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Michaelmas 2025

Topics in Combinatorics IV, Michaelmas Term[1]

1 Catalan numbers

1.1 Definitions

Definition 1.1 (one of many)**.** The n-th Catalan number C_n is the number of sequences $(\varepsilon_1, \dots, \varepsilon_{2n})$ with $\varepsilon_i = \pm 1$ such that

- $\sum_{i=1}^{2n} \varepsilon_i = 0$;
- $\sum_{i=1}^{k} \varepsilon_i \geq 0$ for every $k \leq 2n$.

Remark. Sequences in the definition above are called *ballot sequences.*

Example 1.2. $n = 2$: the only sequences of length $2n = 4$ are $(1, 1, -1, -1)$ and $(1, -1, 1, -1)$, so $C_2 = 2$.

$n = 3$: there are five sequences (list them!), so $C_3 = 5$.

Two equivalent definitions of C_n:

- the number of "bracketings" of a non-associative product of $n + 1$ variables;
- the number of triangulations of a convex $(n + 2)$-gon on a plane (here by a triangulation we mean a maximal collection of non-crossing diagonals, it automatically subdivides the polygon into triangles).

Example 1.3. For $n = 3$, there are precisely five bracketings:

$$((a_1a_2)a_3)a_4 \quad (a_1(a_2a_3))a_4 \quad (a_1a_2)(a_3a_4) \quad a_1((a_2a_3)a_4) \quad a_1(a_2(a_3a_4)).$$

There are also precisely five triangulations of a pentagon: two non-crossing diagonals share a vertex, and there are five vertices to choose from.

Exercise 1.4. Show that the two definitions above are equivalent to Definition 1.1.

Ballot sequences can be represented by *Dyck paths*: these are paths in a $n \times n$ square going along the grid from one corner (say, $(0, n)$) to the opposite (i.e., $(n, 0)$) and staying above the main diagonal. The bijection with ballot sequences is obvious: $\varepsilon_i = 1$ becomes a move to the right, and $\varepsilon_i = -1$ becomes a move down.

One can draw the top right half of the square only, in this case a Dyck path can be understood as a path from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ never going below x-axis. This is the model we will usually use in the sequel.

Remark. Paths whose all steps are vectors of $\mathbb{Z}^d$ are called *lattice paths.* The two models above for Dyck paths sit in $\mathbb{Z}^2$ generated, respectively, by $(1, 0), (0, -1)$, and $(1, 1), (1, -1)$.

[1]Based on previous versions by Pavel Tumarkin and Alex Postnikov.

Example 1.5.

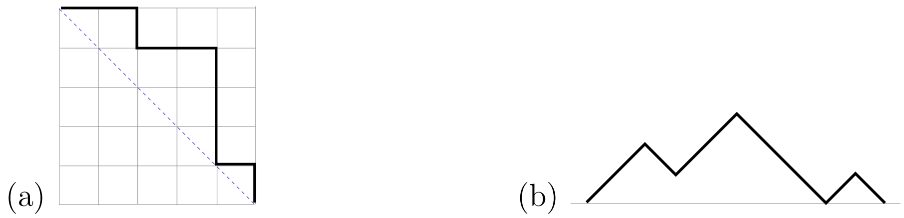


Figure 1: Dyck paths with steps (a) $(1,0)$ and $(0,-1)$ and (b) $(1,1)$ and $(1,-1)$.

1.2 Explicit formula for C_n

Theorem 1.6. $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

Example 1.7. $C_3 = \frac{1}{4}\binom{6}{3} = \frac{1}{4}\frac{6!}{3!3!} = \frac{6\cdot5\cdot4}{4\cdot6} = 5$; $\quad C_4 = \frac{1}{5}\binom{8}{4} = \frac{1}{5}\frac{8!}{4!4!} = \frac{8\cdot7\cdot6\cdot5}{5\cdot8\cdot3} = 14$; $\quad C_5 = \frac{1}{6}\binom{10}{5} = 42$.

We will look at three different proofs of the theorem.

Proof 1: by reflection. The number of all lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ is equal to $\binom{2n}{n}$ – choose n steps "up" out of $2n$ steps. We will compute the number of ones which go below the x-axis (call them *bad*) and subtract.

Observe that a path is bad if and only if it intersects the line $y=-1$. Find the first point where it touches the line $y=-1$, and reflect the part of the path to the right of this point with respect to the line $y=-1$. We get a new lattice path, it goes from $(0,0)$ to $(2n,-2)$. We now claim that this map establishes a bijection between *bad* paths from $(0,0)$ to $(2n,0)$ and *all* lattice paths from $(0,0)$ to $(2n,-2)$. Indeed, this map is injective, and there is an inverse: take any lattice path from $(0,0)$ to $(2n,-2)$, take the first point where it touches the line $y=-1$, and reflect the right part – we get a bad path.

Now, the paths from $(0,0)$ to $(2n,-2)$ contain $n-1$ steps up and $n+1$ steps down, so the number of all paths from $(0,0)$ to $(2n,-2)$ is $\binom{2n}{n+1}$. Thus,

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{n!n!} - \frac{n}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}. \qquad \square$$

Proof 2: by cyclic shifts. First, observe that

$$\frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{2n+1}\frac{(2n+1)!}{n!(n+1)!} = \frac{1}{2n+1}\binom{2n+1}{n}.$$

Interpret $\binom{2n+1}{n}$ as all lattice paths from $(0,0)$ to $(2n+1,-1)$, i.e. paths with n steps up and $n+1$ steps down. Then C_n is the number of those paths (call them *good*) that go below the x-axis at the last step only. We want to show that good paths constitute precisely $1/(2n+1)$ part of all paths.

Let us switch to sequences of ±1. Take any sequence $(\varepsilon_1,\ldots,\varepsilon_{2n+1})$ of length $2n+1$ adding up to -1, and consider all its $2n+1$ *cyclic shifts*:

$$(\varepsilon_1,\ldots,\varepsilon_{2n+1}),\ (\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{2n+1},\varepsilon_1),\ (\varepsilon_3,\ldots,\varepsilon_{2n+1},\varepsilon_1,\varepsilon_2),\ldots,(\varepsilon_{2n+1},\varepsilon_1,\ldots,\varepsilon_{2n}).$$

Now the proof follows from the following statement which we leave as an exercise

Exercise. (1) Show that all cyclic shifts are distinct.
(2) Show that out of $2n+1$ shifts of one sequence precisely one is good. (Hint: consider the leftmost lowest point of the path). □

Definition 1.8. Let $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of non-negative integers. A formal power series $\sum\limits_{k=0}^{\infty} a_k x^k$ is called a *generating function* of (a_n).

Example 1.9. Let $a_n = 1$ for every n. The the generating function is $A(x) = 1+x+\cdots = 1/(1-x)$.

In general, how to find a closed formula for the generating function $A(x)$ of a sequence (a_n)? One can follow the following plan:

- write a recurrence relation on a_n;
- interpret the recurrence relation as an equation on $A(x)$;
- solve the equation and get an explicit expression for a_n.

We will proceed along this plan to get a *third proof* of the theorem.

Lemma 1.10 (Recurrence on Catalan numbers)**.**

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0 = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

Proof. Take any Dyck path of length $2n$, and consider the first point $2k > 0$ where the path touches the x-axis. Then on the right there is a Dyck path of length $2n-2k = 2(n-k)$. On the left there is a Dyck path of length $2k$ which stays strictly above x-axis, which means that we can think of it as a Dyck path of length $2k-2$ between points $(1,1)$ and $(2k-1,1)$. Both Dyck paths on the left and on the right are arbitrary, so there are precisely $C_{k-1}\cdot C_{n-k}$ Dyck paths which touch x-axis for the first time at $2k$, and the result follows. □

Defining $C_0 = 1$, we can now recursively compute any Catalan number.

Example 1.11.

$C_1 = C_0\cdot C_0 = 1;\quad C_2 = C_0\cdot C_1 + C_1\cdot C_0 = 1+1 = 2;\quad C_3 = C_0\cdot C_2 + C_1\cdot C_1 + C_2\cdot C_0 = 2+1+2 = 5.$

Lemma 1.12. *The generating function $C(x)$ satisfies the equation $xC(x)^2 - C(x) + 1 = 0$.*

Proof. We know that $C_n = \sum\limits_{k=1}^{n} C_{k-1}C_{n-k}$. Multiplying by x^n, we get

$$C_nx^n = \sum_{k=1}^{n} C_{k-1}C_{n-k}x^n = x\sum_{k=1}^{n}(C_{k-1}x^{k-1})(C_{n-k}x^{n-k})$$

Now, summing on $n > 0$, we get on the left $\sum\limits_{n=1}^{\infty} C_nx^n = C(x) - C_0 = C(x) - 1$. On the right, we get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} x(C_{k-1}x^{k-1})(C_{n-k}x^{n-k}) = x\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(C_ix^i)(C_{n-i-1}x^{n-i-1})$$
$$= x\sum_{m=0}^{\infty}\sum_{i=0}^{m}(C_ix^i)(C_{m-i}x^{m-i}) = x\cdot C(x)\cdot C(x),$$

so we obtain the equation $C(x) - 1 = xC(x)^2$, which is precisely what we wanted. □

Lemma 1.13.

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

Proof. Solving quadratic equation with respect to $C(x)$ (while considering x as a parameter), we see that $C(x) = \frac{1\pm\sqrt{1-4x}}{2x}$. The reason to choose the sign is the following: we want $\lim_{x\to 0} C(x) = C(0)$, where we already know $C(0) = C_0 = 1$. Now, if we consider the positive sign, then $\lim_{x\to 0} C(x) = \infty$, while the negative sign gives the required limit. □

We are left to extract the explicit expression for C_n from the generating function. For this, we recall the definition of a generalization of binomial coefficients.

Definition 1.14. Given $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, the *binomial coefficient* $\binom{\alpha}{k}$ is given by

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}$$

Exercise. Show that

$$(1+y)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} y^k$$

Proof 3: via generating function. We now apply the exercise above to $\alpha = 1/2$, $y = -4x$. We get

$$\sqrt{1-4x} = \sum_{k\geq 0} \binom{\frac{1}{2}}{k}(-4x)^k = 1 + \binom{\frac{1}{2}}{1}(-4x) + \binom{\frac{1}{2}}{2}(-4x)^2 + \dots,$$

so

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = -\frac{1}{2}\left(\binom{\frac{1}{2}}{1}(-4) + \binom{\frac{1}{2}}{2}(-4)^2 x + \dots\right) = -\frac{1}{2}\left(\sum_{n\geq 0}\binom{\frac{1}{2}}{n+1}(-4)^{n+1}x^n\right).$$

Therefore,

$$C_n = -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1} = -\frac{1}{2}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-n)}{(n+1)!}(-4)^{n+1}$$

$$= -\frac{1}{2}\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2n-1}{2})}{(n+1)!}(-4)^{n+1} = \frac{1\cdot 3\cdot\dots\cdot(2n-1)}{(n+1)!}\frac{2^{2n+2}}{2^{n+2}} = \frac{(2n)!}{(2^n n!)(n+1)!}\cdot 2^n = \frac{1}{n+1}\binom{2n}{n}$$

as required. □

1.3 Further examples and Applications

Example 1.15 (Drunkard's walk)**.** A drunkard walks randomly along a line in steps of length 1 to the left or right with probability 1/2. The starting point is $x = 1$. The walk is terminated when the drunkard reaches the point $x = 0$. Question: what is the probability the walk terminates?

Reaching $x = 0$ for the first time on step $2k+1$ is equivalent to staying greater than or equal to $x = 1$ for the first $2k$ steps, returning to $x = 1$ on step $2k$ step and then going left, so this can be understood as a Dyck path. The probability of any individual path is $(1/2)^{2k+1}$, the number of such paths of length $2k+1$ is clearly C_k. Thus,

$$P = \sum_{k=0}^{\infty} C_k \frac{1}{2^{2k+1}} = \frac{1}{2}\sum_{k=0}^{\infty} C_k \frac{1}{2^{2k}} = \frac{1}{2}\sum_{k=0}^{\infty} C_k \left(\frac{1}{4}\right)^k = \frac{1}{2}C\left(\frac{1}{4}\right) = \frac{1}{2}\frac{1-\sqrt{1-4\frac{1}{4}}}{2\frac{1}{4}} = 1$$

We list below some further interpretations of Catalan numbers C_n.

- The number of *plane binary trees* with n vertices.

  A plane binary tree can be defined recursively: if not empty, it has a root vertex, a left subtree, and a right subtree, both of which are binary trees (either of them can be empty). When drawing such trees, the root is drawn on the top, with an edge drawn from it to the root of each of its subtrees. All plane binary trees with 3 vertices are shown in Fig. 2.

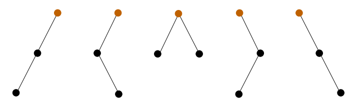


Figure 2: Plane binary trees with 3 vertices

- The number of *complete plane binary* trees with $n+1$ leaves.

  A plane binary tree is complete if it has no vertices of valence 2 (except for the root which is of valence 2), i.e., for every vertex both left and right subtrees are either simultaneously nonempty or empty (in the latter case the vertices are called *leaves*). All complete binary trees with 4 leaves are shown in Fig. 3.

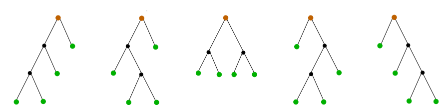


Figure 3: Complete plane binary trees with 4 leaves

- The number of *plane trees* with $n+1$ vertices.

  A plane tree can also be defined recursively: it has a root vertex, and, in the case the whole tree is not a single vertex, it has a sequence $(T_1, \ldots, T_k)$ of subtrees T_i, $1 \leq i \leq k$, each of which is also a plane tree. In particular, the subtrees of each vertex are ordered; when drawing such trees, the subtrees are drawn from left to right. The root is on the top, with an edge drawn from it to the root of each of its subtrees. All plane trees with 4 vertices are shown in Fig. 4.

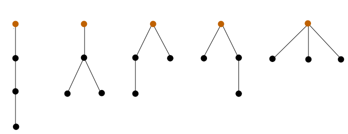


Figure 4: Plane trees with 4 vertices

· The number of *non-crossing matchings* on $2n$ vertices.

A matching on $2n$ vertices can be drawn as a way to connect $2n$ nodes on the x-axis by arcs, with every node connected to precisely one other node by an arc drawn in the upper halfplane. A matching is *non-crossing* if all arcs can be drawn in a way such that no pair of arcs intersects.

All non-crossing matchings on 6 vertices are shown in Fig. 5.

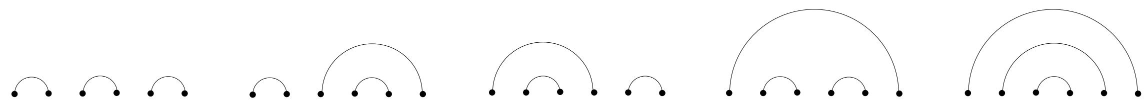


Figure 5: Non-crossing matchings on 6 vertices

· The number of *non-nesting matchings* on $2n$ vertices.

A matching is *non-nesting* if all the arcs can be drawn in a way such that no arc is above the other, or, equivalently, there is no pair of arcs with ends at points (a_1, b_1) and (a_2, b_2) such that $a_1 < a_2 < b_2 < b_1$.

All non-nesting matchings on 6 vertices are shown in Fig. 6.

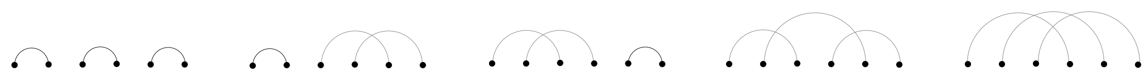


Figure 6: Non-nesting matchings on 6 vertices

2 Partitions and Young Diagrams

2.1 Definitions

Definition 2.1. A *partition* of $n \in \mathbb{N}$ is a sequence of integers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$. If $(\lambda_1, \dots, \lambda_k)$ is a partition of n, we denote $(\lambda_1, \dots, \lambda_k) \vdash n$.

Example. $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$.

Graphically, a partition can be represented by a *Young diagram* which is a left-justified array of boxes arranged in k rows, such that i-th row contains λ_i boxes, see Fig. 7, left, for an example.

Definition 2.2. A *standard Young tableau* (SYT for short) is a filling of a Young diagram of shape $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ by numbers $1, \dots n$ such that each number appears precisely once, and entries increase in rows (to the right) and in columns (downwards).

See Fig. 7, right, for an example.

Example 2.3. Let us compute the number of SYT of shape (n, n). One can assign to any such SYT a Dyck path: put a step up at every place appearing in the first row, and a step down at every place appearing in the second row. It is easy to see that this is a bijection, so the number of SYT of shape (n, n) is just the Catalan number C_n.

Denote by f_λ the number of SYT of shape λ (for example, $f_{(n,n)} = C_n$ as we have seen above). Our next goal is to obtain a formula for f_λ.

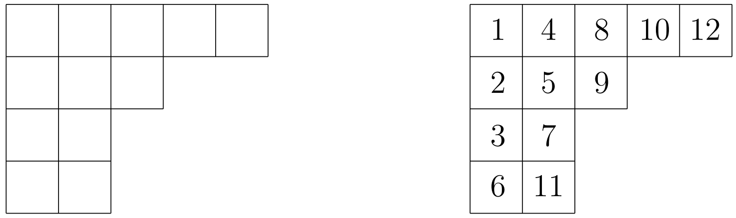


Figure 7: Young diagram of the partition $\lambda = (5, 3, 2, 2) \vdash 12$ and a standard Young tableau of shape λ

2.2 The Hook Length Formula

Definition 2.4. Let (i, j) be a box in a Young diagram λ. The *hook* of (i, j) is the union of all boxes in i-th row to the right of (i, j) and all boxes in j-th column to the bottom of (i, j), i.e., $\{(i, b) \mid b \geq j\} \cup \{(a, j) \mid a \geq i\}$. The *hook length* $h(i, j)$ is the number of boxes in the hook of (i, j).

An example is shown in Fig. 8.

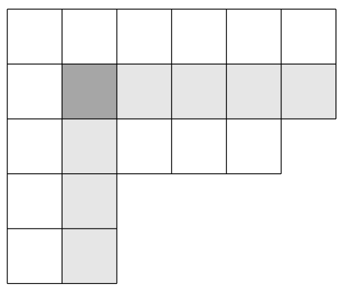


Figure 8: The hook of box $(2, 2)$ is shaded, $h(2, 2) = 8$

Denote $H(\lambda) = \prod_{(i,j)\in\lambda} h(i, j)$.

Theorem 2.5 (The Hook Length Formula)**.** *For every partition $\lambda \vdash n$, $f_\lambda = \frac{n!}{H(\lambda)}$.*

Example 2.6.

- Let $\lambda = (4, 2, 2) \vdash 8$. Then $H(\lambda) = (6 \cdot 5 \cdot 2 \cdot 1) \cdot (3 \cdot 2) \cdot (2 \cdot 1) = 6!$, so $f_\lambda = \frac{8!}{6!} = 56$.

- Let $\lambda = (n, n)$. Then $h(1, j) = n - j + 2$ and $h(2, j) = n - j + 1$. Therefore,

$$H(\lambda) = ((n + 1) \cdot \ldots \cdot 2) \cdot (n \cdot \ldots \cdot 1) = (n + 1)!n!,$$

  and thus

$$f_{(n,n)} = \frac{(2n)!}{(n + 1)!n!} = C_n$$

Let us first prove a recurrence relation on f_λ. Let λ be a Young diagram, and let c be a *corner* of λ, i.e. a box which does not have neighbours from right and bottom. Denote by $\lambda - c$ a Young diagram obtained from λ by removing the box c.

Lemma 2.7. *Define* $f_\emptyset = 1$. *Then*

$$f_\lambda = \sum_{c \text{ is a corner of } \lambda} f_{\lambda - c}$$

Proof. Observe that n is always in the corner of any SYT of shape $\lambda \vdash n$. If c is a corner of $\lambda \vdash n$, there is a clear bijection between SYT of shape λ with n located at box c and SYT of shape $\lambda - c \vdash (n-1)$. Therefore,

$$f_\lambda = \sum_{c \text{ is a corner of } \lambda} (\text{number of SYT of shape } \lambda \text{ with } n \text{ at } c) = \sum_{c \text{ is a corner of } \lambda} f_{\lambda - c}$$

as required.

□

To prove the Hook Length Formula, we will use induction on the size of the Young diagram: the induction step will be to show that the numbers $n!/H(\lambda)$ satisfy the same recursion as in Lemma 2.7. The base is easy: for a one-box Young diagram, all numbers in the equation are equal to 1. Thus, we need to prove the following.

Lemma 2.8. *For every Young diagram* $\lambda \vdash n$, *we have*

$$\sum_{c \text{ is a corner of } \lambda} \frac{(n-1)!}{H(\lambda - c)} = \frac{n!}{H(\lambda)}.$$

Theorem 2.5 then follows: once we prove Lemma 2.8, we have

$$f_\lambda = \sum_{c \text{ is a corner of } \lambda} f_{\lambda - c} = \sum_{c \text{ is a corner of } \lambda} \frac{(n-1)!}{H(\lambda - c)} = \frac{n!}{H(\lambda)},$$

where the first equality follows from Lemma 2.7, the second one is the induction assumption (applied to every corner of λ), and the last one is Lemma 2.8.

Observe that the equality in Lemma 2.8 can be reformulated as

$$\sum_{c \text{ is a corner of } \lambda} \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)} = 1$$

Therefore, we can try to interpret the summands in the left hand side as probabilities of some events. In other words, we will proceed according to the following plan: given $\lambda \vdash n$, find a random process, such that the space of outcomes is the set of corners of λ, and the probability of the outcome c is precisely $\frac{1}{n}\frac{H(\lambda)}{H(\lambda-c)}$. Then the sum of probabilities is automatically equal to 1, and thus we get the statement of the lemma.

The random process we will use is called a *hook walk* and is defined as follows.

(1) Choose uniformly at random a box b_1 in λ; the probability of this is $p(b_1) = 1/n$.

(2) Choose uniformly at random a box $b_2 \neq b_1$ in the hook of b_1; the probability of this is $1/(h(b_1) - 1)$.

(3) Repeat Step (2) until reach a corner.

Denote by $P(b,c)$ the probability of the walk starting at a box b ends in a corner c. The probability of an individual hook walk $b = b_1 \to b_2 \to \cdots \to b_k = c$ is equal to

$$\frac{1}{h(b_1)-1} \cdots \frac{1}{h(b_{k-1})-1},$$

which implies that

$$P(b,c) = \sum_{\text{all walks } b=b_1\to b_2\to\cdots\to b_k=c} \frac{1}{h(b_1)-1} \cdots \frac{1}{h(b_{k-1})-1}$$

Define

$$P(c) = \frac{1}{n} \sum_{b\in\lambda} P(b,c),$$

then $\sum_{c \text{ is a corner of } \lambda} P(c) = 1$. Therefore, we need to prove that

$$P(c) = \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)},$$

or, equivalently,

$$\sum_{b\in\lambda} P(b,c) = \frac{H(\lambda)}{H(\lambda - c)}$$

Example 2.9. Let $\lambda = (3,2,1) \vdash 6$. There are four hook paths ending in the corner located in the first row (denote it by c_1), same for the corner c_3 in the third row, and five hook paths ending in the remaining corner c_2. Then $P(c_1) = P(c_3) = 5/16$, and $P(c_2) = 3/8$.

Let us make some observations.

- Let $c = (u,v)$ be a corner. Then any hook walk ending in c is contained in the rectangle with vertices at $(1,1),(1,v),(u,1)$ and (u,v).

- If $p \le u, q \le v$, then $h(p,q) + h(u,v) = h(p,v) + h(u,q)$. If $c = (u,v)$ is a corner, then $h(u,v) = 1$, so this implies the equality

  $$h(p,q) - 1 = (h(p,v) - 1) + (h(u,q) - 1)$$

  Define a *weight* $\mathrm{wt}\,(p,q)$ of a box $(p,q) \ne (u,v)$ by

  $$\mathrm{wt}\,(p,q) = \frac{1}{h(p,q)-1}$$

  Then, if we denote $1/x = \mathrm{wt}\,(u,q)$ and $1/y = \mathrm{wt}\,(p,v)$, the equality above says that $\mathrm{wt}\,(p,q) = 1/(x+y)$. Then the weights of the boxes in the rectangle with vertices at $(1,1),(1,v),(1,u)$ and (u,v) are determined by the weights of boxes that are either in row u and to the left of box (u,v) or in column v and above box (u,v).

- Setting the weight of the corner box $\mathrm{wt}\,(u,v) = 1$, define, for any path, its weight as the product of weights of all its boxes. Then we can write $P(b,c)$ as the sum of weights of all hook walks from b to c.

To complete the proof, we need further two technical lemmas.

Lemma 2.10. *Consider all lattice paths in a rectangle $(l+1)\times(k+1)$ from the top left box to the bottom right box. Denote*

$$\frac{1}{x_j} = \mathrm{wt}\,(l+1,j), j=1,\ldots,k, \qquad \frac{1}{y_i} = \mathrm{wt}\,(i,k+1), i=1,\ldots,l, \qquad \mathrm{wt}\,(k+1,l+1)=1.$$

Then

$$\sum_{\gamma \textit{ is a lattice path}} \mathrm{wt}\,(\gamma) = \frac{1}{x_1\ldots x_k y_1 \ldots y_l}$$

Example. Let $l=1, k=2$, so we consider paths in a 2×3 rectangle, see Fig. 9. There are

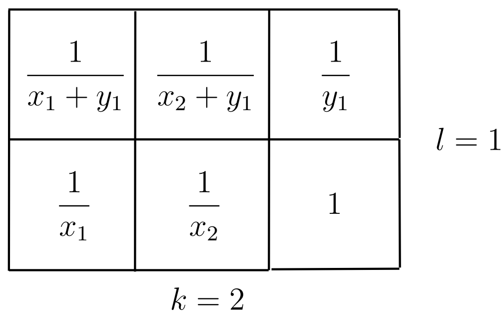


Figure 9: Weights of boxes in a 2×3 rectangle

$\binom{k+l}{k} = \binom{3}{2} = 3$ lattice paths from top left box to the bottom right one. Then we can compute the sum of weights of the three paths:

$$\sum_{\gamma \text{ is a lattice path}} \mathrm{wt}\,(\gamma) = \frac{1}{x_1+y_1}\cdot\frac{1}{x_2+y_1}\cdot\frac{1}{y_1}\cdot 1 + \frac{1}{x_1+y_1}\cdot\frac{1}{x_2+y_1}\cdot\frac{1}{x_2}\cdot 1 + \frac{1}{x_1+y_1}\cdot\frac{1}{x_1}\cdot\frac{1}{x_2}\cdot 1 = \frac{1}{x_1x_2y_1}$$

Proof of Lemma 2.10. We use induction on $(k,l) : k\geq 0,\ l\geq 0$ (this is valid since $\{(k,l) : k\geq 0,\ l\geq 0\}$ is in bijection with the natural numbers.) If either $k=0$ or $l=0$, then the rectangle either has a single row or single column and the sum has a single term ($\frac{1}{x_1\ldots x_k}$ if $l=0$, or $\frac{1}{y_1\ldots y_l}$ if $k=0$). To prove the induction step, observe that for $k\geq 1,\ l\geq 1$ every path starts either from a horizontal step or from a vertical one, i.e., the second box b_2 in the path has coordinates $(1,2)$ or $(2,1)$ respectively. In both cases the remaining part of the path lies in a smaller rectangle, so we may use the induction assumption. More precisely, we have

$$\begin{aligned}\sum_{\gamma \text{ is a lattice path}} \mathrm{wt}\,(\gamma) &= \sum_{\gamma\,|\,b_2=(2,1)} \mathrm{wt}\,(\gamma) + \sum_{\gamma\,|\,b_2=(1,2)} \mathrm{wt}\,(\gamma)\\ &= \frac{1}{x_1+y_1}\frac{1}{x_1\ldots x_k y_2\ldots y_l} + \frac{1}{x_1+y_1}\frac{1}{x_2\ldots x_k y_1\ldots y_l} = \frac{1}{x_1+y_1}\frac{1}{x_2\ldots x_k y_2\ldots y_l}\frac{x_1+y_1}{x_1y_1}\\ &= \frac{1}{x_1\ldots x_k y_1\ldots y_l}\end{aligned}$$

□

In the next lemma we will consider hook walks, i.e., we are allowed to miss steps in a lattice path.

Lemma 2.11. *Consider all hook walks in a rectangle $(l+1)\times(k+1)$ ending in the bottom right box. Define x_j and y_i as in Lemma 2.10. Then*

$$\sum_{\gamma \text{ is a hook walk}} \mathrm{wt}\,(\gamma) = \left(1+\frac{1}{x_1}\right)\dots\left(1+\frac{1}{x_k}\right)\left(1+\frac{1}{y_1}\right)\dots\left(1+\frac{1}{y_l}\right)$$

Proof. The proof follows from an easy observation: all hook walks are precisely lattice paths is some subrectangles (i.e., a rectangle constructed from boxes lying in the intersection of some of rows and some of columns of the initial rectangle). Now, choosing 1 or $1/x_j$ in the product is equivalent to the column j to be either absent or present in the subrectangle, respectively. The same holds for choosing 1 or $1/y_i$ and presence of row i. Then every such term has the form as in Lemma 2.10.

□

We can now complete the proof of Lemma 2.8. Recall that the corner c has coordinates (u,v). By Lemma 2.11, we have

$$\begin{aligned}\sum_{b\in\lambda} P(b,c) &= \sum_{\gamma \text{ is a hook walk}} \mathrm{wt}\,(\gamma)\\ &= \left(1+\frac{1}{x_1}\right)\dots\left(1+\frac{1}{x_k}\right)\left(1+\frac{1}{y_1}\right)\dots\left(1+\frac{1}{y_l}\right) = \prod_{\substack{b=(u,j)\;|\;j<v\\ b=(i,v)\;|\;i<u}} \left(1+\frac{1}{h(b)-1}\right)\\ &= \prod_{\substack{b=(u,j)\;|\;j<v\\ b=(i,v)\;|\;i<u}} \left(\frac{h(b)}{h(b)-1}\right) = \frac{H(\lambda)}{H(\lambda-c)},\end{aligned}$$

where the last equality holds since the only boxes for which the hook lengths differ in λ and $\lambda-c$ are precisely those located at row u or column v, and their hook lengths in $\lambda-c$ are one less than in λ.

2.3 Set partitions

In the previous sections we considered "unlabelled" partitions. We will now consider the "labelled" version.

Denote by $[n]$ the set of integers $1,\dots,n$.

Definition 2.12. A *set partition* of $[n]$ is a subdivision of $[n]$ into a disjoint union of non-empty subsets (*blocks*).

Example 2.13. Let $n=7$, we can write $[7]=\{1,7\}\cup\{2,3,5\}\cup\{4,6\}$ – this is one of partitions of shape $(3,2,2)\vdash 7$. Notation: if we call this partition π, we write $\pi=(1\,7\mid 2\,3\,5\mid 4\,6)$ (note that the order of blocks does not matter).

Graphically, we can draw set partitions as arc diagrams: vertices correspond to set elements $1,\dots,n$, consecutive elements in one block are joined by an arc, see Fig. 10, left.

The arc diagrams representing set partitions are characterized as follows: every vertex is incident to at most one arc from the left and at most one arc from the right.

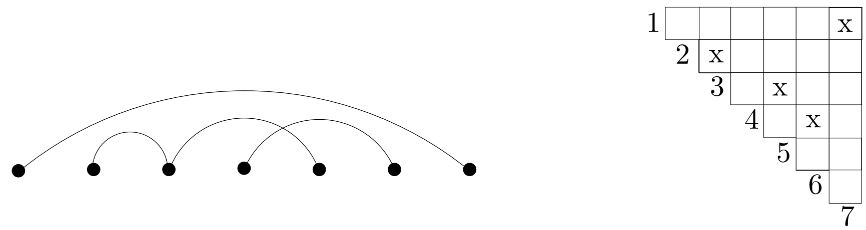


Figure 10: Partition $(1\,7 \mid 2\,3\,5 \mid 4\,6)$ of $[7]$: arc diagram and rook placement

Example 2.14. Another graphical interpretation of set partitions is a *non-attacking rook placement*. If we have a partition of $[n]$, consider a chess "half-board" obtained from the square $n \times n$ by taking all boxes (i, j) with $i < j$. Then place a rook in the box (i, j) if the corresponding arc diagram contains an arc connecting i and j, see Fig. 10, right.

Exercise. Show that rooks placed as described above do not attack each other. Show that this map has an inverse: any non-attacking rook placement gives rise to a set partition.

Note that the number of rooks (which is also the number of arcs) is equal to $n - k$, where k is the number of blocks.

Definition 2.15. The number of set partitions of $[n]$ is called *Bell number* $B(n)$. The number of set partitions of $[n]$ into k blocks is called *Stirling number of second kind* $S(n, k)$, i.e., $B(n) = \sum_{k=1}^{n} S(n, k)$.

Example 2.16. There are five set partitions of $[3]$, namely

$$(1 \mid 2 \mid 3), \quad (1\,2 \mid 3), \quad (1 \mid 2\,3), \quad (1\,3 \mid 2), \quad (1\,2\,3),$$

so $B(3) = 5$, $S(3, 1) = S(3, 3) = 1$, $S(3, 2) = 3$.

Definition 2.17. A set partition is *non-crossing* if the arc diagram is non-crossing. A set partition is *non-nesting* if the arc diagram is non-nesting.

Example 2.18. For $n = 1, 2, 3$ all set partitions are non-crossing and non-nesting. For $n = 4$, there is precisely one set partition which is not non-crossing and one which is not non-nesting.

Exercise. Compute $B(4)$ by listing all arc diagrams.

Note that arc diagrams of non-crossing (non-nesting) partitions are not the same as of non-crossing (non-nesting) matchings. However, the following result holds.

Theorem 2.19. *The number of non-crossing set partitions of $[n]$ is equal to the number of non-nesting set partitions of $[n]$ and is equal to the Catalan number C_n.*

Definition 2.20. Given a Dyck path, a *peak* is a local maximum (i.e., a step up followed by a step down), and a *valley* is a local minimum (i.e., a step down followed by a step up). Clearly, the number of peaks exceeds the number of valleys by one.

Proof of Theorem 2.19. First we construct a bijection between non-nesting set partitions of $[n]$ and Dyck paths of length $2n$. A partition is not non-nesting if and only if there are two arcs $(a_i, b_i), i = 1, 2$, such that $a_1 < a_2 < b_2 < b_1$. This is equivalent to the rook (a_1, b_1) in the corresponding rook placement being located in the positive quadrant with respect to the origin at (a_2, b_2) (here by "positive" we mean standard coordinates in $\mathbb{R}^2$). Therefore, the equivalent criterion for a set partition being non-nesting is the following: the positive quadrant with respect to every rook does not contain any other rook.

Now we can construct the map: take a set partition, consider the corresponding rook placement, draw the union of all positive quadrants centered at rooks, and then the boundary of this domain will be a Dyck path. The inverse map is constructed as follows: put rooks in all valleys of a Dyck path.

Given a Dyck path, a non-crossing partition can be constructed via a "shelling algorithm", see Fig. 11 for an example.

□

Figure 11: (a) non-nesting partition (1 3 5 6 | 2 4 | 7) of [7]: rook placement and Dyck path; (b) non-crossing partition (1 7 | 2 3 | 4 6 | 5) of [7] and its Dyck path

Exercise 2.21. Fill in the details of the proof.

Definition 2.22. *Narayana number* $N(n, k)$ is the number of Dyck paths of length $2n$ with k peaks.

Corollary 2.23 (of the proof of Thm. 2.19)**.** *The number of non-nesting partitions of* $[n]$ *with* k *blocks is equal to* $N(n, n-(k-1))$.

Example 2.24. Let us compute $N(4, 2)$, i.e., the number of Dyck paths of length 8 with 2 peaks. By Cor. 2.23, $N(4, 2)$ is equal to the number of non-nesting partitions of [4] with 3 blocks, which, in its turn, is equal to the number of rook placements, where the number of rooks is equal to $4 - 3 = 1$. The number of boxes in the upper half of the 4×4 board is equal to 6, so there are 6 ways to place one rook, and thus $N(4, 2) = 6$.

2.4 Generating functions

Recall: if we have a sequence (a_n), we can define a generating function of (a_n), which is a formal power series $a_0 + a_1 x + a_2 x^2 + \dots$. This is an *ordinary* generating function. We can also define an *exponential* generating function.

Definition 2.25. *Exponential generating function* of a sequence $(a_n)_{n\geq 0}$ is a formal power series $\sum\limits_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

Usually, ordinary generating functions are used when the objects are unlabelled, and exponential generating functions are used when the objects are labelled.

Example 2.26. The number p_n counts *integer partitions* of n, i.e., the number of Young diagrams $\lambda \vdash n$. Bell number $B(n)$ counts partitions of $[n]$. Bell numbers are "labelled versions" of $p(n)$, so it is reasonable to look for an ordinary generating function for (p_n) and an exponential generating function for (B_n).

Our next goal is to write the generating functions for $p(n)$ and $B(n)$.

2.4.1 Integer partitions

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ be a partition. We can write it as $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \ldots$, where m_i is the number of $\lambda_j = i$.

Example. $\lambda = (4, 2, 1, 1, 1) \vdash 9$ can be written as $\lambda = 1^3 2^1 3^0 4^1$ (formally, there are infinitely many zero powers at the end, but we omit all factors r^0 for $r > \lambda_1$).

The notation above provides a bijection between all partitions of natural numbers and all sequences $(m_i)_{i>0}$ of non-negative integers with finite number of positive entries. Given a sequence (m_i), it corresponds to a partition of $n = 1 \cdot m_1 + 2 \cdot m_2 + 3 \cdot m_3 \ldots$. If $\lambda \vdash n$, denote $|\lambda| = n$.

Then the (ordinary) generating function of $p(n)$ is

$$\begin{aligned}\sum_{n\geq 0} p(n)x^n &= \sum_{n\geq 0} \underbrace{x^n + x^n + \cdots + x^n}_{p(n) \text{ summands}} = \sum_{n\geq 0,\, \lambda\vdash n} x^{|\lambda|} = \sum_{\lambda \text{ any partition}} x^{|\lambda|} \\ &= \sum_{m_1, m_2, \cdots \geq 0} x^{1\cdot m_1 + 2\cdot m_2 + 3\cdot m_3 + \ldots} = \left(\sum_{m_1\geq 0} x^{m_1}\right)\left(\sum_{m_2\geq 0} x^{2m_2}\right)\left(\sum_{m_3\geq 0} x^{3m_3}\right)\cdots \\ &= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots = \prod_{k\geq 1}\frac{1}{1-x^k}\end{aligned}$$

Note that although the product is infinite, the number of terms contributing to any coefficient is finite.

2.4.2 The exponential formula

Let numbers c_n count some objects on n labelled nodes (call them *c-objects*), where $n > 0$. Let $c(x) = \sum_{n\geq 1} c_n \frac{x^n}{n!}$ be the exponential generating function of (c_n).

Define *d-objects* on n nodes as collections of c-objects on $n_1, \ldots, n_k$ nodes such that $n_1 + \cdots + n_k = n$. Let $d(x) = \sum_{n\geq 0} d_n \frac{x^n}{n!}$ be the exponential generating function of (d_n).

Example 2.27. Let c-objects be linear graphs on n nodes (or, equivalently, sequences of n distinct numbers $1, \ldots, n$). Then $c_n = n!$, and thus $c(x) = x + x^2 + \cdots = \frac{x}{1-x}$.

Then d-objects are graphs on n nodes, such that any connected component is a linear graph (or, equivalently, unions of sequences of total length n). How to compute $d(x)$?

Lemma 2.28 (Exponential formula)**.**

$$d(x) = e^{c(x)}$$

Proof. A d-object on n nodes consists of a set partition of $[n]$, $n = n_1 + \cdots + n_k$, and then a c-object on every block. Let us fix k and $n_1, \ldots, n_k$, and compute d-objects.

The number of ways to subdivide n into k blocks of size $n_1, \dots, n_k$ is equal to the multinomial coefficient

$$\binom{n}{n_1\, n_2\, \dots\, n_k} = \frac{n!}{n_1!\, n_2!\, \dots\, n_k!}$$

We have a c-object on every block, so the number of d-objects with fixed k and $n_1, \dots, n_k$ is equal to

$$\frac{1}{k!}\binom{n}{n_1\, n_2\, \dots\, n_k} c_{n_1} \dots c_{n_k},$$

where we divide by $k!$ as we are not interested in the order of the blocks.

Now take a sum over $k \geq 0$ and $n_1 + \dots + n_k = n$:

$$d_n = \sum_{k \geq 0} \sum_{\substack{n_1 + \dots + n_k = n \\ n_1, \dots, n_k \geq 1}} \frac{1}{k!} \frac{n!}{n_1!\, n_2!\, \dots\, n_k!} c_{n_1} \dots c_{n_k},$$

which implies that

$$\frac{d_n}{n!} = \sum_{k \geq 0} \sum_{\substack{n_1 + \dots + n_k = n \\ n_1, \dots, n_k \geq 1}} \frac{1}{k!} \frac{c_{n_1}}{n_1!} \frac{c_{n_2}}{n_2!} \dots \frac{c_{n_k}}{n_k!},$$

and thus

$$d(x) = \sum_{n \geq 0} d_n \frac{x^n}{n!} = \sum_{n \geq 0} \sum_{k \geq 0} \sum_{\substack{n_1 + \dots + n_k = n \\ n_1, \dots, n_k \geq 1}} \frac{1}{k!} \frac{c_{n_1}}{n_1!} \frac{c_{n_2}}{n_2!} \dots \frac{c_{n_k}}{n_k!} x^n =$$

$$= \sum_{k \geq 0} \frac{1}{k!} \sum_{n_1, \dots, n_k \geq 1} \frac{c_{n_1} x^{n_1}}{n_1!} \frac{c_{n_2} x^{n_2}}{n_2!} \dots \frac{c_{n_k} x^{n_k}}{n_k!} = \sum_{k \geq 0} \frac{1}{k!} \left(\sum_{m \geq 1} \frac{c_m x^m}{m!} \right)^k = \sum_{k \geq 0} \frac{1}{k!} c(x)^k = e^{c(x)}.$$

□

Corollary 2.29. *The exponential generating function for Bell numbers is*

$$B(x) = e^{e^x - 1}$$

Proof. Let c-objects be just sets (so, $c_n = 1$), and d-objects be collections of sets of total cardinality n. Then d-objects are precisely partitions of $[n]$. Since $c_n = 1$, we have $c(x) = e^x - 1$, and thus $B(x) = d(x) = e^{e^x - 1}$ by the exponential formula.

□

3 Permutations

3.1 Definitions and notation

Recall that a *permutation* is a bijection $w : [n] \to [n]$. Permutations form a group (called *symmetric group*) S_n with respect to composition, the order of the group is $n!$.

There are several notations for $w \in S_n$.

- 2-*line notation*: if $w(i) = w_i$, we write

$$w = \begin{pmatrix} 1 & 2 & \dots & n \\ w_1 & w_2 & \dots & w_n \end{pmatrix}, \qquad \text{e.g.} \qquad w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 6 & 1 & 5 & 7 & 4 \end{pmatrix} \in S_7;$$

- 1-*line notation*: we simply write $w = w_1, w_2, \dots, w_n$, or even $w = w_1\, w_2 \dots w_n$, e.g. $w = 3\,2\,6\,1\,5\,7\,4$;

- *cycle notation*: recall that every permutation can be decomposed as a product of disjoint cycles. For the permutation above, we have $w = (1\ 3\ 6\ 7\ 4)(2)(5)$. Note that the order of cycles is irrelevant, as well as the starting point of each cycle: we could also write $w = (2)(6\ 7\ 4\ 1\ 3)(5)$.

3.2 Statistics on permutations

Definition 3.1. A *statistic* on permutations is a function $\sigma : S_n \to \mathbb{Z}_{\geq 0}$. Its *generating function* is a polynomial $f_\sigma(x) = \sum\limits_{w \in S_n} x^{\sigma(w)}$.

Two statistics σ and μ are *equidistributed* if $f_\sigma(x) = f_\mu(x)$, i.e., the number of permutations with the same value of statistics is the same.

Definition 3.2. Let $w \in S_n$.

- An *inversion* is a pair (i, j) such that $i < j$, $w_i > w_j$;

- A *descent* is $i \in [n-1]$ such that $w_{i+1} < w_i$.

We can now look at some statistics on S_n.

- $\operatorname{inv}(w)$ is the number of inversions in w;

- $\operatorname{des}(w)$ is the number of descents in w;

- $\operatorname{cyc}(w)$ is the number of cycles in w.

Example 3.3. Let $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$. Then $\operatorname{inv}(w) = 8$, $\operatorname{des}(w) = 3$, $\operatorname{cyc}(w) = 3$.

Definition 3.4. A statistic on S_n is called *Mahonian* if it is equidistributed with inv, and *Eulerian* if it is equidistributed with des.

Example 3.5. Let $n = 3$, order permutations by the number of inversions. Then we have the following.

w	123	132	213	231	312	321
$\operatorname{inv}(w)$	0	1	1	2	2	3
$\operatorname{des}(w)$	0	1	1	1	1	2
$\operatorname{cyc}(w)$	3	2	2	1	1	2

In particular, we can see that all three statistics are different.

We can also compute the generating functions:

$$\begin{aligned} f_{\text{inv}}(x) &= 1 \cdot x^0 + 2 \cdot x^1 + 2 \cdot x^2 + 1 \cdot x^3 = (1+x)(1+x+x^2) \\ f_{\text{des}}(x) &= 1 \cdot x^0 + 4 \cdot x^1 + 1 \cdot x^2 = 1 + 4x + x^2 \\ f_{\text{cyc}}(x) &= 0 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 1 \cdot x^3 = x(x+1)(x+2) \end{aligned}$$

Theorem 3.6. *For every $n \in \mathbb{N}$,*

$$f_{\text{inv}}(x) = (1+x)(1+x+x^2)\dots(1+x+\dots+x^{n-1}) = \frac{\prod_{k=1}^{n}(1-x^k)}{(1-x)^n}$$

Proof. We use induction on n. For $n=1$ the result is clear (as for $n=2,3$). Assume that the theorem holds for $n-1$. Take any permutation $w' \in S_{n-1}$ and "insert" n in all possible n places. Depending on the place where n is inserted, we add from 0 to $n-1$ inversions, where all numbers show up. Therefore, the generating function is

$$f_{\text{inv}}(x) = \sum_{w \in S_n} x^{\text{inv}(w)} = \sum_{w' \in S_{n-1}} x^{\text{inv}(w')} + x \sum_{w' \in S_{n-1}} x^{\text{inv}(w')} + x^2 \sum_{w' \in S_{n-1}} x^{\text{inv}(w')} + \dots + x^{n-1} \sum_{w' \in S_{n-1}} x^{\text{inv}(w')} =$$

$$= \left(\sum_{w' \in S_{n-1}} x^{\text{inv}(w')} \right) (1+x+\dots+x^{n-1}) = (1+x)(1+x+x^2)\dots(1+x+\dots+x^{n-1}),$$

where the last equality follows from the induction assumption.

□

Definition 3.7. The *Major index* of $w \in S_n$ is $\text{maj}(w) = \sum_{k \text{ is a descent of } w} k$.

Example 3.8. For $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$, we have $\text{maj}(w) = 1+3+6 = 10$.

For $n=3$, we have

w	1 2 3	1 3 2	2 1 3	2 3 1	3 1 2	3 2 1
$\text{maj}(w)$	0	2	1	2	1	3

and thus $f_{\text{maj}}(x) = 1 + 2x + 2x^2 + x^3 = (1+x)(1+x+x^2) = f_{\text{inv}}(x)$.

Exercise. Show that $f_{\text{maj}}(x) = f_{\text{inv}}(x)$ for every $n \in \mathbb{N}$, i.e., maj is Mahonian.

Definition 3.9. Let $w = w_1\, w_2\, \dots\, w_n \in S_n$, w_i is a *record* of w if $w_i > w_j$ for all $j < i$. Denote by $\text{rec}(w)$ the number of records of w.

Example. for $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$, we have $\text{rec}(w) = 3$.

For $n=3$, we have

w	1 2 3	1 3 2	2 1 3	2 3 1	3 1 2	3 2 1
$\text{rec}(w)$	3	2	2	2	1	1

and thus $f_{\text{rec}}(x) = 2x + 3x^2 + x^3 = f_{\text{cyc}}(x)$.

Theorem 3.10. $f_{\text{rec}}(x) = f_{\text{cyc}}(x)$ *for every* $n \in \mathbb{N}$.

Proof. We construct a bijection $g : S_n \to S_n$, such that if $w' = g(w)$ then $\text{rec}(w') = \text{cyc}(w)$.

First, write the cycle decomposition in a standard way: if $w = (a_1 \dots)(a_2 \dots)\dots(a_k \dots)$ then every a_i is maximal in its cycle, and $a_1 < a_2 < \dots < a_k$.

Now, the map g takes w written in the standard way above to w' by erasing all brackets (and considering the result as a 1-line notation for w'). Clearly, records are first elements of cycles, so $\text{rec}(w') = \text{cyc}(w)$.

The map g has an inverse: take any permutation w' in 1-line notation, put brackets in the beginning and at the end, and then put closing and opening brackets before every record. We get a cycle decomposition of some permutation w. Then $g(w) = w'$.

□

Example 3.11. The preimage of $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$ under g is $u = (3\,2)(6\,1\,5)(7\,4)$.

Now, $w = 3\,2\,6\,1\,5\,7\,4 = (1\,3\,6\,7\,4)(2)(5) = (2)(5)(7\,4\,1\,3\,6)$, so $g(w) = 2\,5\,7\,4\,1\,3\,6$.

Definition 3.12. Let $w = w_1\, w_2\, \ldots\, w_n \in S_n$, $i \in [n]$ is an *excedance* of w if $i < w_i$. Denote by $\operatorname{exc}(w)$ the number of excedances of w.

Example. For $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$, we have $\operatorname{exc}(w) = 3$.

For $n = 3$, we have

w	123	132	213	231	312	321
$\operatorname{exc}(w)$	0	1	1	2	1	1

and thus $f_{\text{exc}}(x) = 1 + 4x + x^2 = f_{\text{des}}(x)$.

Theorem 3.13. *$f_{\text{exc}}(x) = f_{\text{des}}(x)$ for every $n \in \mathbb{N}$, so exc is Eulerian.*

Proof. Define *anti-excedance* of w as $i \in [n]$ such that $i > w_i$. Observe, that anti-excedances of w are precisely excedances of w^{-1}, so anti-excedances and excedances are equidistributed.

Now, we claim that the map g from the proof of Thm 3.10 takes anti-excedances of w to descents of $w' = g(w)$.

□

Exercise. Complete the proof of the theorem.

4 Posets and lattices

4.1 Definitions

Definition 4.1. A *partially ordered set*, or *poset*, P is a set with a binary relation $\leq$ (or $\leq_P$ if there is an ambiguity) satisfying the following axioms:

- $a \leq a$ (reflexivity);
- if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity);
- if $a \leq b$ and $b \leq a$ then $a = b$ (symmetry).

We say $a < b$ if $a \leq b$ and $a \neq b$, and the notation $a \geq b$ means that $b \leq a$.

Two elements are *incomparable* if $a \not\leq b$ and $b \not\leq a$.

Example 4.2. Any ordered field (e.g. $\mathbb{R}$) is a poset, where one considers usual order relation. (Such posets are *totally ordered*, i.e., they do not contain incomparable elements).

The power set of any set (i.e., the set of all subsets), with the order being inclusion, is also a poset (see below). Two subsets are incomparable if neither of them is a subset of the other.

Definition 4.3. A *covering relation* $\lessdot$ in a poset P is defined as follows: $a \lessdot b$ if $a < b$ and there is no element $c \in P$ such that $a < c < b$.

Definition 4.4. A *Hasse diagram* of a poset P is an oriented graph, where vertices are elements of P, and there is an edge $a \to b$ if $a \lessdot b$. Hasse diagram is usually drawn as an undirected graph with b positioned above a if $a \lessdot b$.

Note that any finite poset is uniquely defined by its Hasse diagram.

Definition 4.5. A *chain* in a poset P is a sequence of elements $a_1 < a_2 < \cdots < a_k$. A chain is *saturated* if $a_i \lessdot a_{i+1}$ for every $i = 1, \ldots, k-1$. An *antichain* is a subset of P consisting of mutually incomparable elements.

Example 4.6. The Hasse diagram shown in Fig. 12 defines a poset. We have $x \leq y$ if and only if there is a path from x to y every step of which goes up. For example, $a \leq c, d, e, f$ and is not comparable to b. Overall, there are five pairs of incomparable elements. The set $\{c, f\}$ is an antichain, and the sequences (a, d, f) and (b, e) are chains.

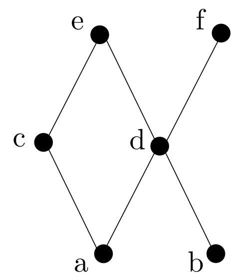


Figure 12: Hasse diagram of a poset.

Definition 4.7. A *Boolean lattice* B_n is the poset of subsets of $[n]$ ordered by inclusion: $A \leq B$ if $A \subseteq B$. The set of subsets (or the power set) of $[n]$ is denoted by $2^{[n]}$.

Example 4.8. Hasse diagrams for B_2 and B_3 are shown in Fig. 13. Note that for every n the Hasse diagram for B_n is a 1-skeleton of a cube.

We are interested in the following questions: what are the maximal sizes of chains and antichains in posets? For example, in the poset defined in Fig. 12, these are 3 and 2 respectively. In B_n, the maximal size of chain is $n+1$, but what is the maximal size of an antichain?

4.2 Sperner's Theorem

In this section we will answer the question above by proving the following result.

Theorem 4.9 (Sperner's Theorem)**.** *Let $S_1, \ldots, S_N$ be different subsets of $[n]$ such that $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$ for any $i \neq j$. Then $N \leq \binom{n}{\lfloor n/2 \rfloor}$, where $\lfloor x \rfloor$ is the maximal integer not exceeding x.*

Figure 13: Hasse diagram of B_2 and B_3.

Remark. Note that the bound in the theorem is sharp: the number of $\lfloor n/2 \rfloor$-subsets of $[n]$ is precisely $\binom{n}{\lfloor n/2 \rfloor}$, and they are all incomparable.

Definition 4.10. A poset P is *ranked* if there is a function $\rho : P \to \mathbb{Z}$ such that $a \lessdot b$ implies $\rho(b) = \rho(a) + 1$. If P is finite, we assume that the minimal rank is zero. If P is finite and ranked, the *rank numbers* r_i are defined as the numbers of elements of P of rank i.

A finite ranked poset P with non-zero rank numbers $r_0, \dots, r_l$ is: *rank-symmetric* if $r_i = r_{l-i}$, for all $i = 0, \dots, l$; *unimodal* if there is k such that $r_0 \le r_1 \le \dots \le r_k \ge r_{k+1} \ge \dots \ge r_l$; and *Sperner* if the maximal size of antichain is the maximum of r_i (i.e., the maximal size of antichain is as small as possible).

Roughly speaking, a poset is ranked if its Hasse diagram can be drawn "on levels".

Example 4.11. The left poset shown in Fig. 14 is ranked, and the right is not.

Figure 14: Ranked (left) and not ranked (right) posets.

Example 4.12. The Hasse diagram in Fig. 15 defines a non-Sperner poset: both rank numbers are four, but there is an antichain of size five.

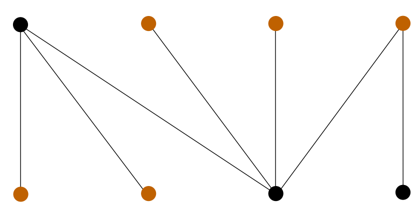


Figure 15: Non-Sperner rank-symmetric poset. Coloured vertices form an antichain of size 5.

Remark. The Boolean lattice B_n is ranked, where $\rho(A) = |A|$, with rank numbers $r_i = r_i(B_n) = \binom{n}{i}$. Since $\binom{n}{i} = \binom{n}{n-i}$, B_n is rank-symmetric. It is also unimodal as

$$\binom{n}{0} \le \binom{n}{1} \le \dots \le \binom{n}{\lfloor n/2 \rfloor} \ge \dots \ge \binom{n}{n}$$

Then Theorem 4.9 can be reformulated as: B_n is Sperner.

Definition 4.13. Let P be a finite ranked poset, ρ is the rank function with values $0, \dots, l$. A *symmetric chain decomposition* (or SCD for short) of P is a decomposition of P as a disjoint union $P = C_1 \sqcup C_2 \sqcup \dots \sqcup C_k$, such that each C_i is a saturated chain, and if $C_i = a_0^i \lessdot a_1^i \lessdot \dots \lessdot a_s^i$ then $\rho(a_0^i) + \rho(a_s^i) = l$.

Example. B_2 has a SCD: $C_1 = \emptyset \lessdot \{1\} \lessdot \{1,2\}$ and $C_2 = \{1\}$.

Lemma 4.14. *If a finite ranked poset P has a SCD, then it is rank-symmetric, unimodal and Sperner.*

Proof. P is clearly rank-symmetric and unimodal, where the maximal rank number is $r_{\lfloor l/2 \rfloor}$, as this property holds for every chain in the SCD. We need to show that P is Sperner.

Take any antichain A, then $|A \cap C_i| \leq 1$ for $i = 1, \dots, k$, so $|A| = \sum_{i=1}^{k} |A \cap C_i| \leq k$. Since every chain C_i intersects the middle level, the number of chains k equals $r_{\lfloor l/2 \rfloor} = \max_i r_i$, so the size of any antichain does not exceed the maximal rank number, which is precisely the definition of Sperner poset. □

Due to Lemma 4.14, to prove Thm 4.9 we are left to show that B_n has a SCD.

Definition 4.15. Let P_1, P_2 be posets, then $P_1 \times P_2$ is also a poset: we can define the order by $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. Similarly, one can define a product $P_1 \times \cdots \times P_n$ of any number of posets: $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ iff $a_i \leq b_i$ for all $i = 1, \dots, n$.

Observe that $B_n = [2]^n = [2] \times \cdots \times [2]$. Indeed, we can assign to every element A of B_n a sequence of 0 and 1 of length n, where $a_i = 0$ if $i \notin A$ and $a_i = 1$ if $i \in A$. Then $A \subseteq B$ is equivalent to $a_i \leq b_i$ for all i, and thus to $A \leq B$.

We are now left to prove the following statement.

Theorem 4.16. *Any product of chains has a SCD.*

Proof. We proceed by induction. First, a product of two chains has a SCD (prove this!). Now take $P = P' \times C = (C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k) \times C$. As a product of two chains has a SCD, every $C_i \times C$ has a SCD. Note that the "middle rank" of all $C_i \times C$ is the same – this is guaranteed by the fact $P' = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$ is a SCD. Thus, taking the union of all SCD for $C_i \times C$, $i \in [k]$, we obtain a SCD for P.

□

4.3 Lattices

Definition 4.17. Let P be a poset, $x, y \in P$.

- $z \in P$ is a *join* of x and y (notation $z = x \vee y$) if the following hold:

  (1) $z \geq x, y$;

  (2) if $v \in P$ and $v \geq x, y$ then $v \geq z$.

  Note that (2) implies that if join exists then it is unique.

- $w \in P$ is a *meet* of x and y (notation $w = x \wedge y$) if the following hold:

  (1) $w \leq x, y$;

  (2) if $v \in P$ and $v \leq x, y$ then $v \leq w$.

  Again, meet is unique if exists.

- P is a *lattice* if every two elements have a join and a meet.

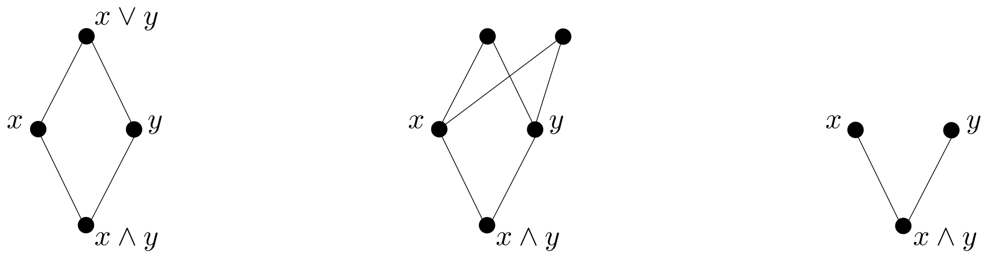


Figure 16: The poset on the left is a lattice, the other two are not.

Example. Out of the three posets whose Hasse diagrams shown in Fig. 16 only the left one is a lattice. In the other two posets elements x, y have no join.

Note that if $x \leq y$, then $x \vee y = y$ and $x \wedge y = x$.

Exercise. Every finite lattice has a unique *minimal* and a unique *maximal* element (i.e., elements c and d such that $c \leq x \leq d$ for all x).

Lemma 4.18. *Boolean lattice B_n is indeed a lattice.*

Proof. Observe that if $X, Y \in B_n$, then $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$. □

Example 4.19. *Young lattice* $\mathbb{Y}$ is the poset of all Young diagrams ordered by inclusion. The covering relation in $\mathbb{Y}$ is defined as follows: $\lambda \lessdot \mu$ if μ has precisely one extra box.

Exercise. Check that union and intersection of Young diagrams is again a Young diagram. Show that $\mathbb{Y}$ is a lattice.

Example. *Partition lattice* Π_n consists of all set partitions of $[n]$ ordered by refinement: $\lambda \leq \mu$ if μ is obtained by joining together some blocks of λ. The covering relation is then following: $\lambda \lessdot \mu$ is μ is obtained by combining two blocks of λ.

Exercise. Show that Π_n is a lattice.

Definition 4.20. Let P be a poset. An *order ideal* of P is a set $I \subseteq P$ such that if $x \in I$ and $y \leq x$ then $y \in I$. Define an *order* on order ideals: $I \leq J$ if $I \subseteq J$. This defines a *poset of order ideals* $J(P)$.

Example. A poset and its poset of order ideals are shown in Fig. 17.

Lemma 4.21. *$J(P)$ is a lattice.*

Proof. Since the order is defined as in the Boolean lattice, meet and join are the intersection and the union (which are also order ideals – check this!). □

Example 4.22.
- Consider a poset on $\mathbb{Z}_{\geq 0}$ with usual order. Then $J(\mathbb{Z}_{\geq 0}) \cong \mathbb{Z}_{\geq 0} \cup \{\emptyset\} \cong \mathbb{Z}_{\geq 0}$ (a non-empty order ideal corresponds to its maximal element; we should also add the empty set below all other order ideals).
- $J(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}) \cong \mathbb{Y}$ (Young lattice) – here we also need to add infinite Young diagrams.
- We can also interpret B_n as $J([n])$, where all elements of $[n]$ are incomparable.

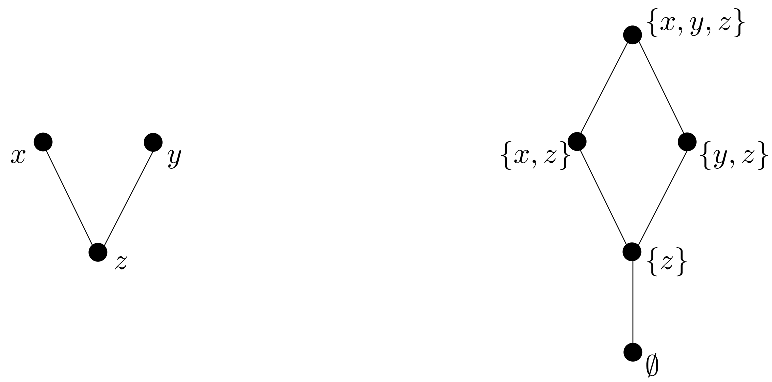


Figure 17: Hasse diagrams of a poset (left) and of its poset of order ideals (right).

Definition 4.23. A lattice is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \text{and} \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Exercise. $J(P)$ is a distributive lattice.

Theorem 4.24 (Fundamental Theorem of Finite Distributive Lattices, Birkhoff)**.** *Any finite distributive lattice is a poset of order ideals for some finite poset.*

Proof. For L a lattice, we call $z \in L$ *join-irreducible* if $z \neq x \vee y$ for all $x < z$ and $y < z$, and z is not a minimal element of L.

Now, let L be a finite distributive lattice. Define P to be the poset of all join-irreducible elements of L (with the order inherited from L). We will prove that $L \cong J(P)$.

Take $x \in L$, and consider $I_x = \{y \in P \mid y \leq x\} \subseteq P$. Clearly, $I_x \in J(P)$: if $y \in I_x$, $z \in P$ and $z \leq y$ then $z \leq y \leq x$ and thus $z \in I_x$.

Thus, we constructed a map from L to $J(P)$ taking $x \in L$ to $I_x \in J(P)$. It is clear for the definition of I_x that $x \leq y$ iff $I_x \subseteq I_y$. We need to show that this map is injective and surjective. Injectivity follows from the following exercise:

Exercise. For every $x \in L$, $x = \bigvee\{t \mid t \in I_x\}$.

Indeed, the Exercise implies that if $I_x = I_y$ and $x \neq y$, then the same set have two different joins, which is impossible in a lattice.

To show the surjectivity, we will prove the following claim.

Claim. Let $I \in J(P)$, take $x = \bigvee\{t \mid t \in I\}$. Then $I = I_x$.

The claim explicitly states that the map $x \mapsto I_x$ is surjective: for every order ideal of P we find a preimage. Therefore, we are left to prove the claim. □

Proof of the claim. First, $I \subseteq I_x$: since $x = \bigvee\{t \mid t \in I\}$, $t \leq x$ for all $t \in I$, and thus every $t \in I$ also lies in I_x by the definition of I_x. We are left to prove that $I_x \subseteq I$.

Take any $s \in I_x$, we want to show that $s \in I$. Take the meet $s \wedge x$, and expand both sides of the definition of x above by the distributive law. Then we get

$$x \wedge s = \bigvee\{t \wedge s \mid t \in I\}$$

Since $s \in I_x$, we have $s \le x$ by the definition of I_x, and thus $x \wedge s = s$. Therefore, the RHS of the above equality is equal to s. Since $s \in I_x$, s is join-irreducible, which means that at least one of the elements of the set $\{t \wedge s \mid t \in I\}$ must be equal to s. But $s \wedge t = s$ implies $s \le t$, and since I is an order ideal, $s \le t \in I$ implies $s \in I$.

□

Example. Let λ be a Young diagram. Define a lattice $\mathbb{Y}_\lambda$ consisting of all Young diagrams that fit inside λ (with the order inherited from $\mathbb{Y}$). Then $\mathbb{Y}_\lambda$ is a finite distributive lattice (check this!). Join-irreducible elements of $\mathbb{Y}_\lambda$ are Young diagrams with precisely one corner, i.e., rectangles. Define P_λ as a poset on boxes of λ, with $(i,j) \le (k,l)$ if $i \le k$ and $j \le l$. Then $J(P_\lambda) = \mathbb{Y}_\lambda$ (check this!).

4.4 Linear extensions of posets

Let P be a finite poset, $|P| = n$.

Definition 4.25. A *linear extension* of a poset P is a bijective map $f : P \to [n]$ such that $x \le_P y$ implies $f(x) \le f(y)$. Denote by $\operatorname{ext}(P)$ the number of linear extensions of P.

Example. Let $P = P_\lambda$, where $\lambda = (2,2) \vdash 4$. Then linear extensions can be identified with SYT of shape λ. In particular, $\operatorname{ext}(P) = 2$.

Lemma 4.26. *Let P be a finite poset, and let $J(P)$ be its poset of order ideals. Then $\operatorname{ext}(P)$ is equal to the number of maximal saturated chains in $J(P)$.*

Corollary 4.27. *The number f_λ of SYT of shape λ is equal to the number of saturated chains from $\emptyset$ to λ in $\mathbb{Y}_\lambda$.*

Proof of Lemma 4.26. We want to construct a bijection between extensions of P and maximal saturated chains in $J(P)$. Let $\varphi : P \to [n]$ be an extension, $j \in [n]$. Define an order ideal $I_j = \varphi^{-1}([j])$ (it is indeed an order ideal: if $a \le b \in I_j$, then $\varphi(a) \le \varphi(b) \le j$, and thus $\varphi(a) \le j$, so $a \in I_j$), then we get a saturated chain $\emptyset \lessdot I_1 \lessdot \ldots \lessdot I_n = P$.

Exercise. Show that this map is a bijection.

□

4.5 Greene's Theorem

(This section was only briefly covered in lectures.)

Let P be a finite poset.

Theorem 4.28 (Dilworth)**.** *The maximal size of an antichain of P is equal to the minimal number of chains needed to cover P.*

Remark. Note that one inequality in the theorem is evident: an antichain cannot contain more elements than the number of chains as it intersect every chain at most once.

A dual version also holds.

Theorem 4.29 (Mirsky)**.** *The maximal size of a chain of P is equal to the minimal number of antichains needed to cover P.*

More generally, define l_k to be the maximal size of a union of k chains of P, and m_k to be the maximal size of a union of k antichains of P. In particular, $l_0 = m_0 = 0$, and l_1 is the maximal size of a chain in P.

Theorem 4.30 (Greene). *Denote $\lambda_i = l_i - l_{i-1}$ for $i > 0$, and let $\lambda(P) = \lambda_1, \lambda_2, \dots$ (note that only finitely many λ_i are positive). Similarly, denote $\mu_i = m_i - m_{i-1}$ for $i > 0$, and let $\mu(p) = \mu_1, \mu_2, \dots$. Then $\lambda_i \geq \lambda_{i+1}$ and $\mu_i \geq \mu_{i+1}$ for all $i > 0$ (so λ and μ can be considered as Young diagrams or partitions of $|P|$), and Young diagrams λ and μ are* conjugate *to each other, i.e., they are symmetric with respect to the main diagonal.*

Example. Let P be defined by the Hasse diagram shown in Fig. 18. Then $l_1 = 3$, $l_2 = 5$, and $l_i = 5$ for $i \geq 2$. Also, $m_1 = 2$, $m_2 = 4$, $m_3 = 5$, and $m_i = 5$ for $i \geq 3$. Therefore, $\lambda = (3, 2)$, and $\mu = (2, 2, 1)$ which are clearly conjugated.

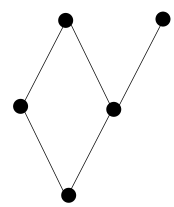


Figure 18: Hasse diagram of a poset.

Remark. Theorems of Dilworth and Mirsky are partial cases of Greene's Theorem. Indeed, the maximal size of antichain is $m_1 = \mu_1$, i.e., the length of the first row of μ. The minimal number of chains needed to cover P is precisely the number of non-zero λ_i, i.e., the number of rows of λ. Greene's Theorem says that Young diagrams λ and μ conjugate, which implies that the first row of μ is equal to the first column of λ, and this is precisely Dilworth's Theorem. Mirsky's Theorem follows similarly.

Given $w \in S_n$, define a poset P_w as follows: elements of P_w are elements of $[n]$, and $w_i <_{P_w} w_j$ if $w_i < w_j$ and $i < j$.

Example. Let $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$. Hasse diagram of P_w is shown in Fig. 19.

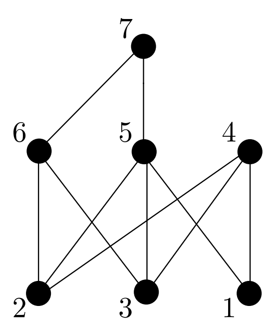


Figure 19: Hasse diagram of the poset P_w for $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$.

Exercise. Show that chains of P_w are precisely increasing subsequences of w_i, and antichains are decreasing subsequences of w_i.

Another corollary of Greene's Theorem is the following statement.

Theorem 4.31 (Erdős–Szekeres)**.** *Let $m, n \geq 1$. Then any permutation of size at least $mn + 1$ contains either an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $n + 1$.*

Proof. According to Greene's Theorem, we can associate a Young diagram λ to P_w, in which the length of the first row is the maximal size of chain in P_w, i.e., the maximal length of an increasing subsequence in w (see the exercise above). Similarly, the length of the first row of the conjugate Young diagram (and thus the length of the first column of λ) is the maximal size of antichain in P_w, i.e., the maximal length of an decreasing subsequence in w. The rest follows from the following elementary statement.

Exercise. Let λ be a Young diagram, $|\lambda| \geq mn+1$. Then either $\lambda_1 \geq n+1$, or $\lambda'_1 \geq m+1$, where λ'_1 is the length of the first column of λ.

□

Exercise. Prove Theorem 4.31 without using Greene's Theorem.

5 Robinson–Schensted correspondence

5.1 The algorithm

Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ be an integer partition. Recall that f_λ is the number of standard Young tableaux (SYT) of shape λ, which is also equal to the number of saturated chains in the Young lattice $\mathbb{Y}$ from $\emptyset$ to λ.

Theorem 5.1.

$$\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$$

Example 5.2. For $n = 3$, there are precisely three Young diagrams. Two of these correspond to a unique SYT, and the other has two SYT. Then $1^2 + 2^2 + 1^2 = 6 = 3!$.

Remark (can be ignored by those not taking Representation Theory)**.** Theorem 5.1 can be interpreted as a fact from the representation theory of symmetric groups. Irreducible representations of S_n are parameterized by integer partitions of n, i.e., by Young diagrams. Then f_λ is precisely the dimension of the irreducible representation V_λ, $n!$ is the order of S_n, and thus the theorem says that sum of squares of dimensions of irreducible representations is equal to the order of the group (which is true for any finite group and can be proved by considering regular representation).

We will prove Theorem 5.1 combinatorially by constructing a bijection between permutations in S_n and pairs (P, Q) of SYT of the same shape $\lambda \vdash n$.

The algorithm is due to Robinson–Schensted, with a generalization due to Knuth, and is usually referred as RSK (Robinson–Schensted–Knuth).

Given $w = w_1 w_2 \ldots w_n \in S_n$, we will construct an *insertion tableau* P and a *recording tableau* Q of the same shape step by step.

Example 5.3. Let $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$. In the beginning, $P = \emptyset$ and $Q = \emptyset$. We start adding w_i in turn. First, let us consider how P is changing. Every new w_i is inserted in the first row.

- At the first step, 3 is inserted in the box $(1, 1)$.
- At the second step, 2 is inserted. It cannot be inserted in the box $(1, 2)$ as $2 > 3$, so it is inserted in the box $(1, 1)$ and thus pushes down 3 into a new box $(2, 1)$.
- At the third step, 6 is inserted in the box $(1, 2)$.
- At the fourth step, 1 is inserted in the box $(1, 1)$, and thus pushes down 2 into the second row. 2, in its turn, pushes down 3 to the third row into a new box $(3, 1)$.
- At the fifth step, 5 is inserted in the box $(1, 2)$, and thus pushes down 6 into the second row, where 6 forms a new box $(2, 2)$.
- At the sixth step, 7 is inserted in the box $(1, 3)$.
- At the last step, 4 is inserted in the box $(1, 2)$, and thus pushes down 5 into the second row. 5, in its turn, pushes down 6 to the third row into a new box $(3, 2)$.

As a result, we get the following SYT of shape $\lambda = (3, 2, 2)$:

$$P = \begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2} \end{array}$$

Now, Q *records* the number of the step each box of P was introduced for the first time. For example, the box $(1, 3)$ was introduced at the sixth step, so there will be 6 in it, and the box $(3, 1)$ has shown up at the fourth step, so there will be 4 there. As a result, we have

$$Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2} \end{array}$$

Let us formalize the procedure. The step (inserting w_i to the tableau P_{i-1} obtained after inserting w_{i-1}) consists of the following:

- if w_i is greater than all entries in the first row of P_{i-1} then w_i is added to the end of the first row;
- otherwise, take the minimal w_j in the first row which is greater than w_i, substitute w_j with w_i, and insert w_j in the tableau obtained by removing the first row from P_{i-1}; i.e., if w_j is greater than all entries in the second row of P_{i-1}, then w_j is added to the end of the second row; otherwise, find the minimal w_k in the second row which is greater than w_j, substitute w_k with w_j, and push w_k into the third row, etc.

Exercise. Show that P and Q are SYT.

Both P and Q are SYT of the same shape λ, which is called the *Schensted shape of* w.

To verify that the procedure works, we need to check that the map $w \mapsto (P, Q)$ is injective and surjective. For this, we can construct the inverse map.

Example 5.4. Let us take $P = P_7$ and $Q = Q_7$ we got in Example 5.3, and try to reconstruct $w \in S_7$. According to Q, the last box to appear was $(3, 2)$ (with 6 in it). It is in row 3, so 6 was pushed down by some element from row 2. The maximal number in row 2 which is less than 6 is 5, which implies that 6 was pushed down by 5, and 5 was pushed down to row 2 by someone from the first row. The maximal number in the first row which is less than 5 is 4, which implies that 5 was pushed down by 4, and thus 4 is precisely the number inserted at the last step.

Therefore, we have found $w_7 = 4$ and reconstructed P_6 (and, obviously, Q_6), where

$$P_6 = \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad Q_6 = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

We now repeat this procedure for the box $(1, 3)$ of (P_6, Q_6) which, according to Q_6, is the last one to appear, etc. After seven steps we will reconstruct the permutation w.

5.2 Properties of RSK

We will now look at some corollaries of the algorithm, as well as at some of its properties.

Theorem 5.5. *Let λ be the Schensted shape of $w \in S_n$, and let (P, Q) be the corresponding SYT. Then*

(1) *λ_1 is the maximal size of an increasing subsequence in w;*

(2) *λ'_1 (= the number of rows in λ) is the maximal size of a decreasing subsequence in w.*

Remark. The Schensted shape of $w \in S_n$ is precisely the Young diagram λ constructed by the poset P_w in the Greene's Theorem. See Appendix 1 (written by Sergey Fomin) to Stanley's "Enumerative Combinatorics, vol. II".

Remark. Theorem 5.5 immediately implies Erdős–Szekeres Theorem.

Exercise 5.6. Denote by $r_x(P)$ the tableau created by an insertion of x in a partial tableau P in the RSK algorithm. Suppose that during the construction of $r_x(P)$ the elements $x_1, \ldots, x_k$ are pushed down from rows $1, 2, \ldots, k$ and columns $j_1, j_2, \ldots, j_k$ respectively. Then

(a) $x < x_1 < \cdots < x_k$;

(b) $j_1 \geq \cdots \geq j_k$;

(c) if $P' = r_x(P)$, then $P'_{i,j} \leq P_{i,j}$ for all i, j.

Statement (1) of Theorem 5.5 immediately follows from the following lemma.

Lemma 5.7. *Let $w = w_1 w_2 \ldots w_n \in S_n$, denote by P_k a partial tableau obtained after insertion of $w_1, \ldots, w_k$. Let w_k enter P_{k-1} in column j. Then the longest increasing subsequence ending in w_k has length j.*

Proof. We use induction on k. The base $(k = 1)$ is obvious.

Let us first prove the existence of an increasing subsequence of length j ending in w_k. Let w_i be the element of P_{k-1} in the box $(1, j - 1)$. Since w_k is inserted in the box $(1, j)$, $w_i < w_k$. By induction, there exists an increasing subsequence ending in w_i of length $j - 1$. By adding w_k, we obtain an increasing subsequence ending in w_k of length j.

Now, let us prove the maximality. Suppose there exists an increasing subsequence ending in w_k of length greater than j. Take this subsequence, and let w_i be the element preceding w_k, $i < k$. By the induction assumption, w_i is inserted in box $(1, m)$, $m \geq j$ (as there exists an increasing subsequence ending in w_i of length at least j). Therefore, for the element w_p in the box $(1, j)$ of P_i we have $w_p \leq w_i < w_k$. Let w_q be the element in the box $(1, j)$ of P_{k-1}. By Exercise 5.6(c), $w_q \leq w_p$ (as $k - 1 \geq i$), and thus $w_q < w_k$. However, during the insertion of w_k it pushes down w_q, which implies that $w_k < w_q$, so we came to a contradiction.

□

We are left to prove statement (2) of the theorem.

Recall that we denote by $r_x(P)$ a step of the RSK algorithm consisting of inserting x in a partial tableau P. Define $c_x(P)$ as a (purely formal) *column-insertion* of x into P. This can be understood as transposing P, then doing $r_x(P^t)$, and then transposing the result again (here by the transpose P^t of P we mean the reflection of P with respect to the main diagonal).

Lemma 5.8. *Let P be a partial tableau, with $x, y \notin P$, and $x \neq y$. Then $c_y r_x(P) = r_x c_y(P)$.*

Proof. The proof is case-by-case, we will consider some and leave others as an exercise.

Assume first that $y > x$ and y is greater than all elements of P. Then c_y places y at the end of the first column, and thus $r_x c_y(P)$ is just $r_x(P)$ with additional y attached to the bottom of the first column. It is clear that $c_y r_x(P)$ does precisely the same.

If we assume that x is the largest of x, y and all elements of P, then the same proof works (just need to transpose the whole picture).

Exercise. Complete the proof of the lemma. (The remaining case is where the largest element of x, y and all elements of P is an element of P.) □

Lemma 5.9. *Let $w = w_1 w_2 \ldots w_n \in S_n$, and let P be the insertion tableau of w. Define $w^r = w_n w_{n-1} \ldots w_2 w_1$. Then the insertion tableau of w^r is P^t.*

Proof. We can write $P = P(w) = r_{w_n} r_{w_{n-1}} \ldots r_{w_2} r_{w_1}(\emptyset)$.

Following this, and by using Lemma 5.8, we have

$$P(w^r) = r_{w_1} r_{w_2} \ldots r_{w_n}(\emptyset) = r_{w_1} r_{w_2} \ldots r_{w_{n-1}} c_{w_n}(\emptyset) = c_{w_n} r_{w_1} r_{w_2} \ldots r_{w_{n-1}}(\emptyset)$$

We can now continue, with

$$P(w^r) = c_{w_n} r_{w_1} r_{w_2} \ldots r_{w_{n-1}}(\emptyset) = c_{w_n} r_{w_1} r_{w_2} \ldots c_{w_{n-1}}(\emptyset) = c_{w_n} c_{w_{n-1}} r_{w_1} r_{w_2} \ldots r_{w_{n-2}}(\emptyset)$$

Applying this transformation n times, we get

$$P(w^r) = r_{w_1} r_{w_2} \ldots r_{w_n}(\emptyset) = c_{w_n} c_{w_{n-1}} \ldots c_{w_2} c_{w_1}(\emptyset) = P^t$$

as required. □

We can now complete the proof of Theorem 5.5(2).

Consider w^r, its increasing subsequences are precisely decreasing subsequences of w. Therefore, the maximal length of a decreasing subsequence of w is equal to the maximal length of an increasing subsequence of w^r, which, by (1), is equal to $(\tilde{\lambda})_1$, where $\tilde{\lambda}$ is the Schensted shape of w^r. By Lemma 5.9, $\tilde{\lambda} = \lambda'$ is conjugate to λ, so $(\tilde{\lambda})_1 = (\lambda')_1$ is precisely the size of the first column of λ.

We will now explore more symmetries of RSK.

Example 5.10. Recall from Example 5.3 that for $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$ we have

$$P = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2}\end{array} \qquad\qquad Q = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array}$$

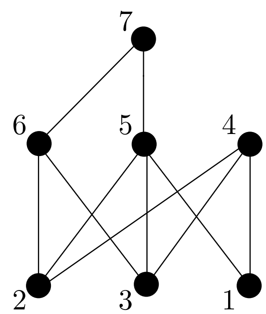
Let's now take the insertion tableau $P' = Q$ and the recording tableau $Q' = P$; which permutation does this pair correspond to? An application of the RSK algorithm leads to $w' = 4\,2\,1\,7\,5\,3\,6 \in S_7$. Now, observe that $w = (1\,3\,6\,7\,4)(2)(5)$ and $w' = (1\,4\,7\,6\,3)(2)(5)$, so $w' = w^{-1}$.

Theorem 5.11. *Suppose the application of RSK takes w to (P, Q). Then w^{-1} is taken to (Q, P).*

Recall that, given $w \in S_n$, one can define a poset P_w on $[n]$ with order $w_i <_{P_w} w_j$ if $w_i < w_j$ and $i < j$. Then chains of P_w are increasing subsequences of w, and antichains are decreasing subsequences of w.

Denote by M_1 the set of minimal elements of P_w, by M_2 the set of minimal elements of $P_w \setminus M_1$, and then by M_i the set of minimal elements of $P_w \setminus \bigcup_{j<i} M_j$. Note that every M_i is an antichain of P_w.

Example. Let $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$, we have already seen that the Hasse diagram of P_w is



Then $M_1 = \{1, 2, 3\}$, $M_2 = \{4, 5, 6\}$, and $M_3 = \{7\}$.

Write w in 2-line notation: $w = \begin{matrix} 1\,2\,3\,4\,5\,6\,7 \\ 3\,2\,6\,1\,5\,7\,4 \end{matrix}$. Then we can interpret elements of P_w as $\begin{matrix} i \\ w_i \end{matrix}$. In particular, we can write every M_i ordering its elements by the top number:

$$\begin{aligned}
M_1 &= \left\{ \begin{matrix}1\\3\end{matrix}, \begin{matrix}2\\2\end{matrix}, \begin{matrix}4\\1\end{matrix} \right\} &= \left\{ \begin{matrix}u_{11}\\v_{11}\end{matrix}, \begin{matrix}u_{12}\\v_{12}\end{matrix}, \begin{matrix}u_{13}\\v_{13}\end{matrix} \right\} \\
M_2 &= \left\{ \begin{matrix}3\\6\end{matrix}, \begin{matrix}5\\5\end{matrix}, \begin{matrix}7\\4\end{matrix} \right\} &= \left\{ \begin{matrix}u_{21}\\v_{21}\end{matrix}, \begin{matrix}u_{22}\\v_{22}\end{matrix}, \begin{matrix}u_{23}\\v_{23}\end{matrix} \right\} \\
M_3 &= \left\{ \begin{matrix}6\\7\end{matrix} \right\} &= \left\{ \begin{matrix}u_{31}\\v_{31}\end{matrix} \right\}
\end{aligned}$$

where $\begin{matrix} u_{ij} \\ v_{ij} \end{matrix}$ denotes j-th element of P_i.

Note that the first row of the insertion tableau P is $1\,4\,7 = v_{13}\,v_{23}\,v_{31}$, and the first row of the recording tableau Q is $1\,3\,6 = u_{11}\,u_{21}\,u_{31}$.

Remark. In general, if $M_j = \left\{ \begin{matrix}u_{j1}\\v_{j1}\end{matrix}, \begin{matrix}u_{j2}\\v_{j2}\end{matrix}, \ldots, \begin{matrix}u_{jn_j}\\v_{jn_j}\end{matrix} \right\}$ (where we always assume that $u_{j1} < u_{j2} < \cdots < u_{jn_j}$), then $v_{j1} > v_{j2} > \cdots > v_{jn_j}$ as all these elements comprise an antichain of P_w.

Remark. While applying RSK to a permutation $w = \begin{pmatrix} 1 & 2 & \dots & n \\ w_1 & w_2 & \dots & w_n \end{pmatrix}$ we insert the elements of the second row into tableau P and the elements of the first row into tableau Q. Note that the algorithm is completely defined by the relations $w_i < w_j$ for every pair (i, j). Thus, if we substitute the first row by any increasing sequence, and the second row by any n distinct numbers with the same relations, then we will get tableaux P' and Q' of the same shape as of P and Q (these are not SYT anymore, although the numbers will still be increasing to the right and to the bottom). Moreover, we can apply RSK to any two-row array of numbers (such that numbers in every row are distinct): sort the columns so that the top row is increasing, and then apply the procedure described above. We will use this modification of RSK to complete the proof of the theorem.

Lemma 5.12. *The first row of the insertion tableau P is $v_{1n_1}\, v_{2n_2}\, v_{3n_3} \dots$, and the first row of the recording tableau Q is $u_{11}\, u_{21}\, u_{31} \dots$. In other words, the first rows of P and Q consist of minimal v and u from every M_j.*

Proof. Let $w = w_1\, w_2\, \dots\, w_n \in S_n$, we use induction on n. If $n = 1$ then the statement is trivial.

Denote $\widetilde{w} = w_1\, w_2\, \dots\, w_{n-1}$, and let $\widetilde{P}$ and $\widetilde{Q}$ be the corresponding insertion and recording tableaux (see the remark above: here the first row of $\widetilde{w}$ is still $[n-1]$, but the second row may miss any one element of $[n]$). Let $\widetilde{M_j}$ be the corresponding antichains of $P_{\widetilde{w}}$, $\widetilde{M_j} = \left\{ \begin{matrix} \widetilde{u}_{j1} \\ \widetilde{v}_{j1} \end{matrix}, \begin{matrix} \widetilde{u}_{j2} \\ \widetilde{v}_{j2} \end{matrix}, \dots, \begin{matrix} \widetilde{u}_{jm_j} \\ \widetilde{v}_{jm_j} \end{matrix} \right\}$, where $j = 1, \dots, l$. By the induction assumption, the first row of $\widetilde{P}$ is $\widetilde{v}_{1m_1}\, \widetilde{v}_{2m_2}\, \dots\, \widetilde{v}_{lm_l}$, and the first row of $\widetilde{Q}$ is $\widetilde{u}_{11}\, \widetilde{u}_{21}\, \dots\, \widetilde{u}_{l1}$.

Insert w_n in $\widetilde{P}$. If $w_n > \widetilde{v}_{jm_j}$ for all j, then we insert w_n in the $(l+1)$-st column. At the same time, this means that $\begin{matrix} n \\ w_n \end{matrix}$ cannot be added to any of l antichains $\widetilde{M_j}$, so it forms a new antichain M_{l+1} in P_w, and thus we can write $\begin{matrix} n \\ w_n \end{matrix} = \begin{matrix} u_{l+1\,1} \\ v_{l+1\,1} \end{matrix} = \begin{matrix} u_{l+1\,m_{l+1}} \\ v_{l+1\,m_{l+1}} \end{matrix}$, so the first row of P is $v_{1m_1}\, v_{2m_2}\, \dots\, v_{lm_l}\, v_{l+1\,m_{l+1}}$, and the first row of Q is $u_{11}\, u_{21}\, \dots\, u_{l1}, u_{l+1\,1}$, as desired.

Assume now that $w_n < \widetilde{v}_{jm_j}$ for some j. Then $\widetilde{M_j} \cup \left\{ \begin{matrix} n \\ w_n \end{matrix} \right\}$ is an antichain of P_w (note that there may be many such j). One can easily see that $\begin{matrix} n \\ w_n \end{matrix}$ belongs to M_j with minimal j amongst those with $w_n < \widetilde{v}_{jm_j}$. For example, if $w_n < \widetilde{v}_{1m_1}$, then $\begin{matrix} n \\ w_n \end{matrix}$ is incomparable with all minimal elements of P_w, and thus it is a minimal element itself. Therefore, the element $\begin{matrix} n \\ w_n \end{matrix}$ is inserted in column j if and only if $\begin{matrix} n \\ w_n \end{matrix} = \begin{matrix} u_{jm_j} \\ v_{jm_j} \end{matrix}$, so again the first row of P is of the form required in the lemma. Since we do not start the new column, Q stays intact, so it also has the required form.

□

Exercise 5.13. The poset P_w is isomorphic to the poset $P_{w^{-1}}$, with the isomorphism given by $\begin{matrix} i \\ w_i \end{matrix} \mapsto \begin{matrix} w_i \\ i \end{matrix}$.

Proof of Theorem 5.11. Let the insertion and recording tableaux of w^{-1} be P' and Q'. Denote by M'_j the corresponding antichains of the poset $P_{w^{-1}}$. According to Exercise 5.13, $M'_j = \left\{ \begin{matrix} v_{j1} \\ u_{j1} \end{matrix}, \begin{matrix} v_{j2} \\ u_{j2} \end{matrix}, \dots, \begin{matrix} v_{jm_j} \\ u_{jm_j} \end{matrix} \right\} = \left\{ \begin{matrix} v_{jm_j} \\ u_{jm_j} \end{matrix}, \dots, \begin{matrix} v_{j2} \\ u_{j2} \end{matrix}, \dots, \begin{matrix} v_{j1} \\ u_{j1} \end{matrix} \right\}$, so the minimal elements in the second row are u_{j1}, and the minimal elements in the first row are v_{jm_j}. According to Lemma 5.12, the first row of P' is now $u_{11}\, u_{21}\, u_{31} \dots$, and the first row of Q' is $v_{1m_1}\, v_{2m_2}\, v_{3m_3} \dots$, which are precisely first rows of Q and P respectively.

Now the plan is the following: denote by $\bar{P}$ and $\bar{Q}$ the tableaux P and Q with their first rows removed, and find a two-row array $\bar{w}$ (i.e., a "permutation") which results in $(\bar{P}, \bar{Q})$ under RSK. We already know the set of elements in both rows, the question is how to match them.

Consider two elements $\begin{matrix} u_{jq} \\ v_{jq} \end{matrix}$, $q < m_j$, and $\begin{matrix} u_{rs} \\ v_{rs} \end{matrix}$, $s < m_r$, then both v_{jq} and v_{rs} are in $\bar{P}$. Thus, there are two elements which pushed these two down.

Exercise. $\begin{matrix} u_{jq} \\ v_{jq} \end{matrix}$ is pushed down from the first row of P by $\begin{matrix} u_{j\,q+1} \\ v_{j\,q+1} \end{matrix}$. (*Hint*: use induction on j.)

Therefore, v_{jq} enters $\bar{P}$ before v_{rs} if and only if $u_{j\,q+1} < u_{r\,s+1}$. This implies that the following array $\bar{w}$ is taken to $(\bar{P}, \bar{Q})$ (check this!):

$$\bar{w} = \begin{pmatrix} u_{12} \dots & u_{1m_1} & u_{22} \dots & u_{2m_2} & \dots & u_{l2} \dots & u_{lm_l} \\ v_{11} \dots & v_{1\,m_1-1} & v_{21} \dots & v_{2\,m_2-1} & \dots & v_{l1} \dots & v_{l\,m_l-1} \end{pmatrix}$$

for some l, where the columns should be permuted for the first row to be increasing.

Now, performing a similar exercise for w^{-1}, we see that

$$\overline{w^{-1}} = \begin{pmatrix} v_{1\,m_1-1} \dots & v_{11} & v_{2\,m_2-1} \dots & v_{21} & \dots & v_{l\,m_l-1} \dots & v_{l1} \\ u_{1m_1} & \dots u_{12} & u_{2m_2} & \dots u_{22} & \dots & u_{lm_l} & \dots u_{l2} \end{pmatrix}.$$

Observe that $\overline{w^{-1}} = (\bar{w})^{-1}$, and thus, by the induction assumption, $\bar{P}' = \bar{Q}$ and $\bar{Q}' = \bar{P}$. As we have already proved that the first rows also coincide, this implies that $P' = Q$ and $Q' = P$.

□

6 Games on graphs

Non-Examinable material

Let $G = (V, E)$ be a graph with the set of vertices V (where $|V| = n$ and the vertices are identified with elements of $[n]$), and the set of edges E, where $E \subseteq V \times V$ (we remove the diagonal from $V \times V$ and identify (i, j) with (j, i)).

Denote by $N(i)$ the set of *neighbours* of i, i.e., vertices connected to i by an edge. A *configuration* is a non-negative integer vector $\boldsymbol{c} = (c_1, \dots, c_n)$ – this can be understood as we put c_i chips in a vertex i.

6.1 Reflection game

We call a vertex i *unstable* if $2c_i < \sum_{j \in N(i)} c_j$, and *stable* otherwise.

The *initial configuration* is a vector with all $c_j = 0$ except for a single $c_i = 1$.

A *move* consists of choosing any unstable vertex i and changing the configuration as follows: $c_i \mapsto -c_i + \sum_{j \in N(i)} c_j$, while all other c_j stay intact. The *goal* of the game is to make every vertex stable.

Example. For G being a path with three vertices, there are three possible initial configurations, they all lead to a final configuration $(1, 1, 1)$.

We can ask several questions:

- For which graphs does the game stop after *some* sequence of moves?
- For which graphs does the game eventually stop whatever the sequence of moves?
- What are possible final configurations?

Example. For G being a cycle of length three, the game never stops.

Exercise.
- A *subgraph* of $G = (V, E)$ consists of a subset V' of V and all edges from E joining elements of V'. Show that if the game never stops on some subgraph of G, then it never stops on G.
- Show that if G has a cycle, then the game never stops.
- The *valence* of a vertex of G is the number of edges incident to it. Show that if G has a vertex of valence at least four, then the game never stops.
- Show that if G has at least two vertices of valence three, then the game never stops.
- Let G be a tree with a unique vertex v_0 of valence three. Denote the lengths of the "legs" (including v_0) by p, q, r, see Fig. 20. Show that if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ then the game never stops.

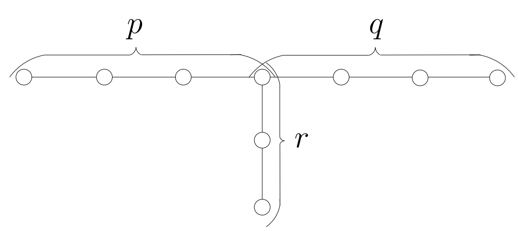


Figure 20: Tree with precisely one vertex of valence at least three

The remaining graphs are shown in Fig. 21.

Exercise. Show that for the graphs in Fig. 21 any sequence of moves terminates, and the final configuration is always the same (such graphs are called *graphs of finite type*).

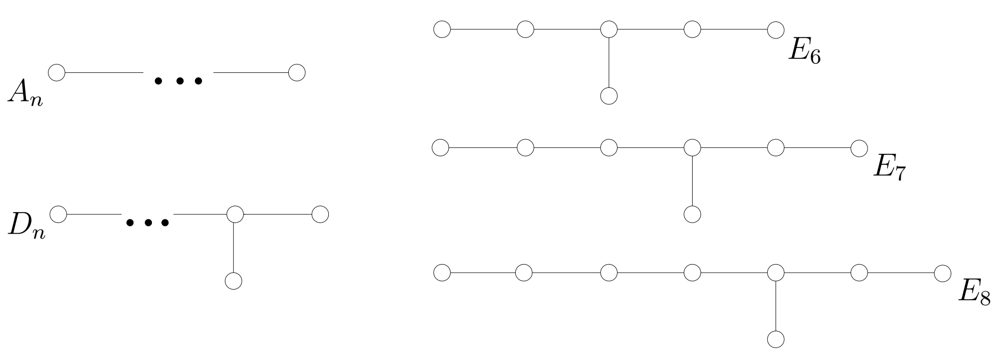


Figure 21: Graphs of finite type

6.2 Cartan firing

We now call a vertex i *unstable* if $c_i > 1$, and *stable* otherwise.

We can start with any initial configuration.

A *move* consists of choosing any unstable vertex i and changing the configuration as follows: $c_i \mapsto c_i - 2$, $c_j \mapsto c_j + 1$ if $j \in N(i)$, and c_j stays intact otherwise. The goal of the game is to make every vertex stable.

Example. For G being a path with three vertices and the initial configuration $(2, 2, 2)$, there is a sequence of moves which terminates.

Given a graph G, define its *Cartan matrix* $A_G = (a_{ij})$ as follows: $a_{ii} = 2$, $a_{ij} = -1$ if $i \in N(j)$, and $a_{ij} = 0$ otherwise. Then the move at vertex i can be understood as $\boldsymbol{c} \mapsto \boldsymbol{c} - A_i$, where A_i is the i-th row of A_G.

Theorem 6.1. *Let G be a graph, and let $A = A_G$ be its Cartan matrix. Then the following are equivalent:*

(1) For any initial configuration and any sequence of moves the game stops.

(2) There exists a positive vector $v = (v_1, \ldots, v_n)$, $v_i > 0$, such that Av also has all coordinates positive.

(3) A is positive definite.

Furthermore, if any of (1)–(3) *holds, then the final configuration does not depend on the sequence of moves.*

The latter statement follows from the following fact: if there are two moves taking $\boldsymbol{c}$ to $\boldsymbol{c}'$ and $\boldsymbol{c}''$ respectively, then the same two moves applied to $\boldsymbol{c}''$ and $\boldsymbol{c}'$ respectively lead to the same configuration $\boldsymbol{c}'''$.