APTS Applied Stochastic Processes: Exercises

(Notes originally produced by Wilfrid Kendall; some material due to Stephen Connor, Christina Goldschmidt, Matt Roberts and Amanda Turner)

Nicholas Georgiou[1] Hugo Lo[2]

April 25 2024

[1]Department of Mathematical Sciences, Durham University, nicholas.georgiou@durham.ac.uk
[2]Department of Statistical Science, UCL, chak.lo@ucl.ac.uk

Contents

Exercises

Introduction

This file contains exercises written to help you understand the course "Applied Stochastic Processes". It also contains, in the second half of the file, some hints and solutions to many of those exercises. We strongly recommend that you attempt the exercises first, then look at a hint if you cannot get started, and only refer to the solutions to check whether your answers are correct. Please let us know of any mistakes. Hints and solutions are not provided for all questions, for various reasons.

Markov chains and reversibility

1. Show that a discrete-time Markov chain run backwards in time (from some time n and state i) is again a Markov chain (until time n).
2. Suppose that $p_{x,y}$ are transition probabilities for a discrete state-space Markov chain satisfying detailed balance. Show that if the system of probabilities given by π_x satisfy the detailed balance equations then they must also satisfy the equilibrium equations.
3. Show that unconstrained simple symmetric random walk has period 2. Show that simple symmetric random walk subject to "prohibition" boundary conditions must be aperiodic.
4. Solve the equilibrium equations $\pi P = \pi$ for simple symmetric random walk on $\{0, 1, \ldots, k\}$ subject to "prohibition" boundary conditions.
5. Suppose that $X_0, X_1, \ldots,$ is a simple symmetric random walk with "prohibition" boundary conditions as above.
   - Use the definition of conditional probability to compute
   $$\overline{p}_{y,x} = \frac{\mathbb{P}\left(X_{n-1} = x\,,\ X_n = y\right)}{\mathbb{P}\left(X_n = y\right)}\,,$$
   - then show that
   $$\frac{\mathbb{P}\left(X_{n-1} = x\,,\ X_n = y\right)}{\mathbb{P}\left(X_n = y\right)} = \frac{\mathbb{P}\left(X_{n-1} = x\right) p_{x,y}}{\mathbb{P}\left(X_n = y\right)}\,,$$
   - now substitute, using $\mathbb{P}\left(X_n = i\right) = \frac{1}{k+1}$ for all i so $\overline{p}_{y,x} = p_{x,y}$.
   - Use the symmetry of the kernel ($p_{x,y} = p_{y,x}$) to show that the backwards kernel $\overline{p}_{y,x}$ is the same as the forwards kernel $\overline{p}_{y,x} = p_{y,x}$.
6. Show that if $X_0, X_1, \ldots,$ is a simple *asymmetric* random walk with "prohibition" boundary conditions, running in equilibrium, then it also has the same statistical behaviour as its reversed chain (i.e. solve the detailed balance equations!).
7. Show that detailed balance doesn't work for the 3-state chain with transition probabilities $\frac{1}{3}$ for $0 \to 1$, $1 \to 2$, $2 \to 0$ and $\frac{2}{3}$ for $2 \to 1$, $1 \to 0$, $0 \to 2$.
8. Work through the Random Chess example to compute the mean return time to a corner of the chessboard.
9. Verify for the Ising model that
   $$\mathbb{P}\left(\mathbf{S} = \mathbf{s}^{(i)} \middle| \mathbf{S} \in \{\mathbf{s}, \mathbf{s}^{(i)}\}\right) = \frac{\exp\left(-J \sum_{j:j\sim i} s_i s_j\right)}{\exp\left(J \sum_{j:j\sim i} s_i s_j\right) + \exp\left(-J \sum_{j:j\sim i} s_i s_j\right)}\,.$$
   Determine how this changes in the presence of an external field. Confirm that detailed balance holds for the heat-bath Markov chain.
10. Write down the transition probabilities for the Metropolis-Hastings sampler. Verify that it has the desired probability distribution as an equilibrium distribution.

Renewal processes and stationarity

1. Suppose that X is a simple symmetric random walk on $\mathbb{Z}$, started from 0. Show that
$$T = \inf\{n \geq 0 : X_n \in \{-10, 10\}\}$$
is a stopping time (i.e. show that the event $\{T \leq n\}$ is determined by $X_0, X_1, \ldots, X_n$). What is the value of $\mathbb{P}(T < \infty)$? What is the distribution of X_T?
2. For an irreducible recurrent Markov chain $(X_n)_{n\geq 0}$ on a discrete state-space S, fix $i \in S$ and let $H_0^{(i)} = \inf\{n \geq 0 : X_n = i\}$. For $m \geq 0$, let
$$H_{m+1}^{(i)} = \inf\{n > H_m^{(i)} : X_n = i\}.$$
Show that $H_0^{(i)}, H_1^{(i)}, \ldots$ is a sequence of stopping times.
3. Check that it follows from the strong Markov property that $(H_{m+1}^{(i)} - H_m^{(i)}, m \geq 0)$ is a sequence of i.i.d. random variables, independent of $H_0^{(i)}$.
4. Suppose that $(N(n))_{n\geq 0}$ is a delayed renewal process with inter-arrival times $Z_0, Z_1, \ldots$ where Z_0 is a non-negative random variable, independent of $Z_1, Z_2, \ldots$ which are i.i.d. strictly positive random variables with common mean μ. Use the Strong Law of Large Numbers for $T_k = \sum_{i=0}^k Z_i$ to show that
$$\frac{N(n)}{n} \to \frac{1}{\mu} \quad \text{a.s. as } n \to \infty.$$
Hint: note that $T_{N(n)} \leq n < T_{N(n)+1}$ so that $N(n)/n$ can be sandwiched between $N(n)/T_{N(n)+1}$ and $N(n)/T_{N(n)}$. Use this and the fact that $N(n) \to \infty$ as $n \to \infty$.
5. Let $(Y(n))_{n\geq 0}$ be the auxiliary Markov chain associated to a delayed renewal process $(N(n))_{n\geq 0}$ i.e. $Y(n) = T_{N(n-1)} - n$. Check that you agree with the transition probabilities given in the lecture notes.
6. Let
$$\nu_i = \frac{1}{\mu}\,\mathbb{P}(Z_1 \geq i+1), \quad i \geq 0.$$
Check that $\nu = (\nu_i)_{i\geq 0}$ defines a probability mass function.
7. Suppose that Z^* has the *size-biased distribution* associated with the distribution of Z_1, defined by
$$\mathbb{P}(Z^* = i) = \frac{i\,\mathbb{P}(Z_1 = i)}{\mu}, \quad i \geq 1.$$
   (a) Verify that this is a probability mass function.
   (b) Let $L \sim \mathrm{U}\{0, 1, \ldots, Z^* - 1\}$. Show that $L \sim \nu$.
   Note that you can generate L starting from Z^* by letting $U \sim U[0,1]$ and then setting $L = \lfloor UZ^* \rfloor$.

(c) What is the size-biased distribution associated with $\mathrm{Po}(\lambda)$?

8. Show that ν is stationary for Y.
   Hint: Y is clearly not reversible, so there's no point trying detailed balance!
9. Check that if $\mathbb{P}(Z_1 = k) = (1-p)^{k-1}p$, for $k \geq 1$, the stationary distribution ν for the time until the next renewal is $\nu_i = (1-p)^i p$, for $i \geq 0$. (In other words, if we flip a biased coin with probability p of heads at times $n = 0, 1, 2, \ldots$ and let $N(n) = \#\{0 \leq k \leq n :$ we see a head at time $k\}$ then $(N(n), n \geq 0)$ is a stationary delayed renewal process.)

Martingales and martingale convergence

1. Let X be a martingale. Use the tower property for conditional expectation to deduce that

$$\mathbb{E}\left[X_{n+k}|\mathcal{F}_n\right] = X_n\,, \quad k = 0, 1, 2, \ldots .$$

2. Recall Thackeray's martingale: let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables, with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = 1/2$. Define the Markov chain M by

$$M_0 = 0; \qquad M_n = \begin{cases} 1 - 2^n & \text{if } Y_1 = Y_2 = \cdots = Y_n = -1, \\ 1 & \text{otherwise.} \end{cases}$$

   (a) Compute $\mathbb{E}[M_n]$ from first principles.
   (b) What should be the value of $\mathbb{E}\left[\widetilde{M}_n\right]$ if $\widetilde{M}$ is computed as for M but stopping play if M hits level $1 - 2^N$?

3. Consider a branching process Y, where $Y_0 = 1$ and Y_{n+1} is the sum $Z_{n+1,1} + \ldots + Z_{n+1,Y_n}$ of Y_n independent copies of a non-negative integer-valued family-size r.v. Z.

   (a) Suppose $\mathbb{E}[Z] = \mu < \infty$. Show that $X_n = Y_n/\mu^n$ is a martingale.
   (b) Show that Y is itself a supermartingale if $\mu < 1$ and a submartingale if $\mu > 1$.
   (c) Suppose $\mathbb{E}\left[s^Z\right] = G(s)$. Let η be the smallest non-negative root of the equation $G(s) = s$. Show that η^{Y_n} defines a martingale.
   (d) Let $H_n = Y_0 + \ldots + Y_n$ be the total of all populations up to time n. Show that $s^{H_n}/(G(s)^{H_{n-1}})$ is a martingale.
   (e) How should these three expressions be altered if $Y_0 = k \geq 1$?

4. Consider asymmetric simple random walk, stopped when it first returns to 0. Show that this is a supermartingale if jumps have non-positive expectation, a submartingale if jumps have non-negative expectation (and therefore a martingale if jumps have zero expectation).

5. Consider Thackeray's martingale based on asymmetric random walk. Show that this is a supermartingale or submartingale depending on whether jumps have negative or positive expectation.

6. Show, using the conditional form of Jensen's inequality, that if X is a martingale then $|X|$ is a submartingale.

7. A shuffled pack of cards contains b black and r red cards. The pack is placed face down, and cards are turned over one at a time. Let B_n denote the number of black cards left *just before*

the n^{th} card is turned over. Let

$$Y_n = \frac{B_n}{r+b-(n-1)}\,.$$

(So Y_n equals the proportion of black cards left just before the n^{th} card is revealed.) Show that Y is a martingale.

8. Suppose $N_1, N_2, \ldots$ are independent identically distributed normal random variables of mean 0 and variance σ^2, and put $S_n = N_1 + \ldots + N_n$.

   (a) Show that S is a martingale.
   (b) Show that $Y_n = \exp\left(S_n - \frac{n}{2}\sigma^2\right)$ is a martingale.
   (c) How should these expressions be altered if $\mathbb{E}\left[N_i\right] = \mu \neq 0$?

9. Let X be a discrete-time Markov chain on a countable state-space S with transition probabilities $p_{x,y}$. Let $f : S \to \mathbb{R}$ be a bounded function. Let $\mathcal{F}_n$ contain all the information about $X_0, X_1, \ldots, X_n$. Show that

$$M_n = f(X_n) - f(X_0) - \sum_{i=0}^{n-1} \sum_{y \in S} (f(y) - f(X_i)) p_{X_i,y}$$

defines a martingale. (Hint: first note that $\mathbb{E}\left[f(X_{i+1}) - f(X_i)|X_i\right] = \sum_{y\in S}(f(y) - f(X_i))p_{X_i,y}$. Using this and the Markov property of X, check that $\mathbb{E}\left[M_{n+1} - M_n|\mathcal{F}_n\right] = 0$.)

10. Let Y be a discrete-time birth-death process absorbed at zero:

$$p_{k,k+1} = \frac{\lambda}{\lambda+\mu}, \quad p_{k,k-1} = \frac{\mu}{\lambda+\mu}, \qquad \text{for } k > 0, \text{ with } 0 < \lambda < \mu.$$

   (a) Show that Y is a supermartingale.
   (b) Let $T = \inf\{n : Y_n = 0\}$ (so $T < \infty$ a.s.), and define

$$X_n = Y_{\min\{n,T\}} + \left(\frac{\mu-\lambda}{\mu+\lambda}\right) \min\{n, T\}\,.$$

   Show that X is a non-negative supermartingale, converging to

$$Z = \left(\frac{\mu-\lambda}{\mu+\lambda}\right) T\,.$$

   (c) Deduce that

$$\mathbb{E}\left[T|Y_0 = y\right] \leq \left(\frac{\mu+\lambda}{\mu-\lambda}\right) y\,.$$

11. Let $L(\theta; X_1, X_2, \ldots, X_n)$ be the likelihood of parameter θ given a sample of independent and identically distributed random variables, $X_1, X_2, \ldots, X_n$.

   (a) Check that if the "true" value of θ is θ_0 then the likelihood ratio

$$M_n = \frac{L(\theta_1; X_1, X_2, \ldots, X_n)}{L(\theta_0; X_1, X_2, \ldots, X_n)}$$

   defines a martingale with $\mathbb{E}\left[M_n\right] = 1$ for all $n \geq 1$.

(b) Using the strong law of large numbers and Jensen's inequality, show that

$$\frac{1}{n} \log M_n \to -c \text{ as } n \to \infty.$$

12. Let X be a simple symmetric random walk absorbed at boundaries $a < b$.

(a) Show that

$$f(x) = \frac{x-a}{b-a} \qquad x \in [a, b]$$

is a bounded harmonic function.

(b) Use the martingale convergence theorem and optional stopping theorem to show that

$$f(x) = \mathbb{P}\left(X \text{ hits } b \text{ before } a | X_0 = x\right) .$$

Recurrence and rates of convergence

1. Recall that the total variation distance between two probability distributions μ and ν on $\mathcal{X}$ is given by

$$\operatorname{dist}_{\mathrm{TV}}(\mu, \nu) \;=\; \sup_{A \subseteq \mathcal{X}} \{\mu(A) - \nu(A)\}\,.$$

   Show that this is equivalent to the distance (note the absolute value signs!)

$$\sup_{A \subseteq \mathcal{X}} |\mu(A) - \nu(A)|\,.$$

2. Show that if $\mathcal{X}$ is discrete, then

$$\operatorname{dist}_{\mathrm{TV}}(\mu, \nu) = \tfrac{1}{2} \sum_{y \in \mathrm{X}} |\mu(y) - \nu(y)|\,.$$

   (Here we *do* need to use the absolute value on the RHS!)
   Hint: consider $A = \{y : \mu(y) > \nu(y)\}$.

3. Suppose now that μ and ν are density functions on $\mathbb{R}$. Show that

$$\operatorname{dist}_{\mathrm{TV}}(\mu, \nu) = 1 - \int_{-\infty}^{\infty} \min\{\mu(y), \nu(y)\} dy\,.$$

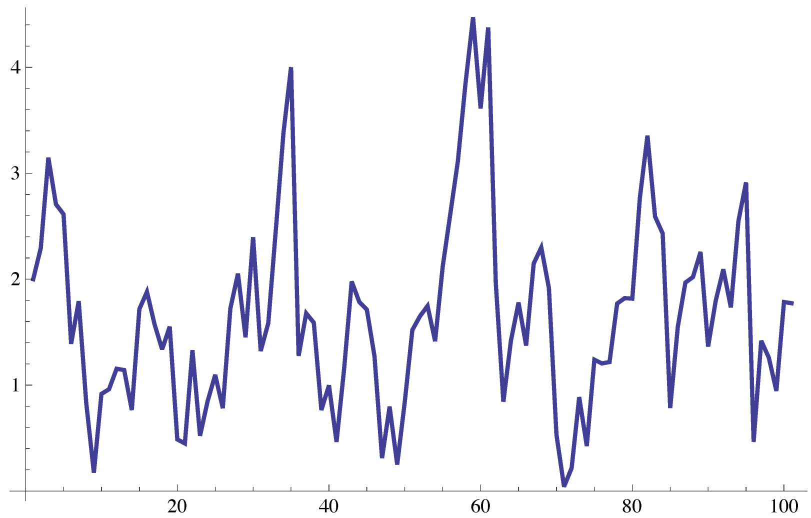
   Hint: remember that $|\mu - \nu| = \mu + \nu - 2\min\{\mu, \nu\}$.

4. Consider a Markov chain X with continuous transition density kernel. Show that it possesses *many* small sets of lag 1.

5. Consider a Vervaat perpetuity X, where

$$X_0 = 0; \qquad X_{n+1} = U_{n+1}(X_n + 1)\,,$$

   and where $U_1, U_2, \ldots$ are independent Uniform$(0, 1)$ (simulated below).



Find a small set for this chain.

6. Recall the idea of regenerating when our chain hits a small set: suppose that C is a small set for a ϕ-irreducible chain X, i.e. for $x \in C$,

$$\mathbb{P}\left(X_1 \in A | X_0 = x\right) \geq \alpha\nu(A).$$

Suppose that $X_n \in C$. Then with probability α let $X_{n+1} \sim \nu$, and otherwise let it have transition distribution $\frac{p(x,\cdot)-\alpha\nu(\cdot)}{1-\alpha}$.

(a) Check that the latter expression really gives a probability distribution.

(b) Check that X_{n+1} constructed in this manner obeys the correct transition distribution from X_n.

7. Define a reflected random walk as follows: $X_{n+1} = \max\{X_n + Z_{n+1}, 0\}$, for $Z_1, Z_2, \ldots$ i.i.d. with continuous density $f(z)$,

$$\mathbb{E}\left[Z_1\right] < 0 \quad \text{and} \quad \mathbb{P}\left(Z_1 > 0\right) > 0\,.$$

Show that the Foster-Lyapunov criterion for positive recurrence holds, using $\Lambda(x) = x$.

Hints and solutions

Markov chains and reversibility

You will get more from the exercises by trying them ***before*** *looking at the solutions!*

1. Show that a discrete-time Markov chain run backwards in time (from some time n and state i) is again a Markov chain (until time n).

Hint: Given a Markov chain $(X_j)_{j\geq 0}$, fix n and let $Y_j = X_{n-j}$ for $j = 0, \dots, n$. We want to show that Y is a Markov chain up to time n, i.e. that for any $j < n$ and $y_0, y_1, \dots, y_j, y_{j+1}$,

$$\mathbb{P}(Y_{j+1} = y_{j+1} | Y_0 = y_0, Y_1 = y_1, \dots, Y_j = y_j) = \mathbb{P}(Y_{j+1} = y_{j+1} | Y_j = y_j).$$

Solution: We start with the left-hand side. We have

$$\begin{aligned}
&\mathbb{P}(Y_{j+1} = y_{j+1} | Y_0 = y_0, Y_1 = y_1, \dots, Y_j = y_j) \\
&= \mathbb{P}\left(X_{n-(j+1)} = y_{j+1} | X_n = y_0, X_{n-1} = y_1, \dots, X_{n-j} = y_j\right) \\
&= \frac{\mathbb{P}\left(X_{n-(j+1)} = y_{j+1}, X_n = y_0, X_{n-1} = y_1, \dots, X_{n-j} = y_j\right)}{\mathbb{P}(X_n = y_0, X_{n-1} = y_1, \dots, X_{n-j} = y_j)} \\
&= \frac{\mathbb{P}\left(X_{n-(j+1)} = y_{j+1}\right) p_{y_{j+1},y_j} p_{y_j,y_{j-1}} \cdots p_{y_1,y_0}}{\mathbb{P}(X_{n-j} = y_j) p_{y_j,y_{j-1}} p_{y_{j-1},y_{j-2}} \cdots p_{y_1,y_0}} \\
&= \frac{\mathbb{P}\left(X_{n-(j+1)} = y_{j+1}\right) p_{y,y_j}}{\mathbb{P}(X_{n-j} = y_j)} \\
&= \frac{\mathbb{P}\left(X_{n-(j+1)} = y_{j+1}, X_{n-j} = y_j\right)}{\mathbb{P}(X_{n-j} = y_j)} \\
&= \mathbb{P}(Y_{j+1} = y_{j+1} | Y_j = y_j).
\end{aligned}$$

2. Suppose that $p_{x,y}$ are transition probabilities for a discrete state-space Markov chain satisfying detailed balance. Show that if the system of probabilities given by π_x satisfy the detailed balance equations then they must also satisfy the equilibrium equations.

Hint: Start from the detailed balance equations $\pi_x p_{xy} = \pi_y p_{yx}$ and sum over x.

Solution: Doing as the hint suggests, for any y,

$$\sum_x \pi_x p_{xy} = \sum_x \pi_y p_{yx} = \pi_y \sum_x p_{yx} = \pi_y$$

which is exactly the equilibrium equations $\pi P = \pi$.

3. Show that unconstrained simple symmetric random walk has period 2. Show that simple symmetric random walk subject to "prohibition" boundary conditions must be aperiodic.

Hint: "Unconstrained" means no boundary, i.e. on the whole of $\mathbb{Z}$, whereas "prohibition" boundary conditions means that the random walk moves on $\{0, 1, \ldots, k\}$ for some k, and when it reaches 0 or k, it stays where it is with probability $1/2$ (rather than moving to -1 or $k+1$).

Solution: If the unconstrained random walk starts on an even integer, then it will always be on even integers at even times and odd integers at odd times; and vice versa if it starts on an odd integer.

Now consider the "prohibition" random walk. For any sites $x, y \in \{0, 1, \ldots, k\}$ and any time $n > 2k$, the random walk started from site x has positive probability of taking its first x steps in the negative direction, then remaining at 0 for the next $n - x - y$ steps, and then taking its last y steps in the positive direction. (In fact it has probability $1/2^n$ of doing so.) Thus the random walk has positive probability of being at any of the $k+1$ sites after n steps, so it is aperiodic.

4. Solve the equilibrium equations $\pi P = \pi$ for simple symmetric random walk on $\{0, 1, \ldots, k\}$ subject to "prohibition" boundary conditions.

Hint: Recall the equilibrium equations $\pi P = \pi$, i.e. $\sum_x \pi_x p_x y = \pi_y$ for all y. You will also need to use $\sum_x \pi_x = 1$.

Solution: When $y = 0$, then $\sum_x \pi_x p_{x0} = \pi_0/2 + \pi_1/2$, so setting this equal to π_0 we see that $\pi_1 = \pi_0$. When $y = 1$, then $\sum_x \pi_x p_{x1} = \pi_0/2 + \pi_2/2$, so setting this equal to π_1 and using that $\pi_0 = \pi_1$ we see that $\pi_2 = \pi_1$. Continuing in this way we get $\pi_0 = \pi_1 = \pi_2 = \ldots = \pi_k$. Finally, $\sum_x \pi_x = 1$ implies that $\pi_x = 1/(k+1)$ for all x.

5. Suppose that $X_0, X_1, \ldots,$ is a simple symmetric random walk with "prohibition" boundary conditions as above.
   - Use the definition of conditional probability to compute
$$\overline{p}_{y,x} = \frac{\mathbb{P}\left(X_{n-1} = x\,,\; X_n = y\right)}{\mathbb{P}\left(X_n = y\right)}\,,$$
   - then show that
$$\frac{\mathbb{P}\left(X_{n-1} = x\,,\; X_n = y\right)}{\mathbb{P}\left(X_n = y\right)} = \frac{\mathbb{P}\left(X_{n-1} = x\right) p_{x,y}}{\mathbb{P}\left(X_n = y\right)}\,,$$
   - now substitute, using $\mathbb{P}\left(X_n = i\right) = \frac{1}{k+1}$ for all i so $\overline{p}_{y,x} = p_{x,y}$.
   - Use the symmetry of the kernel ($p_{x,y} = p_{y,x}$) to show that the backwards kernel $\overline{p}_{y,x}$ is the same as the forwards kernel $\overline{p}_{y,x} = p_{y,x}$.
6. Show that if $X_0, X_1, \ldots,$ is a simple *asymmetric* random walk with "prohibition" boundary conditions, running in equilibrium, then it also has the same statistical behaviour as its reversed chain (i.e. solve the detailed balance equations!).

Hint: Recall that the detailed balance equations are $\pi_x p_{xy} = \pi_y p_{yx}$ for all x and y. Note that p_{xy} is zero unless x and y are neighbours. Start from $x = 0$ and $y = 1$ and work your way up.

Solution: Say we jump from x to $x+1$ with probability p, and $x+1$ to x with probability $1-p$. For each $x \in \{0, \ldots, k-1\}$ we have $\pi_x p = \pi_{x+1}(1-p)$ so, chaining together, $\pi_j = (\frac{p}{1-p})^j \pi_0$. All the other detailed balance equations are trivially satisfied since $p_{xy} = 0$ whenever $x \neq y$ are not neighbours.

7. Show that detailed balance doesn't work for the 3-state chain with transition probabilities $\frac{1}{3}$ for $0 \to 1$, $1 \to 2$, $2 \to 0$ and $\frac{2}{3}$ for $2 \to 1$, $1 \to 0$, $0 \to 2$.

Hint: Proceed as in the previous question, using detailed balance to get π_1 in terms of π_0 and then π_2 in terms of π_1. But then since 2 is also a neighbour of 1, we have a third detailed balance equation to get π_0 in terms of π_2. Show that these three equations have no non-trivial solution.

8. Work through the Random Chess example to compute the mean return time to a corner of the chessboard.

The solution is in the lecture notes.

9. Verify for the Ising model that

$$\mathbb{P}\left(\mathbf{S} = \mathbf{s}^{(i)} \middle| \mathbf{S} \in \{\mathbf{s}, \mathbf{s}^{(i)}\}\right) = \frac{\exp\left(-J \sum_{j:j\sim i} s_i s_j\right)}{\exp\left(J \sum_{j:j\sim i} s_i s_j\right) + \exp\left(-J \sum_{j:j\sim i} s_i s_j\right)}.$$

Determine how this changes in the presence of an external field. Confirm that detailed balance holds for the heat-bath Markov chain.

10. Write down the transition probabilities for the Metropolis-Hastings sampler. Verify that it has the desired probability distribution as an equilibrium distribution.

Solution: For $y \neq x$, the probability that we move from x to y is the probability that y is proposed from x, and then that the proposal is accepted. That is, $p_{xy} = q(x,y)\alpha(x,y)$. We then have $p_{xx} = 1 - \sum_y q(x,y)\alpha(x,y)$ for each x.

To show that π is an equilibrium distribution for this chain, we check that it solves detailed balance. Suppose first that $\pi(y)q(y,x) \leq \pi(x)q(x,y)$. Then from the above and the definition of α,

$$\begin{aligned}
\pi(x)p_{xy} &= \pi(x)q(x,y)\alpha(x,y) \\
&= \pi(x)q(x,y) \cdot \frac{\pi(y)q(y,x)}{\pi(x)q(x,y)} \\
&= \pi(y)q(y,x) \\
&= \pi(y)q(y,x)\alpha(y,x) \\
&= \pi(y)p_{yx}
\end{aligned}$$

where for the fourth equality we used the fact that $\alpha(y,x) = 1$.

If on the other hand $\pi(y)q(y,x) > \pi(x)q(x,y)$, the same calculation works with x and y exchanged. Thus detailed balance holds, and the desired probability distribution π is an equilibrium distribution for this Markov chain.

Renewal processes and stationarity

You will get more from the exercises by trying them ***before*** *looking at the solutions!*

1. Suppose that X is a simple symmetric random walk on $\mathbb{Z}$, started from 0. Show that

$$T = \inf\{n \geq 0 : X_n \in \{-10, 10\}\}$$

is a stopping time (i.e. show that the event $\{T \leq n\}$ is determined by $X_0, X_1, \ldots, X_n$). What is the value of $\mathbb{P}(T < \infty)$? What is the distribution of X_T?

Hint: It may be easier to describe $\{T > n\}$ (why is this enough?). Will X eventually make 20 consecutive jumps to the right? For X_T, think symmetry.

Solution: The event $\{T > n\} = \{X_1 \notin \{-10, 10\}, X_2 \notin \{-10, 10\}, \ldots, X_n \notin \{-10, 10\}\}$ is clearly determined by $X_1, X_2, \ldots, X_n$, and therefore so is the complementary event $\{T \leq n\} = \{T > n\}^{\mathsf{c}}$. Since a simple symmetric random walk must eventually leave any bounded interval containing its starting point, the stopping time is almost surely finite: $\mathbb{P}(T < \infty) = 1$. To prove rigorously, consider X walking on the whole of $\mathbb{Z}$, and define events $A_k = \{\text{jumps } 20k+1, 20k+2, \ldots, 20k+20 \text{ are all to the right}\}$, for $k = 0, 1, \ldots$. Observe that: (i) the A_k are independent, (ii) if A_k occurs then $T < 20k + 20$ (either X already left $(-10, 10)$ by time $20k$, or $X_{20k} \in (-10, 10)$ and the next 20 steps are to the right, meaning $X_{20k+20} > 10$), (iii) $\mathbb{P}(A_k) = 2^{-20} > 0$. Hence the smallest k for which A_k occurs is a Geometric random variable with positive success probability, which is almost surely finite, and hence T is too. The distribution of X_T follows from symmetry of the random walk; since the walk starts at the mid point of the interval $[-10, 10]$ is it equally likely to hit either end first: $\mathbb{P}(X_T = 10) = \mathbb{P}(X_T = -10) = 1/2$.

2. For an irreducible recurrent Markov chain $(X_n)_{n\geq 0}$ on a discrete state-space S, fix $i \in S$ and let $H_0^{(i)} = \inf\{n \geq 0 : X_n = i\}$. For $m \geq 0$, let

$$H_{m+1}^{(i)} = \inf\{n > H_m^{(i)} : X_n = i\}.$$

Show that $H_0^{(i)}, H_1^{(i)}, \ldots$ is a sequence of stopping times.

Hint: You might find it helpful to view $H_m^{(i)}$ as the $(m+1)$-th visit to state i.

Solution: Guided by the hint, observe that $H_{m+1}^{(i)}$ is the first time **after** $H_m^{(i)}$ that X visits state i. Hence an induction argument shows that $H_m^{(i)}$ is the time of the $(m+1)$-th visit to i. Then, $\{H_m^{(i)} \leq n\}$ is just the event that at least $m+1$ of the random variables $X_0, X_1, \ldots, X_n$ are equal to i, which is clearly determined by $X_0, \ldots, X_n$.

3. Check that it follows from the strong Markov property that $(H_{m+1}^{(i)} - H_m^{(i)}, m \geq 0)$ is a sequence of i.i.d. random variables, independent of $H_0^{(i)}$.

Hint: Use the strong Markov property at the times $H_0^{(i)}, H_1^{(i)}, \ldots$. Use the fact that $X_{H_m^{(i)}}$ is (almost surely) constant.

Solution: We apply the strong Markov property sequentially to the stopping times $H_0^{(i)}, H_1^{(i)}, \ldots$. First, we remark that $H_0^{(i)}$ is finite almost surely, since X is irreducible and recurrent. Moreover $X_{H_0^{(i)}} \equiv i$ almost surely, by definition. Hence the strong Markov property implies that $(X_{H_0^{(i)}+n})_{n\geq 0}$ has the same distribution as $(X_n)_{n\geq 0}$ started from $X_0 = i$, **whatever the value** of $H_0^{(i)}$. This implies that $H_1^{(i)} - H_0^{(i)}$ is finite and has the same distribution as the first return time to state i of the walk when started in state i, i.e., the distribution of $R_i = \inf\{n > 0 : X_n = i\}$ given $X_0 = i$, and $H_1^{(i)} - H_0^{(i)}$ is independent of $H_0^{(i)}$. Now, suppose that $H_m^{(i)}$ is finite; applying the Strong Markov property to $H_m^{(i)}$ implies that $(X_{H_m^{(i)}+n})_{n\geq 0}$ has the same distribution as $(X_n)_{n\geq 0}$ started from $X_0 = i$, whatever the value of $H_m^{(i)}$. So, as before, $H_{m+1}^{(i)} - H_m^{(i)}$ is finite, has the same distribution as R_i given $X_0 = i$, and $H_{m+1}^{(i)} - H_m^{(i)}$ is independent of $H_m^{(i)}$. Moreover, the strong Markov property also shows that $(X_{H_m^{(i)}+n})_{n\geq 0}$ and $(X_n)_{0\leq n<H_m^{(i)}}$ are independent, so $H_{m+1}^{(i)} - H_m^{(i)}$ is independent of $H_k^{(i)}$ for all $k = 0, 1, \ldots, m-1$. Hence, by induction the random variables $H_{m+1}^{(i)} - H_m^{(i)}$, $m = 0, 1, \ldots$ are independent and identically distributed and also independent of $H_0^{(i)}$.

4. Suppose that $(N(n))_{n\geq 0}$ is a delayed renewal process with inter-arrival times $Z_0, Z_1, \ldots$ where Z_0 is a non-negative random variable, independent of $Z_1, Z_2, \ldots$ which are i.i.d. strictly positive random variables with common mean μ. Use the Strong Law of Large Numbers for $T_k = \sum_{i=0}^k Z_i$ to show that
$$\frac{N(n)}{n} \to \frac{1}{\mu} \quad \text{a.s. as } n \to \infty.$$
Hint: note that $T_{N(n)-1} \leq n < T_{N(n)}$ so that $N(n)/n$ can be sandwiched between $N(n)/T_{N(n)}$ and $N(n)/T_{N(n)-1}$. Use this and the fact that $N(n) \to \infty$ as $n \to \infty$.

Solution: Since $N(n) = \#\{k \geq 0 : T_k \leq n\}$, the random variable $N(n)$ takes values in $\{1, 2, \ldots\}$ and $N(n) = \ell$ if and only if $T_{\ell-1} \leq n < T_\ell$, for all $\ell \geq 1$. In other words, $T_{N(n)-1} \leq n < T_{N(n)}$. This implies that $N(n)/T_{N(n)} < N(n)/n \leq N(n)/T_{N(n)-1}$. Since Z_1 is finite almost surely (since it has a finite mean), we have $N(n) \to \infty$ almost surely as $n \to \infty$. Thus $\liminf_{n\to\infty}(N(n)/n) \geq \liminf_{n\to\infty}(N(n)/T_{N(n)}) = \liminf_{n\to\infty} n/T_n = 1/\mu$, and $\limsup_{n\to\infty}(N(n)/n) \leq \limsup_{n\to\infty}\left(\frac{N(n)}{N(n)-1}\frac{N(n)-1}{T_{N(n)-1}}\right) \leq \limsup_{n\to\infty}(n/(n-1)) \cdot \limsup_{n\to\infty} n/T_n = 1/\mu$.

5. Let $(Y(n))_{n\geq 0}$ be the auxiliary Markov chain associated to a delayed renewal process $(N(n))_{n\geq 0}$ i.e. $Y(n) = T_{N(n-1)} - n$. Check that you agree with the transition probabilities given in the lecture notes.

Hint: Note that $T_{N(n-1)}$ agrees with the smallest sum $T_j = \sum_{i=0}^j Z_i$ that is greater than or equal to n, because $T_{N(n-1)-1} \leq n-1 < T_{N(n-1)}$. (This interpretation also holds for $n = 0$, where $N(-1) \equiv 0$.)

Solution: Suppose $Y(n) = k \geq 1$, so that $T_{N(n-1)} = n + k \geq n + 1$. In other words, $T_{N(n-1)}$ is greater than or equal to $n+1$, and it must be the smallest such T_j, since $T_{N(n-1)-1} \leq n-1 < n+1$. Hence $T_{N(n)} = T_{N(n-1)}$, and therefore $Y(n+1) = T_{N(n)} - (n+1) = T_{N(n-1)} - n - 1 = Y(n) - 1$, with probability 1. Otherwise, if $Y(n) = 0$, then $T_{N(n-1)} = n$, so $T_{N(n-1)}$ is strictly less than $n+1$, and $T_{N(n-1)+1} = T_{N(n-1)} + Z_{N(n-1)+1}$ is greater than or equal $n+1$ (since $Z_{N(n-1)+1}$ is strictly positive). Hence $T_{N(n)} = T_{N(n-1)} + Z_{N(n-1)+1}$, and therefore $Y(n+1) = T_{N(n)} - (n+1) = T_{N(n-1)} + Z_{N(n-1)+1} - n - 1 = Z_{N(n-1)+1} - 1$. This means that the probability $\mathbb{P}(Y(n+1) = i \mid Y(n) = 0)$ is equal to $\mathbb{P}\left(Z_{N(n-1)+1} = i+1\right) = \mathbb{P}(Z_1 = i+1)$.

6. Let
$$\nu_i = \frac{1}{\mu}\mathbb{P}(Z_1 \geq i+1), \quad i \geq 0.$$
Check that $\nu = (\nu_i)_{i\geq 0}$ defines a probability mass function.

Hint: Recall the "tail sum formula"" for expectation of a non-negative integer-valued random variable.

Solution: Since Z_1 is non-negative and integer valued, $\mu = \mathbb{E}[Z_1] = \sum_{i\geq 0}\mathbb{P}(Z_1 > i) = \sum_{i\geq 0}\mu\nu_i$ implying $\sum_{i\geq 0}\nu_i = 1$.

7. Suppose that Z^* has the *size-biased distribution* associated with the distribution of Z_1, defined by
$$\mathbb{P}(Z^* = i) = \frac{i\,\mathbb{P}(Z_1 = i)}{\mu}, \quad i \geq 1.$$
   (a) Verify that this is a probability mass function.
   (b) Let $L \sim \mathrm{U}\{0, 1, \ldots, Z^* - 1\}$. Show that $L \sim \nu$.
   Note that you can generate L starting from Z^* by letting $U \sim U[0,1]$ and then setting $L = \lfloor UZ^* \rfloor$.
   (c) What is the size-biased distribution associated with $\mathrm{Po}(\lambda)$?

Hint:

(a) Use the usual definition of expectation of a non-negative integer-valued random variable.
(b) Find $\mathbb{P}(L = j)$ using the partition theorem/law of total probability (partitioning on the possible values of Z^*).
(c) if $Z \sim \mathrm{Po}(\lambda)$ then $\mathbb{P}(Z = i) = \exp(-\lambda)\lambda^i/i!$.

Solution:

(a) We have $\sum_{i\geq 1}\mathbb{P}(Z^* = i) = \frac{1}{\mu}\sum_{i\geq 1} i\,\mathbb{P}(Z_1 = i) = \frac{1}{\mu}\sum_{i\geq 0} i\,\mathbb{P}(Z_1 = i) = \mathbb{E}[Z_1]/\mu = 1$.
(b) By the law of total probability $\mathbb{P}(L = j) = \sum_{i\geq 1}\mathbb{P}(L = j \mid Z^* = i)\,\mathbb{P}(Z^* = i) = \sum_{i\geq j+1}\frac{1}{i}\frac{i\,\mathbb{P}(Z_1=i)}{\mu} = \frac{1}{\mu}\mathbb{P}(Z_1 \geq j+1) = \nu_j$.
(c) If $Z \sim \mathrm{Po}(\lambda)$ then $\mu = \mathbb{E}[Z] = \lambda$, and $\mathbb{P}(Z^* = i) = \frac{i\exp(-\lambda)\lambda^i/i!}{\lambda} = \exp(-\lambda)\lambda^{i-1}/(i-1)! = \mathbb{P}(Z = i-1)$. In other words, the size biased distribution Z^* has the same distribution as $Z+1$.

8. Show that ν is stationary for Y.
Hint: Y is clearly not reversible, so there's no point trying detailed balance!

Additional hint: Show that $\nu P = \nu$ for the transition matrix P for the chain Y.

Solution: Recall (from the lectures, or Exercise 5 above) that the non-zero entries of P are $P_{k,k-1} = 1$ for $k \geq 1$ and $P_{0,i} = \mathbb{P}(Z_1 = i+1)$ for $i \geq 0$. Thus for all $j \geq 0$, we have $\sum_i \nu_i P_{i,j} = \nu_{j+1} + \nu_0\,\mathbb{P}(Z_1 = j+1) = \frac{1}{\mu}(\mathbb{P}(Z_1 \geq j+2) + \mathbb{P}(Z_1 = j+1)) = \nu_j$.

(In fact, it is not difficult to show uniqueness here: if π is a measure satisfying $\pi_j = \sum_i \pi_i P_{i,j} = \pi_{j+1} + \pi_0\,\mathbb{P}(Z_1 = j+1)$ then π is proportional to ν.)

9. Check that if $\mathbb{P}(Z_1 = k) = (1-p)^{k-1}p$, for $k \geq 1$, the stationary distribution ν for the time until the next renewal is $\nu_i = (1-p)^i p$, for $i \geq 0$. (In other words, if we flip a biased coin with probability p of heads at times $n = 0, 1, 2, \ldots$ and let $N(n) = \#\{0 \leq k \leq n :$ we see a head at time $k\}$ then $(N(n), n \geq 0)$ is a stationary delayed renewal process.)

Hint: Calculate $\mathbb{P}(Z_1 \geq i+1) = \mathbb{P}(Z_1 > i)$ using a geometric series (or knowledge about Geometric random variables).

Solution: For all $i \geq 0$, $\mathbb{P}(Z_1 \geq i+1) = \sum_{k \geq i+1}(1-p)^{k-1}p = (1-p)^i \sum_{k \geq 1}(1-p)^{k-1}p = (1-p)^i$. But $\mu = 1/p$ since Z_1 is a Geometric random variable with success probability p. Hence $\nu_i = \frac{1}{\mu}\mathbb{P}(Z_1 \geq i+1) = (1-p)^i p$.

Martingales and martingale convergence

You will get more from the exercises by trying them ***before*** *looking at the solutions!*

1. Let X be a martingale. Use the tower property for conditional expectation to deduce that

$$\mathbb{E}\left[X_{n+k}|\mathcal{F}_n\right] = X_n\,, \quad k = 0, 1, 2, \ldots .$$

Hint: Prove by induction on k, using the martingale property for the base case and both the martingale and tower properties for the inductive step.

Solution: For $k = 1$, $\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] = X_n$ is exactly the martingale property. For $k > 1$, suppose $\mathbb{E}\left[X_{n+k-1}|\mathcal{F}_n\right] = X_n$ holds. The martingale and tower properties imply $\mathbb{E}\left[X_{n+k}|\mathcal{F}_n\right] = \mathbb{E}\left[\mathbb{E}\left[X_{n+k}|\mathcal{F}_{n+k-1}\right]|\mathcal{F}_n\right] = \mathbb{E}\left[X_{n+k-1}|\mathcal{F}_n\right]$, hence $\mathbb{E}\left[X_{n+k}|\mathcal{F}_n\right] = X_n$ also holds.

2. Recall Thackeray's martingale: let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables, with $\mathbb{P}\left(Y_1 = 1\right) = \mathbb{P}\left(Y_1 = -1\right) = 1/2$. Define the Markov chain M by

$$M_0 = 0; \qquad M_n = \begin{cases} 1 - 2^n & \text{if } Y_1 = Y_2 = \cdots = Y_n = -1, \\ 1 & \text{otherwise.} \end{cases}$$

   (a) Compute $\mathbb{E}\left[M_n\right]$ from first principles.
   (b) What should be the value of $\mathbb{E}\left[\widetilde{M}_n\right]$ if $\widetilde{M}$ is computed as for M but stopping play if M hits level $1 - 2^N$?

Hint:

(a) Just calculate the expectation!
(b) Consider $n < N$ and $n \geq N$ separately.

Solution:

(a) By definition M_n takes value $1 - 2^n$ with probability 2^{-n} and value 1 with probability $1 - 2^{-n}$; hence $\mathbb{E}\left[M_n\right] = 0$.
(b) Check that if $n < N$ we *cannot* have hit level $1 - 2^N$ yet, so $\widetilde{M}_n \equiv M_n$; in contrast if $n \geq N$ we *must* have either hit level $1 - 2^N$ (because $Y_1 = Y_2 = \ldots\; Y_N = -1$) or one of $Y_1, Y_2, \ldots, Y_N$ equals 1, meaning $\widetilde{M}_n = M_N$. (Formally speaking, $\widetilde{M}_n$ is the stopped martingale $M_{\min\{n,N\}}$.) Clearly, in either case $\mathbb{E}\left[\widetilde{M}_n\right] = 0$ using (a).

3. Consider a branching process Y, where $Y_0 = 1$ and Y_{n+1} is the sum $Z_{n+1,1} + \ldots + Z_{n+1,Y_n}$ of Y_n independent copies of a non-negative integer-valued family-size r.v. Z.
   (a) Suppose $\mathbb{E}\left[Z\right] = \mu < \infty$. Show that $X_n = Y_n/\mu^n$ is a martingale.
   (b) Show that Y is itself a supermartingale if $\mu < 1$ and a submartingale if $\mu > 1$.

(c) Suppose $\mathbb{E}\left[s^Z\right] = G(s)$. Let η be the smallest non-negative root of the equation $G(s) = s$. Show that η^{Y_n} defines a martingale.

(d) Let $H_n = Y_0 + \ldots + Y_n$ be the total of all populations up to time n. Show that $s^{H_n}/(G(s)^{H_{n-1}})$ is a martingale.

(e) How should these three expressions be altered if $Y_0 = k \geq 1$?

4. Consider asymmetric simple random walk, stopped when it first returns to 0. Show that this is a supermartingale if jumps have non-positive expectation, a submartingale if jumps have non-negative expectation (and therefore a martingale if jumps have zero expectation).

Hint: The martingale property trivially holds if the walk has stopped (why?). Otherwise, consider the conditional expectation $\mathbb{E}\left[X_{n+1} - X_n | X_n\right]$ (where X_n is the position of the walk at time n).

Solution: If the walk X has not yet returned to 0, $\mathbb{E}\left[X_{n+1} - X_n | X_n\right]$ is equal to the expectation of the next jump. Hence, if this expectation is non-positive, then $\mathbb{E}\left[X_{n+1}\right] \leq X_n$, and if it is non-negative, then $\mathbb{E}\left[X_{n+1}\right] \geq X_n$. If the walk has returned to 0 (so $X_n = 0$ for some $n > 0$), then $X_{n+1} = 0$ also, so $\mathbb{E}\left[X_{n+1} - X_n | X_n\right] = 0$. Hence the sub/supermartingale condition is satisfied for all n.

5. Consider Thackeray's martingale based on asymmetric random walk. Show that this is a supermartingale or submartingale depending on whether jumps have negative or positive expectation.

Hint: Again, consider $\mathbb{E}\left[M_{n+1} - M_n | M_n\right]$ (where M_n is your fortune after n steps), and separate the cases having already hit level 1 or not.

Solution: If at time n the Thackeray "martingale" M has hit level 1 (i.e., the asymmetric walk has at least one jump to the right in the first n steps), then $\mathbb{E}\left[M_{n+1} - M_n | M_n\right] = 0$. Otherwise, the asymmetric walk has made n left jumps (losing bets), and the wager is now 2^n, and $\mathbb{E}\left[M_{n+1} - M_n | M_n\right] = 2^n p - 2^n(1-p) = 2^n(2p-1)$ where p is the probability that the asymmetric walk jumps to the right.

6. Show, using the conditional form of Jensen's inequality, that if X is a martingale then $|X|$ is a submartingale.

Hint: What is the convex function?

Solution: The function $\phi(x) = |x|$ is convex, so $\mathbb{E}\left[\phi(X_{n+1}) | \mathcal{F}_n\right] \geq \phi(\mathbb{E}\left[X_{n+1} | \mathcal{F}_n\right]) = \phi(X_n)$.

7. A shuffled pack of cards contains b black and r red cards. The pack is placed face down, and cards are turned over one at a time. Let B_n denote the number of black cards left *just before* the n^{th} card is turned over. Let

$$Y_n = \frac{B_n}{r + b - (n-1)}\,.$$

(So Y_n equals the proportion of black cards left just before the n^{th} card is revealed.) Show that Y is a martingale.

8. Suppose $N_1, N_2, \ldots$ are independent identically distributed normal random variables of mean 0 and variance σ^2, and put $S_n = N_1 + \ldots + N_n$.

(a) Show that S is a martingale.

(b) Show that $Y_n = \exp\left(S_n - \frac{n}{2}\sigma^2\right)$ is a martingale.

(c) How should these expressions be altered if $\mathbb{E}[N_i] = \mu \neq 0$?

9. Let X be a discrete-time Markov chain on a countable state-space S with transition probabilities $p_{x,y}$. Let $f : S \to \mathbb{R}$ be a bounded function. Let $\mathcal{F}_n$ contain all the information about $X_0, X_1, \ldots, X_n$. Show that

$$M_n = f(X_n) - f(X_0) - \sum_{i=0}^{n-1} \sum_{y \in S} (f(y) - f(X_i)) p_{X_i,y}$$

defines a martingale. (Hint: first note that $\mathbb{E}[f(X_{i+1}) - f(X_i)|X_i] = \sum_{y \in S}(f(y) - f(X_i))p_{X_i,y}$. Using this and the Markov property of X, check that $\mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] = 0$.)

Solution:

Guided by the hint, we observe that $f(X_{n+1})$ is an integrable random variable, and $\mathbb{E}[f(X_{n+1})|X_n = x] = \sum_y f(y)p_{x,y}$; equivalently, $\mathbb{E}[f(X_{n+1}) - f(X_n)|X_n] = \sum_y (f(y) - f(X_n))p_{X_n,y}$. Therefore $M_{n+1} - M_n = f(X_{n+1}) - f(X_n) - \mathbb{E}[f(X_{n+1}) - f(X_n)|X_n]$. But, since X is a Markov chain, the Markov property says that $\mathbb{E}[f(X_{n+1}) - f(X_n)|X_n] = \mathbb{E}[f(X_{n+1}) - f(X_n)|\mathcal{F}_n]$, and therefore $\mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] = 0$ (recall the property of conditional expectation that $\mathbb{E}[Z|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}_n]|\mathcal{F}_n]$).

10. Let Y be a discrete-time birth-death process absorbed at zero:

$$p_{k,k+1} = \frac{\lambda}{\lambda+\mu}, \quad p_{k,k-1} = \frac{\mu}{\lambda+\mu}, \qquad \text{for } k > 0, \text{ with } 0 < \lambda < \mu.$$

(a) Show that Y is a supermartingale.

(b) Let $T = \inf\{n : Y_n = 0\}$ (so $T < \infty$ a.s.), and define

$$X_n = Y_{\min\{n,T\}} + \left(\frac{\mu - \lambda}{\mu + \lambda}\right) \min\{n, T\}.$$

Show that X is a non-negative supermartingale, converging to

$$Z = \left(\frac{\mu - \lambda}{\mu + \lambda}\right) T.$$

(c) Deduce that

$$\mathbb{E}[T|Y_0 = y] \leq \left(\frac{\mu + \lambda}{\mu - \lambda}\right) y.$$

Hint: Remember that $p_{0,0} = 1$.

Solution:

(a) If $Y_n = 0$ then $Y_{n+1} = 0$ and $\mathbb{E}[Y_{n+1}|Y_n] = Y_n$. Otherwise, if $Y_n = k \neq 0$, then $\mathbb{E}[Y_{n+1} - Y_n|Y_n] = \frac{\lambda}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} < 0$.

(b) Observe, that if $T \leq n$ then $T \leq n+1$, so $\min\{n, T\} = \min\{n+1, T\}$ so $X_{n+1} - X_n = 0$. Otherwise, $T > n$ and then $T \geq n+1$, so $X_{n+1} - X_n = Y_{n+1} - Y_n + \frac{\mu-\lambda}{\mu+\lambda}$. Then part (a) implies $\mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] \leq 0$. (Formally, we used here that T is a stopping time so $\{T \leq n\} \in \mathcal{F}_n$.) Martingale convergence then implies that X converges almost surely. Since $T < \infty$, a.s., $\min\{n, T\} \to T$ a.s. as $n \to \infty$, and therefore X_n converges to $Y_T + Z = Z$ for Z defined in the question.

(c) The martingale convergence theorem also implies that $\mathbb{E}\left[Z|\mathcal{F}_0\right] \leq X_0 = Y_0$. In other words, $\mathbb{E}\left[\left(\frac{\mu-\lambda}{\mu+\lambda}\right) T | Y_0 = y\right] \leq y$.

11. Let $L(\theta; X_1, X_2, \ldots, X_n)$ be the likelihood of parameter θ given a sample of independent and identically distributed random variables, $X_1, X_2, \ldots, X_n$.
    (a) Check that if the "true" value of θ is θ_0 then the likelihood ratio

$$M_n = \frac{L(\theta_1; X_1, X_2, \ldots, X_n)}{L(\theta_0; X_1, X_2, \ldots, X_n)}$$

    defines a martingale with $\mathbb{E}\left[M_n\right] = 1$ for all $n \geq 1$.
    (b) Using the strong law of large numbers and Jensen's inequality, show that

$$\frac{1}{n} \log M_n \to -c \text{ as } n \to \infty.$$

Hint:

(a) Use independence to write $L(\theta; X_1, X_2, \ldots, X_n)$ as a product of identically distributed terms, each having mean 1.

Solution:

(a) Suppose that $f(\theta; x)$ is the common density of X_i. Then $M_n = \prod_{i=1}^{n} \frac{f(\theta_1; X_i)}{f(\theta_0; X_i)}$, and noting that $\mathbb{E}\left[\frac{f(\theta_1;X_i)}{f(\theta_0;X_i)}\right] = \int f(\theta_0; x) \frac{f(\theta_1;x)}{f(\theta_0;x)} dx = 1$ (computing expectations using $\theta = \theta_0$), yields $\mathbb{E}\left[M_n|\mathcal{F}_{n-1}\right] = \mathbb{E}\left[M_{n-1} \frac{f(\theta_1;X_n)}{f(\theta_0;X_n)} | \mathcal{F}_{n-1}\right] = M_{n-1}$. Since M is a martingale, $\mathbb{E}\left[M_n\right] = \mathbb{E}\left[M_1\right] = 1$.

12. Let X be a simple symmetric random walk absorbed at boundaries $a < b$.
    (a) Show that

$$f(x) = \frac{x-a}{b-a} \qquad x \in [a, b]$$

    is a bounded harmonic function.
    (b) Use the martingale convergence theorem and optional stopping theorem to show that

$$f(x) = \mathbb{P}\left(X \text{ hits } b \text{ before } a | X_0 = x\right).$$

Hint:

(a) Find the non-zero values of $p_{x,y}$.

(b) This is the 'same' example as in the lectures, but on the interval $[a, b]$ rather than $[-a, b]$.

Solution:

(a) $f(x)$ is increasing on $[a, b]$, hence $0 = f(a) \leq f(x) \leq f(b) = 1$. If $x \in \{a, b\}$ then $p_{x,y} = 1$ if and only if $x = y$ (since walk is absorbed); hence $\sum_y f(y) p_{x,y} = f(x)$. Otherwise, $p_{x,x-1} = p_{x,x+1} = 1/2$ and $p_{x,y} = 0$ for all $y \notin \{x-1, x+1\}$; hence

$$\sum_y f(y) p_{x,y} = \frac{x-1-a}{b-a}\frac{1}{2} + \frac{x+1-a}{b-a}\frac{1}{2} = \frac{x-a}{b-a} = f(x).$$

(b) Since f is bounded harmonic function, $f(X_n)$ is a bounded martingale. Hence by the martingale convergence theorem, $f(X_n)$ converges a.s. to a limit Z and $\mathbb{E}\left[Z|\mathcal{F}_0\right] = f(X_0)$.

But Z equals $f(b) = 1$ if X hits b before a and equals $f(a) = 0$ otherwise; hence $\mathbb{E}[Z|\mathcal{F}_0] = \mathbb{P}(X \text{ hits } b \text{ before } a|X_0)$. This can also be deduced from the optional stopping theorem, using the stopping time $T = \min\{n \geq 0 : X_n \in \{a, b\}\}$ which is a.s. finite. On the time interval $[0, T]$ the martingale $f(X_n)$ is bounded, hence $\mathbb{E}[f(X_T)|\mathcal{F}_0] = f(X_0)$.

Recurrence and rates of convergence

You will get more from the exercises by trying them ***before*** *looking at the solutions!*

1. Recall that the total variation distance between two probability distributions μ and ν on $\mathcal{X}$ is given by

$$\text{dist}_{\text{TV}}(\mu, \nu) \;=\; \sup_{A \subseteq \mathcal{X}} \{\mu(A) - \nu(A)\} \,.$$

Show that this is equivalent to the distance (note the absolute value signs!)

$$\sup_{A \subseteq \mathcal{X}} |\mu(A) - \nu(A)| \,.$$

Hint: Consider A^c.

Solution: If $A \subseteq \mathcal{X}$ then $A^c \subseteq \mathcal{X}$. And if $\pi(A) - \nu(A)$ is negative, then $\pi(A^c) - \nu(A^c)$ is positive, since

$$\pi(A^c) - \nu(A^c) = 1 - \pi(A) - (1 - \nu(A)) = \nu(A) - \pi(A).$$

Thus the supremum of $\pi(A) - \nu(A)$ over all subsets A of $\mathcal{X}$ is always achieved when $\pi(A) - \nu(A)$ is positive.

2. Show that if $\mathcal{X}$ is discrete, then

$$\text{dist}_{\text{TV}}(\mu, \nu) = \tfrac{1}{2} \sum_{y \in \text{X}} |\mu(y) - \nu(y)| \,.$$

(Here we *do* need to use the absolute value on the RHS!)
Hint: consider $A = \{y : \mu(y) > \nu(y)\}$.

Further hint: Show that for A as above and any $B \subseteq \mathcal{X}$, we have $\mu(B) - \nu(B) \leq \mu(A) - \nu(A)$, and $\nu(B) - \mu(B) \leq \nu(A^c) - \mu(A^c)$.

Solution: As in the hints, note that since $\mu(y) - \nu(y) > 0$ for all $y \in A$ and $\nu(y) - \mu(y) \geq 0$ for all $y \in A^c$,

$$\mu(B) - \nu(B) = \sum_{y \in B} (\mu(y) - \nu(y)) \leq \sum_{y \in A \cap B} (\mu(y) - \nu(y)) \leq \sum_{y \in A} (\mu(y) - \nu(y)) = \mu(A) - \nu(A).$$

and similarly,

$$\nu(B) - \mu(B) \leq \nu(A^c) - \mu(A^c).$$

Thus

$$\sup_{B \subseteq \mathcal{X}} \{\mu(B) - \nu(B)\} = \mu(A) - \nu(A),$$

and by question 1 above we also have

$$\sup_{B\subseteq\mathcal{X}}\{\mu(B)-\nu(B)\} = \sup_{B\subseteq\mathcal{X}}|\mu(B)-\nu(B)| = \sup_{B\subseteq\mathcal{X}}\{\nu(B)-\mu(B)\} = \nu(A^c)-\mu(A^c).$$

Therefore, averaging the two lines above,

$$\sup_{B\subseteq\mathcal{X}}\{\mu(B)-\nu(B)\} = \frac{1}{2}(\mu(A)-\nu(A)+\nu(A^c)-\mu(A^c))$$

and since every $y\in\mathcal{X}$ is in exactly one of A or A^c, this equals

$$\frac{1}{2}\sum_{y\in\mathcal{X}}|\mu(y)-\nu(y)|.$$

3. Suppose now that μ and ν are density functions on $\mathbb{R}$. Show that

$$\mathrm{dist}_{\mathrm{TV}}(\mu,\nu) = 1-\int_{-\infty}^{\infty}\min\{\mu(y),\nu(y)\}dy\,.$$

   Hint: remember that $|\mu-\nu| = \mu+\nu-2\min\{\mu,\nu\}$.

4. Consider a Markov chain X with continuous transition density kernel. Show that it possesses *many* small sets of lag 1.

Hint: Suppose the transition density kernel of X is $p:\mathbb{R}\times\mathbb{R}\to[0,\infty)$. For any $x\in\mathbb{R}$, there must be some $y\in\mathbb{R}$ such that $p(x,y)>0$, and then since p is continuous, there must be $\varepsilon>0$ such that $p(x',y')>p(x,y)/2$ for all $x'\in[x-\varepsilon,x+\varepsilon]$ and $y'\in[y-\varepsilon,y+\varepsilon]$. Use these objects to create E, α and ν.

Solution: Fix $x\in\mathbb{R}$ and then $\varepsilon>0$ as in the hint. Let $E=[x-\varepsilon,x+\varepsilon]$, and $\alpha=\varepsilon p(x,y)$. Finally let ν be uniform on $[y-\varepsilon,y+\varepsilon]$.

For any $x'\in E$ and $A\subset\mathbb{R}$,

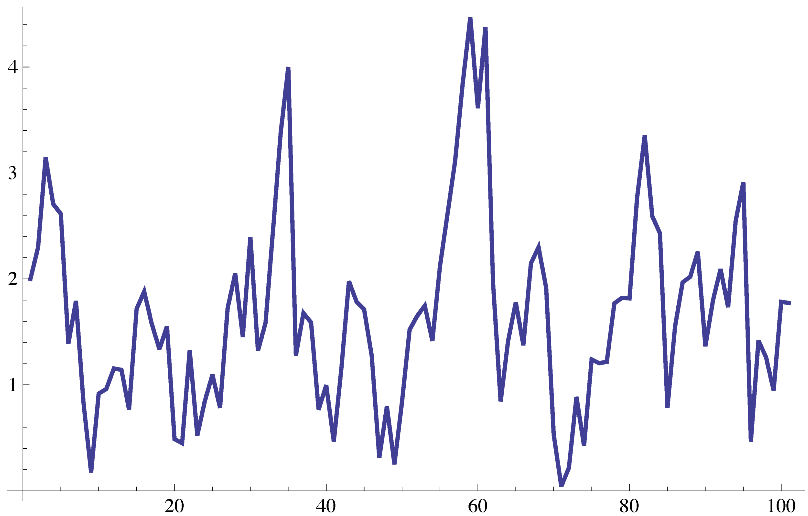
$$\begin{aligned}\mathbb{P}\left(X_1\in A|X_0=x'\right) = \int_A p(x',y')dy' &\geq \int_{A\cap[y-\varepsilon,y+\varepsilon]}p(x',y')dy'\\ &\geq \int_{A\cap[y-\varepsilon,y+\varepsilon]}\frac{p(x,y)}{2}dy'\\ &= \varepsilon p(x,y)\int_{A\cap[y-\varepsilon,y+\varepsilon]}\frac{1}{2\varepsilon}dy'\\ &= \alpha\nu(A).\end{aligned}$$

Thus E is a small set. Since we can do this for any $x\in\mathbb{R}$, and we can choose $\varepsilon>0$ arbitrarily small, we can create arbitrarily many such small sets.

5. Consider a Vervaat perpetuity X, where

$$X_0=0;\qquad X_{n+1}=U_{n+1}(X_n+1)\,,$$

   and where $U_1,U_2,\ldots$ are independent Uniform$(0,1)$ (simulated below).



Find a small set for this chain.

Solution: There are many possible answers. Here is one: let $E = [0,1]$, $\alpha = 1/2$ and ν be uniform on $[0,1]$. Let U be a uniform random variable on $[0,1]$. Then for any $x \in E$ and $A \subset [0,\infty)$,

$$\mathbb{P}(X_1 \in A | X_0 = x) = \mathbb{P}(U(x+1) \in A) = \int_{A\cap[0,x+1]} \frac{1}{x+1} dy$$
$$\geq \frac{1}{2}\int_{A\cap[0,1]} dy = \frac{1}{2}\nu(A).$$

So E is small.

6. Recall the idea of regenerating when our chain hits a small set: suppose that C is a small set for a ϕ-irreducible chain X, i.e. for $x \in C$,

$$\mathbb{P}(X_1 \in A | X_0 = x) \geq \alpha\nu(A).$$

Suppose that $X_n \in C$. Then with probability α let $X_{n+1} \sim \nu$, and otherwise let it have transition distribution $\frac{p(x,\cdot)-\alpha\nu(\cdot)}{1-\alpha}$.
   (a) Check that the latter expression really gives a probability distribution.
   (b) Check that X_{n+1} constructed in this manner obeys the correct transition distribution from X_n.
7. Define a reflected random walk as follows: $X_{n+1} = \max\{X_n + Z_{n+1}, 0\}$, for $Z_1, Z_2, \ldots$ i.i.d. with continuous density $f(z)$,

$$\mathbb{E}[Z_1] < 0 \quad \text{and} \quad \mathbb{P}(Z_1 > 0) > 0\,.$$

Show that the Foster-Lyapunov criterion for positive recurrence holds, using $\Lambda(x) = x$.

Hint: Choose $c \in (0,\infty)$ such that $\mathbb{P}(Z_1 \leq -c) > 0$ and $\mathbb{E}[Z_1 \mathbf{1}_{Z_1 > -c}] < 0$. (To see that there exists such a c, let

$$\tilde{c} = \sup\{x \in \mathbb{R} : \mathbb{P}(Z_1 \leq -x) > 0\} \in (0,\infty].$$

Then since Z_1 has a density, $\mathbb{E}[Z_1 \mathbf{1}_{Z_1 > -\tilde{c}}] = \mathbb{E}[Z_1] < 0$, so $\lim_{x\uparrow\tilde{c}} \mathbb{E}[Z_1 \mathbf{1}_{Z_1 > -x}] = \mathbb{E}[Z_1] < 0$ and $\mathbb{P}(Z_1 \leq -c) > 0$ for all $c < \tilde{c}$.) Then show that $\{x : \Lambda(x) \leq c\}$ is small.

Solution: Take c as in the hint. First we show that $\{x : \Lambda(x) \leq c\} = [0, c]$ is a small set. Let $\nu = \delta_0$, the delta mass at 0, and $\alpha = \mathbb{P}(Z_1 \leq -c)$. Then for $x \in [0, c]$,

$$\mathbb{P}(X_1 \in A | X_0 = x) \geq \mathbf{1}_{0 \in A} \mathbb{P}(X_1 = 0 | X_0 = x) \geq \mathbf{1}_{0 \in A} \mathbb{P}(Z_1 \leq -c) = \alpha \nu(A),$$

so indeed $[0, c]$ is small.

Then we have

$$\mathbb{E}[\Lambda(X_{n+1}) | \mathcal{F}_n] = \mathbb{E}[X_{n+1} | \mathcal{F}_n] = \int_{-X_n}^{\infty} (X_n + z) f(z) dz \leq X_n + \int_{-X_n}^{\infty} z f(z) dz.$$

If $X_n \notin C$, i.e. $X_n > c$, then this is smaller than $X_n - \int_{-c}^{\infty} z f(z) dz$ and we have chosen c such that $\int_{-c}^{\infty} z f(z) dz = \mathbb{E}[Z_1 \mathbf{1}_{Z_1 > -c}] < 0$. Thus we can set $a = -\mathbb{E}[Z_1 \mathbf{1}_{Z_1 > -c}]$.

On the other hand, if $X_n \in C$, then

$$\mathbb{E}[\Lambda(X_{n+1}) | \mathcal{F}_n] \leq X_n + \int_{-X_n}^{\infty} z f(z) dz \leq X_n + \int_0^{\infty} z f(z) dz$$

so we can choose $b = a + \int_0^{\infty} z f(z) dz$. Thus we have shown that X satisfies the Foster-Lyapunov criterion for positive recurrence.