DURHAM UNIVERSITY — Department of Mathematical Sciences
COLLECTION 2015

Name:	College:

Analysis
MATH1051

Time allowed: 45 minutes. Answer all questions . ***Use of electronic calculators is forbidden. You should explicitly refer to (but not necessarily write out the statements of) any of the main results of the course that you use.***

1. Evaluate the following limits.

   (a) (15 marks.) $\lim_{n\to\infty} \dfrac{\sqrt{n-2}-\sqrt{4n+7}}{\sqrt{n}}$.

   (b) (10 marks.) $\lim_{k\to\infty} \dfrac{4\cdot 10^k - 3\cdot 10^{2k}}{10^{k-1}+2\cdot 10^{2k-1}}$.

SOLUTION:

(a) Multiplying numerator and denominator by $\sqrt{n^2-2n}+\sqrt{4n^+7n}$, applying COLT and the continuity of the square root function, yields

$$\lim_{n\to\infty} \frac{\sqrt{n-2}-\sqrt{4n+7}}{\sqrt{n}} = \lim_{n\to\infty} \frac{(n-2)-(4n+7)}{\sqrt{n^2-2n}+\sqrt{4n^2+7n}} = \lim_{n\to\infty} -\frac{-3n-9}{\sqrt{n^2-2n}+\sqrt{4n^2+7n}} =$$
$$-3\lim_{n\to\infty} \frac{1+3/n}{\sqrt{1-2/n}+\sqrt{4+7/n}} = -1.$$

Here is an even shorter solution

$$\lim_{n\to\infty} \frac{\sqrt{n-2}-\sqrt{4n+7}}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{1-\frac{2}{n}} - \sqrt{4+\frac{7}{n}} = 1-\sqrt{4} = -1,$$

where we applied, again, COLT and the continuity of the square root function.

(b) We have,

$$\lim_{k\to\infty} \frac{4\cdot 10^k - 3\cdot 10^{2k}}{10^{k-1}+2\cdot 10^{2k-1}} = \lim_{k\to\infty} \frac{40\cdot 10^{-k} - 3\cdot 10}{10^{-k}+2} = \frac{-30}{2} = -15,$$

using COLT and $\lim_{k\to\infty} 10^{-k} = \lim_{k\to\infty}(1/10)^k = 0$, since $0 < 1/10 < 1$.

2. (a) (5 marks.) Give an example of a convergent sequence which is not monotone increasing and not monotone decreasing.

   (b) (5 marks.) Using quantifiers, give the precise logical formulation that a sequence (x_n) **does not** converge to the value x^*.

   (c) (15 marks.) Let (a_n) and (b_n) be two convergent sequences with $a_n \to a$ and $b_n \to b$, as $n \to \infty$. Give an (ϵ, N)-proof that $(a_n + b_n)$ is also convergent with $a_n + b_n \to a + b$.

SOLUTION:

(a) $x_n = \dfrac{(-1)^n}{n}$.

(b) $\exists\, \epsilon > 0 \quad \forall\, N \in \mathbb{N} \quad \exists\, n \geq N : \qquad |x_n - x^*| \geq \epsilon.$

(c) Let $\epsilon > 0$ be given. Since $a_n \to a$, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \forall\, n \geq N_1.$$

Since $b_n \to b$, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\epsilon}{2} \quad \forall\, n \geq N_2.$$

Therefore, we have for all $n \geq N = \max(N_1, N_2) \in \mathbb{N}$, using the triangle inequality,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that $a_n + b_n \to a + b$.

3. Calculate the following expressions explicitly.

   (a) (10 marks.) $f^{-1}(\,[0,1)\,)$ for $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin x$.

   (b) (15 marks.) $\inf(g)$ and $\sup(g)$ for $g:\mathbb{R}\to\mathbb{R}$ with $g(x)=(2+3x^2\sin x)/(3+x^2)$.

SOLUTION:

(a) We have

$$f^{-1}(\,[0,1)\,)=\{x\in\mathbb{R}\mid 0\le\sin x<1\}=\bigcup_{k\in\mathbb{Z}}\Big(\,\big[2k\pi,2k\pi+\pi/2\big)\cup\big(2k\pi+\pi/2,(2k+1)\pi\big]\,\Big).$$

(b) We have for all $x\in\mathbb{R}$,

$$-3\le -3+\frac{11}{3+x^2}=\frac{2-3x^2}{3+x^2}\le g(x)\le\frac{2+3x^2}{3+x^2}=3-\frac{7}{3+x^2}\le 3.$$

So the guess is $\inf(g)=-3$ and $\sup(g)=3$. Choosing $x_n=\pi/2+2n\pi$, we have

$$\lim_{n\to\infty}g(x_n)=\lim_{n\to\infty}\frac{2+3(\pi/2+2n\pi)^2}{3+(\pi/2+2n\pi)^2}=\lim_{n\to\infty}\frac{3+2/(\pi/2+2n\pi)^2}{1+3/(\pi/2+2n\pi)^2}=3,$$

and therefore $\sup(g)=3$. Similarly, choosing $y_n=3\pi/2+2n\pi$, we have

$$\lim_{n\to\infty}g(y_n)=\lim_{n\to\infty}\frac{2-3(3\pi/2+2n\pi)^2}{3+(3\pi/2+2n\pi)^2}=\lim_{n\to\infty}\frac{-3+2/(3\pi/2+2n\pi)^2}{1+3/(3\pi/2+2n\pi)^2}=-3,$$

and therefore $\inf(g)=3$.

4. (a) (5 marks.) State the Bolzano-Weierstrass Theorem.

   (b) (5 marks.) Let $a, b \in \mathbb{R}$ be two real numbers with $a < b$. Give the contrapositive statement to

   $$A: \qquad \text{If } f : [a, b] \to \mathbb{R} \text{ is a continuous function, then } f \text{ is bounded.}$$

   (c) (5 marks.) Give a reformulation of the unboundedness of a function $f : [a, b] \to \mathbb{R}$ in terms of a sequence (x_n) with particular properties.

   (d) (10 marks.) Use (c) and Bolzano-Weierstrass to prove the contrapositive statement to A.

SOLUTION:

(a) Every bounded real sequence has a convergent subsequence.

(b) The contrapositive statement to A is "If $f : [a, b] \to \mathbb{R}$ is unbounded, then f is not continuous."

(c) $f : [a, b] \to \mathbb{R}$ is unbounded means that there exists a sequence $x_n \in [a, b]$ with $|f(x_n)| \geq n$ for all $n \in \mathbb{N}$.

(d) Let $f : [a, b] \to \mathbb{R}$ be unbounded. Then there exists a sequence $x_n \in [a, b]$ with $|f(x_n)| \geq n$ for all $n \in \mathbb{N}$. Since $a \leq x_n \leq b$, the sequence (x_n) is bounded and has, by Bolzano-Weierstrass, a convergent subsequence (x_{n_j}). Let $x^* = \lim_{j \to \infty} x_{n_j} \in [a, b]$. Since $y_j = |f(x_{n_j})| \geq n_j \geq j$ for all $j \in \mathbb{N}$, we see that (y_j) is unbounded and cannot be convergent to $f(x^*) \in \mathbb{R}$. But this means that f cannot be continuous at x^*.