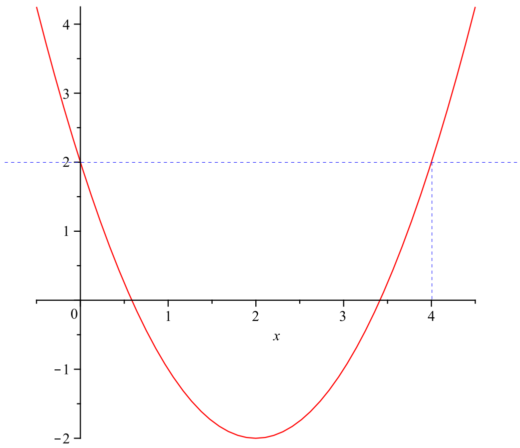

Figure 4: Graph of the parabola $h(x) = x^2 - 4x + 2$. The preimage $h^{-1}((-\infty, 2])$ are all x-values for which the graph stays below the horizontal line $y = 2$.

c) $f^{-1}(Y_0 \cap Y_1) = f^{-1}(Y_0) \cap f^{-1}(Y_1)$

d) $f^{-1}(Y_1 \backslash Y_0) = f^{-1}(Y_1) \backslash f^{-1}(Y_0)$

Proof. We only prove b) and leave the other identities to the reader.

$$\begin{aligned} x \in f^{-1}(Y_0 \cup Y_1) &\Leftrightarrow f(x) \in Y_0 \cup Y_1 \\ &\Leftrightarrow f(x) \in Y_0 \text{ or } f(x) \in Y_1 \\ &\Leftrightarrow x \in f^{-1}(Y_0) \text{ or } x \in f^{-1}(Y_1) \\ &\Leftrightarrow x \in f^{-1}(Y_0) \cup f^{-1}(Y_1) \end{aligned}$$

□

Reader's Task. Find a simple example of a function $f : X \to Y$ and two subsets $X_0, X_1 \subset X$ such that $f(X_0 \cap X_1) \neq f(X_0) \cap f(X_1)$. This shows that taking preimages behaves more consistently under set operations than taking images.

7.2 Limits of a function

Next we introduce properly expressions like $\lim_{x\to\infty} f(x)$ and $\lim_{x\to c} f(x)$. It is intuitively clear what we mean by that. $\lim_{x\to\infty} f(x) = A$ means that if x gets large then $f(x)$ comes closer and closer to the value A. $\lim_{x\to c} f(x) = A$ means that if $x \neq c$ comes closer and closer to c (note that $f(c)$ may not be defined) then $f(x)$ comes closer and closer to A. Here are the precise definitions.

Definition 7.3. *let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. For $c \in \mathbb{R}$, we say that "$f(x) \to A$ as $x \to c$" or "$\lim_{x\to c} f(x) = A$", if the following holds:* $\lim_{x\to c} f(x)$

i) For every $\delta > 0$ *the intersection* $((c-\delta, c) \cup (c, c+\delta)) \cap X$ *is not empty.*

ii) For every $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|f(x) - A| < \epsilon \quad \forall\, x \in ((c-\delta, c) \cup (c, c+\delta)) \cap X.$$

We say that "$f(x) \to A$ *as* $x \to \infty$*" or "*$\lim_{x\to\infty} f(x) = A$*", if the following holds:* $\lim_{x\to\infty} f(x)$

i) $X \subset \mathbb{R}$ *is not bounded above.*

ii) For every $\epsilon > 0$ *there exists* $K \in \mathbb{R}$ *such that*

$$|f(x) - A| < \epsilon \quad \forall\, x \in (K, \infty) \cap X. \tag{17}$$

The expression $\lim_{x\to-\infty} f(x)$ *is defined analogously.*

Note that in each of the two cases of the above definition, i) guarantees that ii) is actually a proper condition. For example, if $X \subset \mathbb{R}$ is bounded, it does not make sense to consider $f(x)$ as $x \to \infty$. Moreover, in this case $(K, \infty) \cap X$ becomes empty if $K \in \mathbb{R}$ is chosen sufficiently large, in which case (17) is not a proper condition. The set $(c-\delta, c) \cup (c, c+\delta)$ carries a name: it is the *punctured open* δ*-interval around* c.
The concept of a limit of a function is very similar to the concept of a limit of a sequence. In fact, the Theorems for limits of sequences presented in Chapter 3 have counterparts for limits of functions and the proofs are very similar. As an example, we only present the counterpart of COLT ("Calculus of Limits Theorem") for sequences.

COLT for limits of functions

Theorem 7.4 (COLT)**.** *Let* $A = \lim_{x\to c} f(x)$ *and* $B = \lim_{x\to c} g(x)$ *with* $c \in \mathbb{R} \cup \{\pm\infty\}$*. Let* a, b *be constants. Then we have*

(i) $af(x) + bg(x) \to aA + bB$ *as* $x \to c$.

(ii) $f(x)g(x) \to AB$ *as* $x \to c$.

(iii) $\frac{f(x)}{g(x)} \to \frac{A}{B}$ *as* $x \to c$*, provided* $B \neq 0$ *and* g *is nowhere zero near* c.

Reader's Task. Give a proof of statement (ii) in Theorem 7.4. You may consult the proof of Theorem 3.7 as a guideline.

The following proposition tells us that if $\lim_{x\to c} f(x) = A$, then we have $f(x_n) \to A$ for every sequence $x_n \to c$ in the domain of f with all its terms x_n different from c.

Proposition 7.5. *Let $X \subset \mathbb{R}$, $f : X \to \mathbb{R}$ and $\lim_{x\to c} f(x) = A$. Let (x_n) be a sequence in X (in other words, $x_n \in X$ for all $n \in \mathbb{N}$) with $\lim_{n\to\infty} x_n = c$ and $x_n \neq c$ for all $n \in \mathbb{N}$. Then we have*

$$\lim_{n\to\infty} f(x_n) = A.$$

Proof. Let f and (x_n) be as in the proposition. We need to show that the sequence $(f(x_n))$ has the limit A. Choose $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$|f(x) - A| < \epsilon \quad \forall\, x \in (c - \delta, c + \delta). \tag{18}$$

Since $x_n \to c$, there exists $N \in \mathbb{N}$ such that

$$|x_n - c| < \delta \quad \forall\, n \geq N. \tag{19}$$

Combining (18) and (19), we obtain

$$|f(x_n) - A| < \epsilon \quad \forall\, n \geq N,$$

finishing the proof of $f(x_n) \to A$. $\square$

Examples. 1. Compute $\lim\limits_{x\to\infty} \dfrac{\log(x^3 + e^{2x})}{x+3}$. Using COLT and the rule "powers beat logarithms", we have

$$\frac{\log(x^3 + e^{2x})}{x+3} = \frac{\log(e^{2x}(1 + x^3 e^{-2x}))}{x+3} = \frac{2x + \log(1 + x^3 e^{-2x})}{x+3}$$
$$= \frac{2 + x^{-1}\log(1 + x^3 e^{-2x})}{1 + 3/x} = 2 \quad \text{as } x \to \infty.$$

2. Compute $\lim\limits_{x\to 1} \dfrac{x^3 + x - 2}{x - 1}$. Here we have $\lim_{x\to 1} x^3 + x - 2 = 0$ and $\lim_{x\to 1} x - 1 = 0$, so we cannot use COLT directly. But this tells us that $x^3 + x - 2$ is divisible by $x - 1$, and we obtain by a polynomial division

$$x^3 + x - 2 \,:\, x - 1 \,=\, x^2 + x + 2.$$

So we have

$$\lim_{x\to 1} \frac{x^3 + x - 2}{x - 1} = \lim_{x\to 1} x^2 + x + 2 = 4.$$

3. Check whether $f(x) = \dfrac{x^2\cos(x)}{2x^3+3}$ has a limit as $x \to -\infty$. We have $|x^2\cos(x)| \le x^2$ and

$$\lim_{x\to-\infty} \frac{x^2}{2x^3+3} = 0.$$

Therefore, by an analogue of the Squeezing Theorem (see Theorem 3.5), we also have

$$\lim_{x\to-\infty} \frac{x^2\cos(x)}{2x^3+3} = 0.$$

4. Check whether $f(x) = \dfrac{x^2}{2x^3\sin^2(x)+1}$ has a limit as $x \to \infty$. We can choose a sequence $x_n \to \infty$ for which $2x_n^3\sin^2(x_n) = 0$, namely, $x_n = n\pi$. Then we have $f(n\pi) = n^2\pi^2$ and see that there is no limit as $x \to \infty$.

5. Check wether the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

has a limit as $x \to 0$. Note that we can find two different sequences $x_n \to 0$ and $y_n \to 0$ with $f(x_n) = 1$ and $f(y_n) = 0$ for all $n \in \mathbb{N}$ (simply choose $x_n = 1/n$ and $y_n = \sqrt{2}/n$). Therefore, a limit does not exist by Proposition 7.5.

Remark. We can also define *one-sided limits* of a function $f : X \to \mathbb{R}$ with $X \subset \mathbb{R}$. Let $c \in \mathbb{R}$. Then we write "$\lim_{x\to c+} f(x) = A$" (or "$f(x) \to A$ as $x \to c$ from the right") if the following holds: $\lim_{x\to c+} f(x)$

i) For every $\delta > 0$ the intersection $(c, c+\delta) \cap X$ is not empty.

ii) For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - A| < \epsilon \quad \forall\, x \in (c, c+\delta) \cap X.$$

Analogously, we define $\lim_{x\to c-} f(x)$ and call it the "limit of $f(x)$ as $x \to c$ from the left". It is an easy exercise to show that if $\lim_{x\to c-} f(x)$

$$\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x) = A$$

then

$$\lim_{x\to c} f(x) = A.$$

Example. let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 2, & \text{if } x \geq 1, \\ -2x, & \text{if } x < 1. \end{cases}$$

Then it is easy to see that $\lim_{x \to 1-} f(x) = -2$ and $\lim_{x \to 1+} f(x) = 1^2 - 2 = -1$.

7.3 Continuity

Note that for $\lim_{x \to c} f(x)$ to be defined, $f(c)$ does not need to exist. But if $f(c)$ exists, we may like it to coincide with $\lim_{x \to c} f(x)$, which leads naturally to the notion of *continuity at c*.

Definition 7.6. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ be a function. Then f is* continuous at $c \in X$ *if we have*

continuous at c

$$f(c) = \lim_{x \to c} f(x).$$

In other words, f is continuous at $c \in X$ *if we have*

$$\forall \epsilon > 0 \, \exists \delta > 0 : |f(x) - f(c)| < \epsilon \quad \forall x \in X \text{ with } |x - c| < \delta.$$

We say that $f : X \to \mathbb{R}$ is continuous *if f is continuous at all points in X.*

continuous function

In the above example we have $f(1) = 1^2 - 2 = -1$ and $\lim_{x \to 1-} f(x) = -2 \neq f(1)$. Therefore, f is not continuous at 1.
An immediate application of Proposition 7.5 is the following.

Corollary 7.7. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ be continuous at $c \in X$. Then we have for every sequence (x_n) with $x_n \in X$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = c$:*

$$\lim_{n \to \infty} f(x_n) = f(c).$$

Remark. In fact, the converse of Corollary 7.7 is also true: If we have for every sequence (x_n) with $x_n \in X$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = c$:

$$\lim_{n \to \infty} f(x_n) = f(c),$$

then f is continuous at c. Some books use this fact about sequences as the definition of continuity. We do not prove the converse here.

Example. We give a direct ϵ-δ proof that $f(x) = \sqrt{|2x+4|}$ is continuous at $x = -2$. We first compute

$$|f(x) - f(-2)| = \sqrt{2}\sqrt{|x+2|}.$$

If we assume that $|x - (-2)| < \delta$, then we obtain

$$|f(x) - f(-2)| < \sqrt{2}\sqrt{\delta}.$$

If $\epsilon > 0$ is given, we need to find $\delta > 0$ such that $\epsilon = \sqrt{2}\sqrt{\delta}$. This is equivalent to $\delta = \epsilon^2/2$. Therefore, for this choice of $\delta > 0$ we have

$$|f(x) - f(-2)| < \epsilon \quad \forall\, |x - (-2)| < \delta.$$

It is easy to show that the functions $f(x) = c$ for any $c \in \mathbb{R}$ and $f(x) = x$ are continuous. From this we can derive continuity of many other functions using the following theorem.

Theorem 7.8. *Let f, g be continuous at $x = c$ and a, b be constant. Then we have*

(i) $af + bg$ is continuous at $x = c$.

(ii) fg is continuous at $x = c$.

(iii) $\frac{f}{g}$ is continuous at $x = c$, provided g is nowhere zero near c.

(iv) $h \circ f$ is continuous at $x = c$, provided h is continuous at $y_0 = f(c)$.

Proof. (i)-(iii) are direct applications of the Calculus of Limits Theorem 7.4. It remains to prove (iv). To avoid notational difficulties with the domains of f and h, we assume that both functions f, h are defined on all of $\mathbb{R}$. Let $\epsilon > 0$ be given. Continuity of h at $y_0 = f(c)$ implies that there exists $\alpha > 0$ such that

$$|h(y) - h(y_0)| < \epsilon \quad \forall\, |y - y_0| < \alpha. \tag{20}$$

Continuity of f at c implies that there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \alpha \quad \forall\, |x - c| < \delta. \tag{21}$$

Combining (20) and (21) and using $y_0 = f(c)$, we conclude that

$$|h \circ f(x) - h \circ f(c)| < \epsilon \quad \forall\, |x - c| < \delta.$$

□

Now we present three theorems about continuous functions on compact intervals which are of fundamental importance. Before doing so, we need to explain the notion of a *compact interval*.

Definition 7.9. *Let $a < b$ be two real numbers. A real interval $I \subset \mathbb{R}$ of the form $I = [a, b]$ is called* closed[23] *and* bounded[24]. *Closed and bounded intervals are also called* compact. *compact interval*

In his "Calculus" book, M. Spivak dedicates a chapter to the following theorems entitled "Three Hard Theorems". He does not prove them there, but he discusses many important consequences. It is worth to have a look at this alternative source.

Theorem 7.10 (Intermediate Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then there exists $c \in [a, b]$ with $f(c) = 0$.* *Intermediate Value Theorem*

Obviously, there are several variations of the Intermediate Value Theorem which hold also true. For example, we may have $f(a) < y < f(b)$ and deduce the existence of $c \in [a, b]$ with $f(c) = y$; or we may have $f(a) > y \geq f(b)$ and deduce the same fact.
The Intermediate Value Theorem states generally the intuitively clear fact that if a continuous function, defined on a real interval, assumes both the real values $A < B$ then it must also assume every value in between A and B. But we like to mention that the *Completeness Axiom for* $\mathbb{R}$ plays a crucial role in the proof below.

Theorem 7.11. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is a bounded function.*

The following third theorem can be rephrased as "Every continuous function on a compact interval assumes its maximum".

Theorem 7.12. *If $f : [a, b] \to \mathbb{R}$ is continuous, then* $\sup(f)$ *exists and there exists $c \in [a, b]$ with $f(c) = \sup(f)$.*

Now we present the proofs. For Theorem 7.10 we present a proof which also contains an iterative method to find the value $c \in [a, b]$ with $f(c) = 0$. There are other shorter proofs for this theorem, but they are not as constructive.

[23]Closed refers here to the fact that both end points a and b belong to the interval I.
[24]Boundedness refers here to the fact that the subset $I \subset \mathbb{R}$ is a bounded set.

Proof of Theorem 7.10. The main idea is to construct two sequences (x_n), (y_n) with $x_1 = a$ and $y_1 = b$, satisfying the following properties

i) (x_n) is monotone increasing with $f(x_n) \le 0$.

ii) (y_n) is monotone decreasing with $f(y_n) > 0$.

iii) We have $x_n \le y_n$ and $y_{n+1} - x_{n+1} = (y_n - x_n)/2$ for all $n \in \mathbb{N}$.

The construction is very easy and called the *bisection procedure*: Having already x_n and y_n, we consider the mid point $\xi = (x_n + y_n)/2$ of the interval $[x_n, y_n]$ and choose

bisection procedure

$$x_{n+1} = \xi \text{ and } y_{n+1} = y_n \quad \text{if } f(\xi) \le 0 \tag{22}$$

or

$$x_{n+1} = x_n \text{ and } y_{n+1} = \xi \quad \text{if } f(\xi) > 0. \tag{23}$$

It is easy to see that this construction provides two sequences (x_n) and (y_n) with the above properties.
Having such a pair of sequences, we know that (x_n) and (y_n) are convergent ((x_n) is monotone increasing and bounded because of $a \le x_n \le b$, therefore convergent by Theorem 6.2; similarly for (y_n)). Moreover, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

by the Squeezing Theorem 3.5, since iii) implies that

$$y_n - x_n \le \frac{1}{2^{n-1}}(b-a)$$

and $(b-a)/(2^{n-1}) \to 0$ as $n \to \infty$.
It remains to show that $f(c) = 0$ for $c = \lim_{n\to\infty} x_n$. We use continuity, property i) and Proposition 3.12 to derive

$$f(c) = \lim_{n\to\infty} \underbrace{f(x_n)}_{\le 0} \le 0.$$

Analogously, we obtain

$$f(c) = \lim_{n\to\infty} \underbrace{f(y_n)}_{>0} \ge 0.$$

Both inequalities together imply that $f(c) = 0$. $\square$

Example. We like to use the bisection procedure to obtain good approximations of $\sqrt{3}$. We choose $f(x) = x^2 - 3$ and have $f(1) = -2$ and $f(2) = 1$. Therefore, we set $x_1 = 1$ and $y_1 = 2$ and we obtain, using the rules (22) and (23):

n	x_n	y_n	$y_n - x_n$
2	1.5	2	0.5
3	1.5	1.75	0.25
4	1.625	1.75	0.125
5	1.6875	1.75	0.0625
6	1.71875	1.75	0.03125
7	1.71875	1.734375	0.015625
8	1.7265625	1.734375	0.0078125

After 8 iterations, we know that $\sqrt{3} = 1.7320508\ldots$ lies between 1.7265625 and 1.734375. We like to mention that this is not a very efficient method to find approximations for $\sqrt{3}$ and that the Newton method (16) converges much faster. Such considerations are obviously of practical importance.

Proof of Theorem 7.11. We prove indirectly that f is bounded above. The fact that f is bounded below is proved analogously. Let us assume that $f : [a, b] \to \mathbb{R}$ is continuous and unbounded above. Then there exists a sequence $y_n \in f([a, b])$ with $y_n \geq n$. Let $x_n \in [a, b]$ such that $y_n = f(x_n)$. Using the Bolzano-Weierstrass Theorem 6.6, we conclude that there exists a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$. Let $c = \lim_{j\to\infty} x_{n_j} \in [a, b]$. Continuity of f yields

$$\lim_{j\to\infty} f(x_{n_j}) = f(c),$$

in other words $(f(x_{n_j})$ is a convergent sequence and, therefore, bounded. But this is in contradiction to $f(x_{n_j}) = y_{n_j} \geq n_j \geq j \to \infty$ as $j \to \infty$. $\square$

Proof of Theorem 7.12. Let $y = \sup(f)$. From property (ii') we conclude that there exists a sequence $y_n \in f(X)$ with $y = \lim_{n\to\infty} y_n$. Let $y_n = f(x_n)$ with $x_n \in [a, b]$. Then x_n has a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$, by the Bolzano-Weierstrass Theorem 6.6. Let $c = \lim_{j\to\infty} x_{n_j} \in [a, b]$. Using continuity of f and Proposition 6.4, we obtain

$$f(c) = \lim_{j\to\infty} f(x_{n_j}) = \lim_{j\to\infty} y_{n_j} = \lim_{n\to\infty} y_n = y = \sup(f).$$

$\square$

Reader's Task. A good method to fully understand a theorem is to check whether all assumptions are really necessary. Find examples that show that the statements in Theorems 7.11 and 7.12 are no longer true if we replace the compact interval $[a, b]$ by the open and bounded interval (a, b).

Important concepts/typical problems in this chapter that you should try without looking anything up:

- Let $f : X \to Y$ be a function and $Y_0 \subset Y_1 \subset Y$. Show that $f^{-1}(Y_1 \backslash Y_0) = f^{-1}(Y_1) \backslash f^{-1}(Y_0)$.
- Compute $\lim\limits_{x \to 2} \dfrac{4 - x^2}{3 - \sqrt{x^2 + 5}}$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $c \in \mathbb{R}$. Use the contrapositive proof technique to show: If we have for every convergent sequence (x_n) with $\lim_{n\to\infty} x_n = c$ that

$$\lim_{n\to\infty} f(x_n) = f(c),$$

  then f is continuous at c.
- Prove the following fact: Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions with $f(a) < g(a)$ and $f(b) > g(b)$. Then there exists $c \in [a, b]$ with $f(c) = g(c)$.