Appendix B : SUBMERGING A PUNCTURED TORUS

This contains verbatim a letter from J. Milnor of October, 1969 , which gives an elementary construction of a submersion of the punctured torus T^n–point into euclidean space R^n . It is used in §3 . A different elementary construction was found by D. Barden [Bar] [Ru] earlier in 1969 , and another by S. Ferry , [Fe] 1973†. Milnor produces a smooth C^∞ (= DIFF) submersion . A secant approximation to it in the sense of J. H. C. Whitehead [Mu_1 , §9] provides a piecewise-linear (= PL) submersion .

"Let M be a smooth compact manifold .

HYPOTHESIS . M has a codimension 1 embedding in euclidean space so that , for some smooth disk $D \subset M$ and some hyperplane P in euclidean space , the orthogonal projection from $M - D$ to P is a submersion .

THEOREM . *If M satisfies this hypothesis , so does* $M \times S^1$.

It follows inductively that every torus satisfies the hypothesis .

PROOF . Suppose that $M = M^{k-1}$ embeds in R^k so that $M - D$ projects submersively to the hyperplane $x_1 = 0$. We will assume that the subset $M \subset R^k$ lies in the half-space $x_k > 0$. Hence , rotating R^k about R^{k-1} in R^{k+1}, we obtain an embedding $(x, \theta) \leftrightarrow (x_1 ,..., x_{k-1} , x_k \cos\theta , x_k \sin\theta)$ of $M \times S^1$ in R^{k+1} . This embedding needs only a mild deformation in order to satisfy the required property .

Let $e_1 ,..., e_{k+1}$ be the standard basis for R^{k+1} . Let r_θ be the rotation

$$e_i \leftrightarrow e_i \text{ for } i < k , \quad \begin{array}{l} e_k \leftrightarrow e_k \cos\theta + e_{k+1} \sin\theta , \\ e_{k+1} \leftrightarrow -e_k \sin\theta + e_{k+1} \cos\theta . \end{array}$$

Let $n(x) = n_1(x)e_1 + \cdots + n_k(x)e_k$ be the unit normal vector to M in R^k.

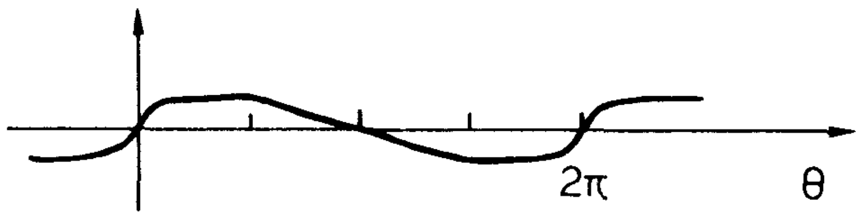
† And still another by A. Gramain [Gra] 1973 .

For $x \in M-D$ we can assume that n_1 is bounded away from zero. Say $n_1 \geqslant 2\alpha > 0$.

Suppose that M lies in the open slab $0 < x_k < \beta$ of R^k. Choose $\epsilon > 0$ so that the correspondence $(x, t) \mapsto x + tn(x)$ embeds $M \times (-\epsilon, \epsilon)$ diffeomorphically in this slab.

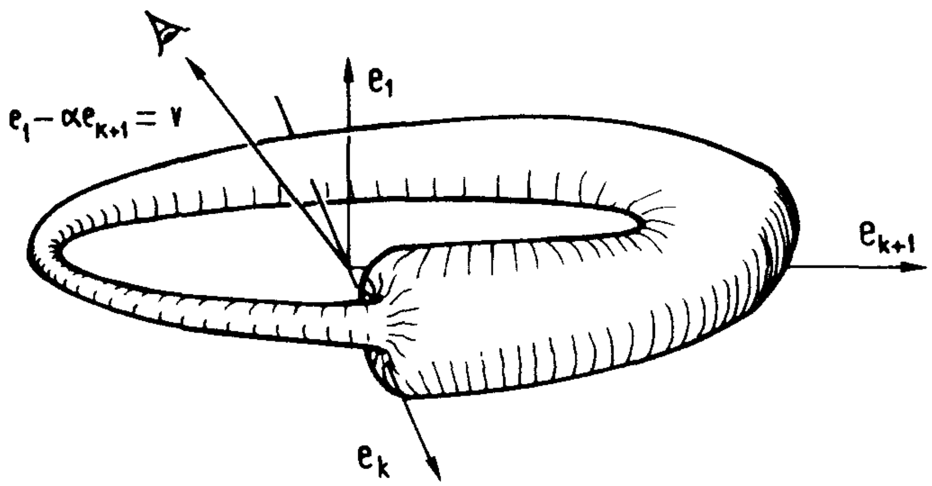
Choose a smooth map $t : S^1 \to (-\epsilon, \epsilon)$ so that

$$\frac{dt}{d\theta} \geqslant 2\beta/\alpha \text{ when } \theta = 0 ; \qquad \cos\theta \frac{dt}{d\theta} \geqslant 0 \text{ always}.$$



The required embedding $M \times S^1 \to R^{k+1}$ is now given by

$$(x, \theta) \mapsto r_\theta(x + t(\theta)n(x)).$$



Computation shows that the normal vector to this embedding is $p/\|p\|$ where

$$p(x, \theta) = (x_k + tn_k)r_\theta(n) - \frac{dt}{d\theta}$$

Let $v = e_1 - \alpha e_{k+1}$. Then $p \cdot v = A + B$ where

$$A = (x_k + tn_k)(n_1 - \alpha \sin\theta\, n_k) \quad \text{and} \quad B = \alpha \cos\theta \frac{dt}{d\theta} \geqslant 0 .$$

Thus if $x \in M - D$ we have

$$A \geqslant (x_k + tn_k)(2\alpha - \alpha) > 0$$

hence $p \cdot v > 0$. On the other hand, for any $x \in M$, if $\theta = 0$, we have

$$A \geqslant -\beta , \qquad B \geqslant \alpha(2\beta/\alpha) .$$

Hence $p \cdot v > 0$ for $\theta = 0$, and therefore $p \cdot v > 0$ for all sufficiently small θ; say for $|\theta| \leqslant \eta$.

It now follows that the complement $(M \times S^1) - (D \times [\eta, 2\pi - \eta])$ projects submersively to the hyperplane $v^\perp$. This completes the proof.
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