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Algebraic Geometry III/IV

Solutions, set 6.

Exercise 9. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a non-singular cubic. As mentioned in the exercise, we can projectively transform C into C_F with

$$F(X,Y,Z) = (\alpha X + \beta Y + \gamma Z)YZ + G(X,Z),$$

where $\beta \neq 0$ and

$$G(X,Z) = aX^3 + bX^2Z + dXZ^2 + gZ^3$$

with $a \neq 0$. Combining both expressions and substituting $c = \alpha$, $e = \beta$ and $f = \gamma$, we end up with

$$F(X,Y,Z) = aX^3 + bX^2Z + cXYZ + dXZ^2 + eY^2Z + fYZ^2 + gZ^3,$$

where $a \neq 0$ and $e \neq 0$.

(a) The substitution yields

$$aX^3 + bX^2Z + cX(Y - \frac{c}{2e}X - \frac{f}{2e}Z)Z + dXZ^2 + e(Y - \frac{c}{2e}X - \frac{f}{2e}Z)^2Z$$
$$+ f(Y - \frac{c}{2e}X - \frac{f}{2e}Z)Z^2 + gZ^3 = aX^3 + bX^2Z + cXYZ - \frac{c^2}{2e}X^2Z - \frac{cf}{2e}XZ^2$$
$$+dXZ^2+eY^2Z+\frac{c^2}{4e}X^2Z+\frac{f^2}{4e}Z^3-cXYZ-fYZ^2+\frac{cf}{2e}XZ^2+fYZ^2-\frac{cf}{2e}XZ^2$$
$$-\frac{f^2}{2e}Z^3 + gZ^3 = aX^3 + b'X^2Z + d'XZ^2 + eY^2Z + g'Z^3,$$

with the same non-zero coefficients a and e of X^3 and Y^2Z.

(b) The substitution yields

$$a'(X - \frac{b'}{3a'}Z)^3 + b'(X - \frac{b'}{3a'}Z)^2Z + d'(X - \frac{b'}{3a'}Z)Z^2 + e'Y^2Z + g'Z^3$$
$$= a'X^3 + (-3a'\frac{b'}{3a'} + b')X^2Z + d''XZ^2 + e'Y^2Z + g''Z^3$$
$$= a'X^3 + d''XZ^2 + e'Y^2Z + g''Z^3,$$

with the same non-zero coefficients a' and e' of X^3 and Y^2Z.

(c) Since $e'' \neq 0$, we can arrange $e'' = 1$ by substituting Z by $\frac{1}{e''}Z$. Since $a'' \neq 0$, we can arrange $a'' = -4$ by rescaling X appropriately. So we end up with

$$Y^2Z = 4X^3 - d''XZ^2 - g''Z^3.$$

Choosing $g_2 = d''$ and $g_3 = g''$, we obtain the desired *Weierstraß normal form*.

(d) Assume that $C = C_F$ with $F(X,Y,Z) = Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3$ is a singular cubic. Then there is a point $P = [a,b,c] \in \mathbb{P}^2_{\mathbb{C}}$ with $F(P) = F_X(P) = F_Y(P) + F_Z(P) = 0$. We have

$$\begin{aligned}
F(X,Y,Z) &= Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3, \\
F_X(X,Y,Z) &= -12X^2 + g_2Z^2, \\
F_Y(X,Y,Z) &= 2YZ, \\
F_Z(X,Y,Z) &= Y^2 + 2g_2XZ + 3g_3Z^2.
\end{aligned}$$

From $F_Y(P) = 0$ we conclude that $b = 0$ or $c = 0$. If $c = 0$, we conclude from $F_X(P) = 0$ that $a = 0$ and from $F_Z(P) = 0$ that also $b = 0$, which is a contradiction. Therefore, we must have $c \neq 0$ and $b = 0$. From $b = 0$ and $F_Z(P) = 0$, we conclude that $2g_2ac + 3g_3c^2 = 0$. Since $c \neq 0$, this implies that $2g_2a + 3g_3c = 0$. We conclude from $0 = F_X(P) = -12a^2 + g_2c^2$ that

$$0 = -12g_2^2a^2 + g_2^3c^2 = -3(2g_2a)^2 + g_2^3c^2 = -3(-3g_3c)^2 + g_2^3c^2 = (-27g_3^2 + g_2^3)c^2.$$

Since $c \neq 0$, we finally conclude that $g_2^3 - 27g_3^2 = 0$.

Conversely, let $g_2^3 - 27g_3^2 = 0$. In the case $(g_2, g_3) = (0,0)$, we choose $P = [0,0,1] \in \mathbb{P}^2_{\mathbb{C}}$, and we see that $F(P) = F_X(P) = F_Y(P) = F_Z(P) = 0$, i.e., C_F is singular. If $(g_2,g_3) \neq (0,0)$, we choose $P = [3g_3, 0, -2g_2] \in \mathbb{P}^2_{\mathbb{C}}$. Then we have

$$\begin{aligned}
F(3g_3,0,-2g_2) &= -108g_3^3 + 12g_2^3g_3 - 8g_2^3g_3 \\
&= 4g_3(g_2^3 - 27g_3^2) = 0, \\
F_X(3g_3,0,-2g_2) &= -108g_3^2 + 4g_2^3 = 4(g_2^3 - 27g_3^2) = 0, \\
F_Y(3g_3,0,-2g_2) &= 0, \\
F_Z(3g_3,0,-2g_2) &= -12g_2^2g_3 + 12g_2^2g_3 = 0.
\end{aligned}$$

This shows that C_F is singular.