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SOME WILD CELLS AND SPHERES IN THREE-DIMENSIONAL SPACE

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A curved polyhedron[1] in spherical n-dimensional space S^n will be said to be *tamely imbedded* if there is a homeomorphism of S^n on itself which transforms the imbedded polyhedron into a Euclidean polyhedron[1]; if no such homeomorphism exists we shall say that the polyhedron is *wildly imbedded*. It is a corollary of classical results in plane topology that every curved polyhedron in 2-dimensional space is tamely imbedded. On the other hand it was shown by Antoine[2] that there are wild imbeddings in 3-dimensional space. A well-known example of this is the Alexander "horned sphere".[3]

In all the known examples the wildness of the imbedding hinges on consideration of the fundamental group of the complement. We present here a series of examples, some of which may be regarded as simplifications of classical examples and some of which are such that their wildness can not be deduced from the fundamental group of the complement.

Our basic examples are wild arcs. For them the projection method of knot theory is available, and this allows descriptions of greater precision than had previously been possible.

1. Simple arcs

To describe our examples we begin with a right circular cylinder C, mark on one base A_- three collinear points r_-, s_-, t_-, and on the other base A_+ three collinear points r_+, s_+, t_+. We construct in C three non-intersecting oriented simple polygonal arcs, K_- joining s_- to r_-, K_0 joining t_- to s_+, and K_+ joining r_+ to t_+. These three arcs, which are to have only their end-points in common with the boundary $\dot{C}$ of C, are to be arranged as indicated in figure 1. Denote the union of K_-, K_0 and K_+ by K.

Divide the ellipsoid of revolution whose equation is $x^2 + 4y^2 + 4z^2 \leqq 4$ into an infinite number of sections by the family of parallel planes $x = \pm(2 - 2^{1-m})$, $m = 0, 1, \cdots$. For each positive integer n denote by D_n the section $2 - 2^{2-n} \leqq x \leqq 2 - 2^{1-n}$, and for each non-positive integer n denote by D_n the section $-2 + 2^{-n} \leqq x \leqq -2 + 2^{1-n}$. The observer is to be so situated that the ellipsoid appears as in figure 2, with D_n to the left of D_{n+1}. Denote by p and q the vertices $(-2, 0, 0)$ and $(2, 0, 0)$ respectively.

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[1] By a Euclidean polyhedron we mean a subset of S^n which is the union of a finite collection of convex cells. By a curved polyhedron we mean any subset of S^n which is homeomorphic to a Euclidean polyhedron. Cf. P. ALEXANDROFF and H. HOPF, Topologie, Berlin (1935) Chapter III.

[2] L. ANTOINE, C.R., Acad. Sci. Paris 171 (1920), p. 661 and Journ. Math. pures appl. (8) 4 (1921), pp. 221–325.

[3] J. W. ALEXANDER, Proc., Nat. acad. sci. 10 (1924) pp. 8–10.

For each integer n choose an orientation-preserving[4] homeomorphism f_n of C upon D_n in such a way that

(i) the base A_- is mapped upon the left face of D_n, the base A_+ upon the right face;

(ii) the three points $f_n(r_+)$, $f_n(s_+)$ $f_n(t_+)$ lie, in ascending order, on a vertical line through the x-axis and coincide with the three points $f_{n+1}(r_-)$, $f_{n+1}(s_-)$, $f_{n+1}(t_-)$ respectively;

(iii) $f_n(K)$ has a regular normed projection in the xz-plane similar to the one indicated in figure 3.

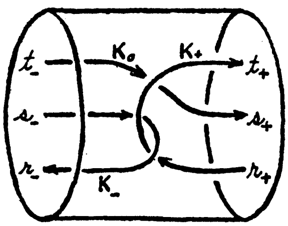

FIG. 1

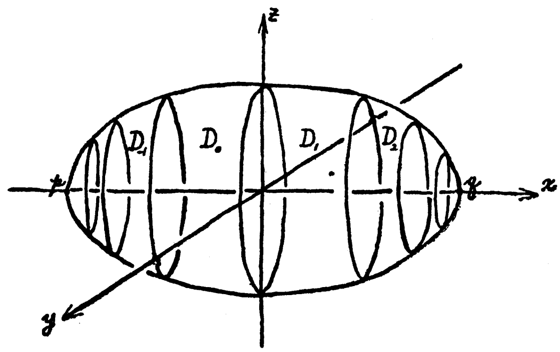

FIG. 2

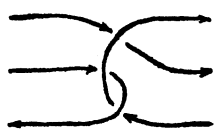

FIG. 3

We shall also have occasion to refer to an orientation-reversing homeomorphism g_n of C upon D_n which has the following properties:

(i) A_- is mapped upon the right face $f_n(A_+)$ of D_n and A_+ upon the left face $f_n(A_-)$;

(ii) $g_n(r_+) = f_n(r_-)$, $g_n(s_+) = f_n(s_-)$, $g_n(t_+) = f_n(t_-)$;

(iii) $g_n(K)$ has a regular normed projection in the xz-plane similar to the mirror image of figure 3.

EXAMPLE 1.1. *A simple arc in 3-space whose complement is not simply connected.*[5]

The simple arc X to be considered is the set $p \cup \bigcup_{n=-\infty}^{\infty} f_n(K) \cup q$. Its projection in the xz-plane is shown in figure 4.

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[4] C and D_n are supposed to be oriented similarly in figures 1 and 2.

[5] An example of this sort is implicit in Alexander, ibid., p. 12.

The fundamental group $\pi(S - X)$ of the complement of X is[6] the direct limit of the direct homomorphism sequence

$$G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow \cdots ,$$

where G_m denotes the fundamental group of the complement of $X \cup \overline{\bigcup_{|n| \geqq m} D_n}$, and $G_m \rightarrow G_{m+1}$ denotes the injection homomorphism. Each group G_m is thus the group of a certain knotted graph. Hence, using a standard method of calculation,[7] a set of generators and defining relations for G_m may be written down explicitly. We find that G_m is generated by elements a_n, b_n, c_n

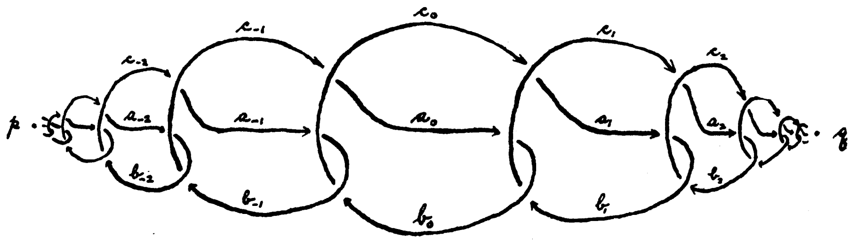

FIG. 4

$(-m \leqq n < m)$ indicated in the usual way in figure 4, and that a set of defining relations is the following:

$$\begin{aligned}
b_{-m}a_{-m}^{-1}c_{-m}^{-1} &= 1, && \text{(relation about the "vertex" } \overline{\textstyle\bigcup_{n \leqq -m} D_n}), \\
c_{m-1}a_{m-1}b_{m-1}^{-1} &= 1, && \text{(relation about the "vertex" } \overline{\textstyle\bigcup_{n \geqq m} D_n}), \\
a_{n+1} &= c_{n+1}^{-1}c_nc_{n+1}, && \\
b_n &= c_{n+1}^{-1}a_nc_{n+1}, && \\
c_{n+1} &= b_n^{-1}b_{n+1}b_n, && (-m \leqq n < m-1).
\end{aligned}$$

The injection homomorphism $G_m \rightarrow G_{m+1}$ maps each generator a_n, b_n, c_n of G_m into the same-named generator of G_{m+1}. Hence[8] $\pi(S - X)$ is generated by

[6] Direct limit is defined for example in N. STEENROD, Am. Jour. of Math. 58 (1936), p. 669. We are making use of the following easily proved theorem: Let $M_1 \subset M_2 \subset \cdots$, suppose that M_m is open in $\bigcup_{n=1}^{\infty} M_n$ and choose a base point in M_1. Then $\pi(M)$ *is the limit group of the direct homomorphism sequence* $\pi(M_1) \rightarrow \pi(M_2) \rightarrow \cdots$, *where the homomorphism* $\pi(M_m) \rightarrow \pi(M_{m+1})$ *is the injection.*

[7] See, for example, K. REIDEMEISTER, Knotentheorie, Berlin (1932), Chapter III, §3, and J. H. C. WHITEHEAD, Fund. Math. 32 (1939), p. 151 and p. 156. Our convention is such that an element g is represented by a path linking the segment marked g in such a way that it penetrates the plane of projection from above on the left-hand side of the (oriented) segment.

[8] *If* $G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \cdots$ *is a direct homomorphism sequence and if* G_m *is generated by elements* $x_{m1}, \cdots, x_{m\lambda_m}$ *subject to defining relations* $R_{m1}(x_{m1}, \cdots, x_{m\lambda_m}) = 1, \cdots, R_{m\mu_m}(x_{m1}, \cdots, x_{m\lambda_m}) = 1$, *then the limit group G is generated by all the elements* x_{mj} $(m = 1, 2, \cdots; i = 1, \cdots, \lambda_m)$ *subject to the defining relations* $R_{mi} = 1$ $(m = 1, 2, \cdots; i = 1, \cdots, \mu_m)$ *and* $x_{mj} = \phi_m(x_{mj})$ $(m = 1, 2, \cdots; j = 1, \cdots, \lambda_m)$.

elements a_n, b_n, c_n $(-\infty < n < \infty)$, represented by loops which represent the same-named elements of G_m, and a set of defining relations is the following:

$$(1_n) \qquad \begin{aligned} c_n a_n b_n^{-1} &= 1, \\ a_{n+1} &= c_{n+1}^{-1} c_n c_{n+1}, \\ b_n &= c_{n+1}^{-1} a_n c_{n+1}, \\ c_{n+1} &= b_n^{-1} b_{n+1} b_n. \end{aligned} \qquad (-\infty < n < \infty)$$

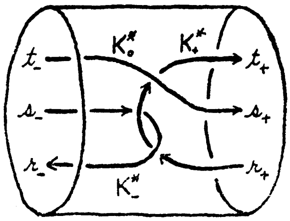

FIG. 5

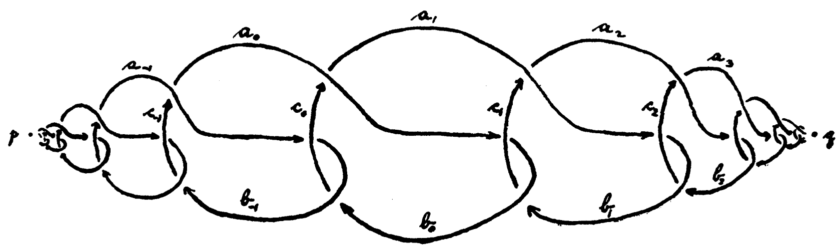

FIG. 6

Upon eliminating a_n and b_n we obtain the single set of relations

$$(2_n) \qquad c_{n-1}c_nc_{n+1} = c_nc_{n+1}c_{n-1}c_n \qquad (-\infty < n < \infty)$$

in the generating set $\cdots, c_{-1}, c_0, c_1, \cdots$. This group is non-trivial because it has the representation

$$\begin{aligned} c_n &\rightarrow (1\ 2\ 3\ 4\ 5) \quad \text{for } n \text{ odd} \\ &\rightarrow (1\ 4\ 2\ 3\ 5) \quad \text{for } n \text{ even} \end{aligned}$$

into the permutation group on five letters.

EXAMPLE 1.1* *Another simple arc whose complement is not simply connected.* A modification of the previous example is obtained by replacing the three arcs K_-, K_0, K_+ by three arcs K_-^*, K_0^*, K_+^* situated in C as shown in figure 5. Proceeding as before, we construct the simple arc $X^* = p \cup \bigcup_{n=-\infty}^{\infty} f_n(K^*) \cup q$, where $K^* = K_-^* \cup K_0^* \cup K_+^*$. Its projection in the xz-plane is shown in figure 6.[9]

[9] This is just the chain stitch of knitting extended indefinitely in both directions. The later examples based on 1.1 could just as well have been based on 1.1*.

The group $\pi(S - X^*)$ is generated by the elements a_n, b_n, c_n $(-\infty < n < \infty)$ indicated in figure 6. A set of defining relations is

$$(1_n^*)\qquad \begin{aligned} a_{n+1}a_nb_n^{-1} &= 1, \\ b_n &= c_{n+1}^{-1}a_nc_{n+1}, \\ c_{n+1} &= b_n^{-1}b_{n+1}b_n, \\ a_{n+2} &= a_{n+1}c_{n+1}a_{n+1}^{-1}. \end{aligned} \qquad (-\infty < n < \infty)$$

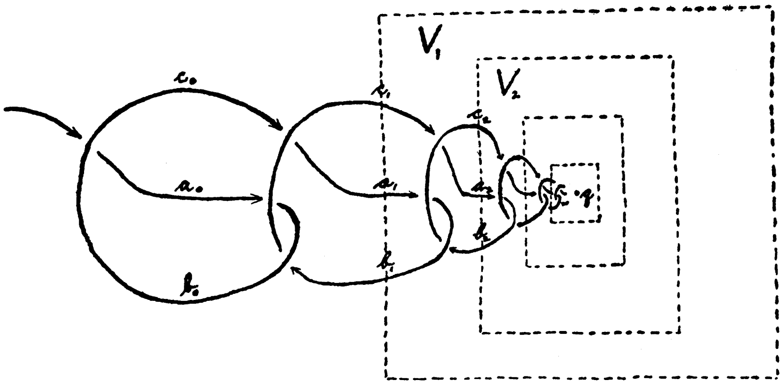

FIG. 7

Elimination of b_n and c_n leads to the single set of relation,

$$(2_n^*)\qquad a_na_{n+1}^2a_{n+2} = a_{n+1}a_{n+2}a_{n+1}^{-1}a_na_{n+1} \qquad (-\infty < n < \infty)$$

in the generating set $\cdots, a_{-1}, a_0, a_1, \cdots$. This group is non-trivial because it has the representation

$$\begin{aligned} a_n &\to (1\ 2\ 3\ 4\ 5) \quad \text{for } n \equiv 0 \pmod 3 \\ &\to (1\ 2\ 5\ 3\ 4) \quad \text{for } n \equiv 1 \pmod 3 \\ &\to (1\ 2\ 4\ 5\ 3) \quad \text{for } n \equiv 2 \pmod 3 \end{aligned}$$

into the permutation group on five letters.

In these two examples the wildness of the imbedding is a consequence of the non-triviality of the fundamental group of the complement. In fact the complement of a tame arc is necessarily an open 3-cell. However, even if the complement of a wild arc is simply connected it need not be an open 3-cell. The next two examples show that such a simply-connected complement of a wild arc may or may not be an open 3-cell.

EXAMPLE 1.2. *A simple arc in 3-space whose imbedding is wild even though its complement is an open 3-cell.*

This simple arc, which will be denoted by Y, is the set $f_0(K_0) \cup f_0(K_+) \cup \bigcup_{n=1}^{\infty} f_n(K) \cup q$. Its projection in the xz-plane is shown in figure 7.

To show that its exterior is an open 3-cell we construct in C non-intersecting closed tubular neighborhoods U_- of K_-, U_0 of K_0, and U_+ of K_+ in such a way that $U_- \cap \dot{C} = S_- \cup R_-$, $U_0 \cap \dot{C} = T_- \cup S_+$, $U_+ \cap \dot{C} = R_+ \cup T_+$, where R_-, S_-, T_-, R_+, S_+, T_+ are closed 2-cells on $A_- \cup A_+$ containing respectively the points r_-, s_-, t_-, r_+, s_+, t_+ in their interiors. Denote $U_- \cup U_0 \cup U_+$ by U. We may suppose that $f_n(R_+) = f_{n+1}(R_-) = g_n(R_-)$, $f_n(S_+) = f_{n+1}(S_-) = g_n(S_-)$, $f_n(T_+) = f_{n+1}(T_-) = g_n(T_-)$.

It is easy to see that $\bigcup_{n=-1}^{\infty} f_n(U) \cup q$ is a 3-cell and that there is a homeomorphism φ of $\bigcup_{n=-1}^{\infty} f_n(U) \cup q$ upon the 3-cell $x^2 + y^2 + z^2 \leqq 1$ which transforms q into the point (1, 0, 0) and Y into the segment $0 \leqq x \leqq 1$, $y = z = 0$. Now the 3-cell $x^2 + y^2 + z^2 \leqq 1$ can be mapped continuously upon itself in such a way that the points of the boundary remain fixed, the segment $\varphi(Y)$ is mapped into the point $\rho(q)$, and the mapping is a homeomorphism over $\{x^2 + y^2 + z^2 \leqq 1\} - \varphi(Y)$. Therefore 3-space can be mapped continuously upon itself in such a way that the points on the boundary and in the exterior of $\bigcup_{n=-1}^{\infty} f_n(U) \cup q$ remain fixed, the mapping is a homeomorphism over the exterior of Y, and Y itself is mapped into q. Thus the exterior of Y is mapped homeomorphically upon the exterior of q, and this is an open 3-cell.

To show that Y is wildly imbedded we first develop a necessary condition for an imbedding of an arc to be tame. The prototype of a tamely imbedded arc is the segment $L = \{0 \leqq x \leqq 1, y = z = 0\}$. Let $\{V_n\}$ be a sequence of closed neighborhoods of the end-point $o = (0, 0, 0)$ of L such that $V_1 \supset V_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} V_n = o$. Choose in succession a positive number ϵ such that the 3-cell $C_\epsilon : x^2 + y^2 + z^2 \leqq \epsilon$ is a subset of V_1 and an index N such that V_N is a subset of C_ϵ. Also choose a point in $V_N - L$ to serve as the base point for the three fundamental groups $\pi(V_N - L)$, $\pi(C_\epsilon - L)$, $\pi(V_1 - L)$. Since the injection of $\pi(V_N - L)$ into $\pi(V_1 - L)$ is compounded of the injection of $\pi(V_N - L)$ into $\pi(C_\epsilon - L)$ and the injection of $\pi(C_\epsilon - L)$ into $\pi(V_1 - L)$ and since $\pi(C_\epsilon - L)$ is trivial it follows that the injection of $\pi(V_N - L)$ into $\pi(V_1 - L)$ is trivial. Thus we have the following theorem:

If o is an end-point of a tamely imbedded arc L and $\{V_n\}$ is any sequence of closed neighborhoods of o such that $V_1 \supset V_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} V_n = o$ then there must exist an index N such that the injection $\pi(V_N - L) \to \pi(V_1 - L)$ is trivial.

Let $\{V_n\}$ be a sequence of closed neighborhoods of the end-point q of the simple arc Y such that $V_1 \supset V_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} V_n = q$. Let us choose V_n in such a way that V_n is convex and intersects the ellipsoid $x^2 + 4y^2 + 4z^2 \leqq 4$ in the set $\bigcup_{n=1}^{\infty} f_n(C) \cup q$. The group $\pi(V_N - Y)$ is generated by the elements c_N, c_{N+1}, $\cdots$ indicated in figure 7 subject to the defining relations (2_n), $N \leqq n < \infty$. The base point for $\pi(V_N - Y)$ is to be chosen in $V_N - Y$ in the same general position with respect to Y as the original observer. Any element $c_n (n \geqq N)$ of $\pi(V_N - Y)$ is mapped by injection into the element c_n of $\pi(V_1 - Y)$. Hence if this injection were trivial the element c_n of $\pi(V_1 - Y)$ would be trivial. However, the group $\pi(V_1 - Y)$ has the representation

$$\begin{aligned} c_n &\to (1\ 2\ 3\ 4\ 5) \quad \text{for } n \text{ odd} \\ &\to (1\ 4\ 2\ 3\ 5) \quad \text{for } n \text{ even} \end{aligned}$$

into the permutation group on five letters, in which no element c_n is represented by the identity permutation. Therefore, according to the criterion developed above, the simple arc Y is wildly imbedded.

We conjecture that every sufficiently small neighborhood of q which is homeomorphic to a 3-cell is such that its boundary 2-sphere has an intersection with Y which consists of at least three components. A further conjecture is that the intersection of the complement of Y with any sufficiently small neighborhood of q has a non-trivial 1-dimensional homology group.

We note however that Y is the intersection of a monotone decreasing sequence of tamely imbedded 3-cell neighborhoods of Y. Clearly every tamely imbedded arc has this property; however the simple arc X of 1.1 obviously does not.

EXAMPLE 1.3. *A simple arc in 3-space whose complement is simply connected but is not an open 3-cell.*[10]

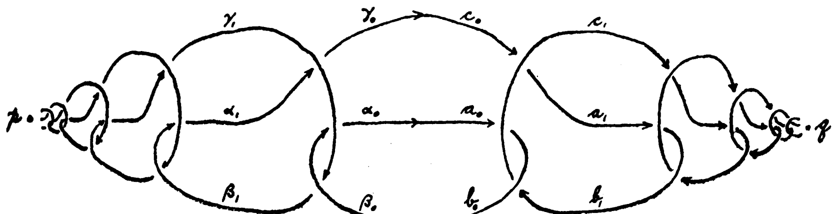

FIG. 8

The simple arc Z now to be considered is the set

$$p \cup \bigcup_{n=-\infty}^{-1} g_n(K) \cup g_0(K_+) \cup g_0(K_0) \cup f_1(K_0) \cup f_1(K_+) \cup \bigcup_{n=2}^{\infty} f_n(K) \cup q.$$

The set $g_0(K_-) \cup f_1(K_-)$ is a simple closed polygonal curve W disjoint to Z. Thus $p \cup \bigcup_{n=-\infty}^{0} g_n(K) \cup \bigcup_{n=1}^{\infty} f_n(K) \cup q$ consists of the simple arc Z and the simple closed curve W. The projection of Z and W in the xz-plane is shown in figure 8.

That Z is wildly imbedded is clear from the previous examples. Although it is clear geometrically that the complement of Z is simply connected, this fact will be given a precise proof below. To show that the complement of Z is not an open 3-cell we shall require an analysis of the fundamental group $\pi(S - W \cup Z)$ of the complement of $W \cup Z$. This group is generated by the elements a_n, b_n, c_n, $(n \geqq 0)$ and elements α_n, β_n, $\gamma_n (n \geqq 0)$ which are represented by loops which are symmetric with respect to the plane $x = 0$ to loops representative of a_n, b_n, c_n. A set of defining relations, read from figure 8 is the following:

$$\begin{aligned} c_n a_n b_n^{-1} &= 1, & \gamma_n \alpha_n \beta_n^{-1} &= 1, \\ a_{n+1} &= c_{n+1}^{-1} c_n c_{n+1}, & \alpha_{n+1} &= \gamma_{n+1}^{-1} \gamma_n \gamma_{n+1}, \end{aligned} \qquad (n \geqq 0)$$

[10] A closed subset of S^3 whose complement is simply connected but is not an open 3-cell was constructed by M. H. A. NEWMAN and J. H. C. WHITEHEAD, Quart. Journal of Math. 8 (1937), p. 14. In their example the closed set is rather pathological (it is not even locally connected).

$$b_n = c_{n+1}^{-1} a_n c_{n+1}, \qquad \beta_n = \gamma_{n+1}^{-1} \alpha_n \gamma_{n+1},$$
$$c_{n+1} = b_n^{-1} b_{n+1} b_n, \qquad \gamma_{n+1} = \beta_n^{-1} \beta_{n+1} \beta_n,$$
$$a_0 = \alpha_0, \qquad b_0 = \beta_0, \qquad c_0 = \gamma_0.$$

Elimination of a_n, $\alpha_n (n \geqq 1)$ and b_n, $\beta_n (n \geqq 0)$ leads to the set of defining relations

$$(3), \qquad \begin{aligned} c_{n-1} c_n c_{n+1} &= c_n c_{n+1} c_{n-1} c_n, \\ \gamma_{n-1} \gamma_n \gamma_{n+1} &= \gamma_n \gamma_{n+1} \gamma_{n-1} \gamma_n, \end{aligned} \qquad (n \geqq 1)$$
$$a_0 c_1 = c_1 c_0 a_0, \qquad \alpha_0 \gamma_1 = \gamma_1 \gamma_0 \alpha_0,$$
$$a_0 = \alpha_0, \qquad c_0 = \gamma_0,$$

in the generating set a_0, α_0, c_0, γ_0, c_1, γ_1, $\cdots$. This group has the representation

$$\begin{aligned} c_n &\rightarrow (1\ 2\ 3\ 4\ 5) && \text{for } n \text{ odd} \\ & (1\ 4\ 2\ 3\ 5) && \text{for } n \text{ even} \\ \gamma_n &\rightarrow (1\ 2\ 3\ 4\ 5) && \text{for } n \text{ odd} \\ & (1\ 4\ 2\ 3\ 5) && \text{for } n \text{ even} \\ a_0 &\rightarrow (2\ 3)(4\ 5) \\ \alpha_0 &\rightarrow (2\ 3)(4\ 5) \end{aligned}$$

in the permutation group on five letters. From (3) we see that the commutator quotient group of $\pi(S - W \cup Z)$ is the abelian group generated by a_0, α_0, c_0, γ_0, c_1, γ_1, $\cdots$ subject to the relations $a_0 = \alpha_0$, $c_n = 1$, $\gamma_n = 1 (n \geqq 0)$. Since we have represented c_n non-trivially in the permutation group on five letters it follows that $\pi(S - W \cup Z)$ is a non-abelian group.

We note that the simple connectivity of the complement of Z can now be proved precisely. In fact the fundamental group of the complement of Z is obtained from $\pi(S - W \cup Z)$ by adjoining the relation $a_0 = 1$. It follows from (3) that, as a consequence of this adjunction, $a_0 = \alpha_0 = 1$ and $c_n = \gamma_n = 1 (n \geqq 0)$. Thus $\pi(S - Z)$ is a trivial group.

We now prove that the complement of Z is not an open 3-cell.

Any compact subset of an open 3-cell is contained in a closed 3-cell whose complement is simply connected. Hence we need only prove that

W is contained in no 3-cell subset J of $S - Z$ whose complement in $S - Z$ is simply connected.

Suppose then that we had such a 3-cell J. Choose the base point for fundamental groups in the complement of $J \cup Z$ and so close to the point $f_1(t_-)$ (cf. Fig. 1) that there is a loop in $S - J \cup Z$ which represents the element $c_0 = \gamma_0$ of $\pi(S - W \cup Z)$. Since $S - J \cup Z$ is simply connected this loop can be shrunk to the base point in the complement of $J \cup Z$ and hence in the complement of $W \cup Z$. This is impossible as we have seen that the element c_0 of $\pi(S - W \cup Z)$

is not trivial. Since no such 3-cell J can be found $S - Z$ cannot be an open 3-cell.

We may clarify the structure of the open 3-dimensional manifold $S - Z$ by noting that any point in the complement of Z is contained in the interior of a 3-cell disjoint to Z whose complement in $S - Z$ is simply connected. Hence the remarkable property of W, proved above, may be restated as follows:

There is a simple closed curve W in the open 3-dimensional manifold $S - Z$ which is such that no homeomorphism of $S - Z$ upon itself can transform W into a sufficiently small neighborhood of any point.

It seems to us that the existence or non-existence of a *closed* simply connected 3-dimensional manifold with this property would be a decisive point for the settling of the Poincaré conjecture.

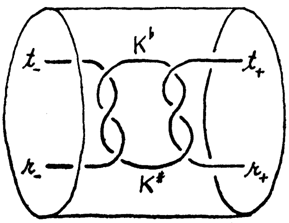

FIG. 9

Since $S - Z$ is an open subset of S it is locally connected in dimensions 1, 2 and 3 and hence is an absolute neighborhood retract. Since $S - Z$ is furthermore simply connected and acyclic in dimensions 1, 2 and 3 it follows that $S - Z$ is contractible.[11]

Another way to prove that $S - Z$ is not an open 3-cell is to prove that the hyperspace Σ of the decomposition of S into the closed set Z and the individual points of $S - Z$ is not a 3-sphere. This can be made to follow from the non-abelian character of $\pi(S - W \cup Z)$.

EXAMPLE 1.4. *A wildly imbedded arc which is the union of two tamely imbedded arcs.*[12]

Denote by $K^{\sharp}$ and $K^{\flat}$ two arcs situated in C as shown in figure 9, joining t_- to t_+ and r_+ to r_- respectively. The two arcs $H^{\sharp} = \bigcup_{n=1}^{\infty} f_n(K^{\sharp}) \cup q$ and $H^{\flat} = \bigcup_{n=1}^{\infty} f_n(K^{\flat}) \cup q$ intersect in their common end-point q. Their union is the simple arc $H^{\natural} = H^{\sharp} \cup H^{\flat}$. The projection of these three arcs on the xz-plane is shown in figure 10. From the fact that this projection of the arc $H^{\sharp}$

[11] A proof of this theorem will be found in the forthcoming book Topology of Deformations by W. HUREWICZ and J. DUGUNDJI.

[12] An example which is virtually equivalent to 1.4 is furnished by a simple arc in which an infinite sequence of knots have been tied in such a way that they converge to a mid-point of the arc (cf. SEIFERT and THRELFALL, Lehrbuch der Topologie, p. 224, fig. 113). R. L. WILDER has considered such an example a number of times in lectures and has used it (Trans., Am. Math. Soc. 32 (1930), p. 634 footnote‡) to disprove several conjectures of R. L. MOORE (Bull., Am. Math. Soc. 29 (1923), p. 302). It is easy to show that the arc would be tame if the sequence of knots converged to an end-point instead of a mid-point

has no double points it follows easily that it is tamely imbedded. Similarly the simple arc $H^\flat$ is seen to be tamely imbedded.

The point q is an interior point of the arc H. Proceeding as in 1.2 it is easy to prove the following theorem:

If o is an interior point of a tamely imbedded arc L and $\{V_n\}$ is any sequence of closed neighborhoods of o such that $V_1 \supset V_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} V_n = o$ then there must exist an index N such that the injection $\pi(V_N - L) \to \pi(V_1 - L)$ has an abelian image group.

As before choose $\{V_n\}$, a sequence of closed neighborhoods of q to satisfy $V_1 \supset V_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} V_n = q$ and such that V_n is convex and intersects the ellipsoid $x^2 + 4y^2 + 4z^2 \leqq 4$ in the set $\bigcup_{n=1}^{\infty} f_n(C) \cup q$. The group $\pi(V_N - H^\natural)$

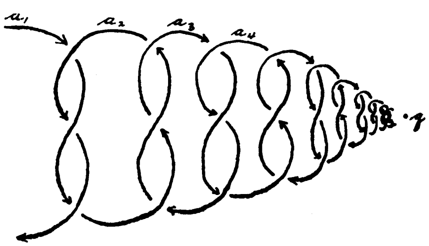

FIG. 10

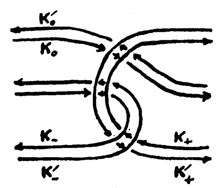

FIG. 11

is generated by the elements $a_{2N-1}, a_{2N}, \cdots$ indicated in figure 10 with the defining relations

$$(4_N) \qquad a_n a_{n+1} a_n = a_{n+1} a_n a_{n+1} \qquad (2N - 1 \leqq n < \infty).$$

Any element a_n of $\pi(V_N - H^\natural)$ is mapped by injection into the element a_n of $\pi(V_1 - H^\natural)$. Hence if the image of this injection were abelian the elements $a_{2N-1}, a_{2N}, \cdots$ would commute in $\pi(V_1 - H^\natural)$. But it would then follow from (4_1) that $a_{2N-1} = a_{2N} = \cdots$. This is not possible because the group $\pi(V_1 - H^\natural)$ has the representation

$$\begin{aligned} a_n &\to (1\ 2) \quad \text{for } n \text{ odd} \\ &\to (1\ 3) \quad \text{for } n \text{ even} \end{aligned}$$

into the permutation group on three letters. Thus the simple arc $H^\natural$ must be wildly imbedded.

2. Simple Closed Curves.

The two wildly imbedded simple closed curves considered in this section have a certain significance in connection with the (still unproved) Dehn's lemma.

EXAMPLE 2.1. *A simple closed curve which bounds a 2-cell although the fundamental group of its complement is non-abelian.*[13]

[13] An example of this sort was constructed by ALEXANDER, loc. cit., p. 12.

Construct in the cylinder C three arcs K'_-, K'_0, K'_+ on the boundaries of U'_-, U'_0, U'_+ respectively, and parallel to and oppositely oriented to the three arcs K_-, K_0, and K_+ respectively. We may suppose that the end-points, r'_- and s'_- of K'_-, s'_+ and t'_- of K'_0, and t'_+ and r'_+ of K'_+, are so placed that $f_n(r'_+) = f_{n+1}(r'_-) = g_n(r'_-)$, $f_n(s'_+) = f_{n+1}(s'_-) = g_n(s'_-)$, $f_n(t'_+) = f_{n+1}(t'_-) = g_n(t'_-)$ and the projection of $K \cup K'$, where $K' = K'_- \cup K'_0 \cup K'_+$, is as shown in figure 11. Clearly $X \cup X'$, where $X' = p \cup \bigcup_{n=-\infty}^{\infty} f_n(K') \cup q$, is a simple closed curve. It is equally clear that $X \cup X'$ is the boundary of a 2-cell in $p \cup \bigcup_{n=-\infty}^{\infty} f_n(U) \cup q$. The fundamental group of the complement of $X \cup X'$ maps homomorphically by injection onto the fundamental group of the complement of X and is therefore non-abelian.

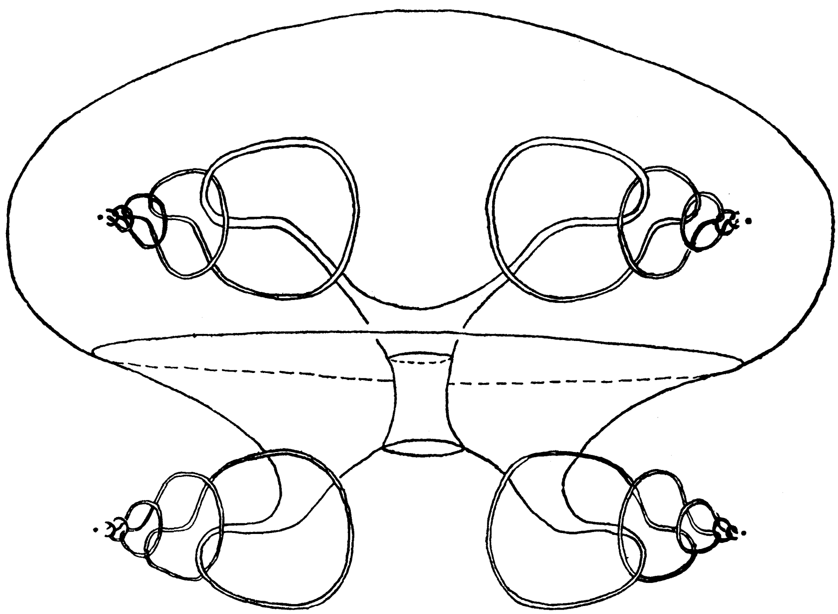

FIG. 12

EXAMPLE 2.2. *A simple closed curve whose imbedding is wild even though it bounds a 2-cell and the fundamental group of its complement is infinite cyclic.*

The simple closed curve to be considered is obtained from the arc

$$f_0(K_0) \cup f_0(K_+) \cup \bigcup_{n=1}^{\infty} f_n(K) \cup q \cup \bigcup_{n=1}^{\infty} f_n(K') \cup f_0(K'_+) \cup f_0(K'_0)$$

by joining the two end-points $f_0(t'_-)$ and $f_0(t_-)$ by a segment on $f_0(A_-)$. It is clearly wildly imbedded and the boundary of a 2-cell. It is easy to calculate the fundamental group of its complement and check that it is infinite cyclic.

A simple closed curve 2.3 with the same properties may be obtained by applying the same process to example 1.3. We do not know whether the complementary domain of either 2.2 or 2.3 is an open tubular manifold.

3. 2-spheres.

EXAMPLE 3.1. *A 2-sphere whose exterior is not simply connected.*[14]

Such a 2-sphere is the boundary X^0 of the 3-cell $p \cup \bigcup_{n=-\infty}^{\infty} f_n(U) \cup q$. It would be very simple to obtain from this a 2-sphere with both interior and exterior non-simply connected (cf. below).

EXAMPLE 3.2. *A 2-sphere which is wildly imbedded even though both complementary domains are open 3-cells.*

Such a 2-sphere is the boundary Y^0 of the 3-cell

$$f_0(U_0) \cup f_0(U_+) \cup \bigcup_{n=1}^{\infty} f_n(U) \cup q.$$

The proofs in 1.2 apply with a few mild changes in the wording. This example shows that *the 3-sphere S may be decomposed into a closed 3-cell and a complementary open 3-cell in several essentially distinct ways.*

EXAMPLE 3.3. *A 2-sphere whose exterior though simply connected is not an open 3-cell.*

For this example we choose Z^0, the boundary of the 3-cell

$$p \cup \bigcup_{n=-\infty}^{-1} g_n(U) \cup g_0(U_+ \cup U_0) \cup f_1(U_0 \cup U_+) \cup \bigcup_{n=2}^{\infty} f_n(U) \cup q.$$

Its exterior is homeomorphic to the complement of Z. From this example it is easy to construct a 2-sphere both of whose complementary domains are simply connected (and hence contractible) open manifolds not homeomorphic to an open 3-cell. Such a one is shown in figure 12.

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[14] Such a 2-sphere was constructed by ALEXANDER, loc. cit., p. 11 and pp. 8–10. Our example, which is a much simpler one, has only two singular points while the Alexander examples have an infinity of singular points.