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Structures on $M \times R$

BY WILLIAM BROWDER
Institute for Advanced Study, Princeton

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1. *Introduction.* A central role in the theory of smoothing combinatorial manifolds is played by the Cairns–Hirsch Theorem, which may be expressed (in a weak form) as follows:

If M is a combinatorial manifold and if $M \times R$ has a differentiable structure α compatible with its combinatorial structure then M has a differentiable structure γ, such that $(M \times R)_\alpha$ is diffeomorphic with $M_\gamma \times R$.

The analogous theorem for topological manifolds would be extremely useful for the theory of smoothing topological manifolds.

We will prove the following:

THEOREM. *Let M^n be a closed* 1*-connected topological manifold of dimension $n \geqslant 5$. If $M \times R$ is a differentiable (combinatorial) manifold, then M is h-cobordant to a smooth (combinatorial) manifold N, and $M \times R$ is diffeomorphic (combinatorially equivalent) to $N \times R$. Further, such an N is unique.*

COROLLARY 1. *If M is a closed,* 1*-connected manifold of dimension $\geqslant 5$ which is not the homotopy type of a smooth manifold, (e.g. see* (6) *or* (12)*), then $M \times R$ does not have a differentiable structure.*

COROLLARY 2. *If M^n is a closed,* 1*-connected n-manifold, $n \geqslant 5$, and if $M \times R^k$ is differentiable (combinatorial), $k > 2$, then $M \times S^{k-1}$ is h-cobordant to a differentiable (combinatorial) manifold.*

For if $M \times R^k$ is differentiable then $M \times S^{k-1} \times R$ is differentiable, being an open subset of $M \times R^k$ (similarly for combinatorial), and the theorem applies.

We now proceed to the proof of the theorem. We will consider the combinatorial case, and indicate any changes necessary for the differentiable case. The differentiable case may actually be derived from the combinatorial one, using results of (3) and (8).

Consider the projection map form $M \times R \to R$ and choose a simplicial approximation f with respect to the combinatorial structures of $M \times R$ and R. Consider a point $\alpha \in R$ between vertices of R and let $K = f^{-1}(\alpha) \subset M \times R$. Then K is a combinatorial manifold, piecewise linearly embedded in $M \times R$ with a product neighbourhood $K \times R \subset M \times R$ (see, for example, (19)). (In the differentiable case choose a regular point of a smooth approximation f, using Sard's Theorem; see (11).) By choosing an appropriate component of K in case K is not connected, we may assume that K is connected, compact (since f is proper being an approximation to a proper map) and K divides $M \times R$ into two manifolds A and B with boundary $K = A \cap B$.

Now the idea of the proof is as follows:

The inclusion of $K \subset M \times R$ followed by the projection $M \times R \to M$ is shown to be a map of degree $+1$. It follows that this map g has the property that kernel g_* is a direct summand of $H_*(K)$ which satisfies Poincaré duality. We proceed to reduce kernel g_* by doing surgery on K inside $M \times R$ (cf. (1), (9) and (16)).

More precisely assuming g_* is an isomorphism for $i < k$, onto for $i = k$, $k < \frac{1}{2}n$, we show that we may find for each $x \in (\text{kernel } g_*)_k$ a handle $H = D^{k+1} \times D^{n-k}$ contained either in A or B (say A) with $H \cap K = S^k \times D^{n-k}$ and with $S^k \times 0$ representing the homology class x. Then we define $A' = A -$ interior of H, $B' = B \cup H$, and $A' \cap B' = K'$ where K' is obtained from K by the surgery associated with $S^k \times D^{n-k} \subset K$. We show that this process kills $(\text{kernel } g_*)_k$ for $k < \frac{1}{2}n$. We call this process 'exchanging handles'.

If $n = 2k+1$, g_* is then an isomorphism for $i \leqslant k$, hence by Poincaré duality for all i. It follows that K, A, B and $M \times R$ are all homotopy equivalent, and then it follows from the h-cobordism theorem ((13)) or from the engulfing theorem ((14)) that $M \times R$ is equivalent to $K \times R$, and the theorem follows.

In case $n = 2k$, the above process makes $g_*: H_i(K) \to H_i(M)$ isomorphic for $i < k$. Then we consider $\bar{A} =$ closure of A in $M \times I$, where R is considered as the interior of I. Then $\bar{A}$ is a cobordism between K and M, and we make it an h-cobordism by attaching handles to $\bar{A}$ along K. Then from the engulfing theorem ((14)) it follows that $K' \times R$ is equivalent to $M \times R$ and the result follows.

The details follow.

2. *Algebra.* In this section we deduce some facts about the homology of our situation which we will need.

Let $K \subset M \times R = A \cup B$, $K = A \cap B$, everything connected, A and B combinatorial (smooth) manifolds with boundary K. Let $M \times R =$ interior of $M \times I$, $\bar{A}$, $\bar{B}$ the closures of A and B in $M \times I$, so that $\bar{A}$ and $\bar{B}$ are cobordisms between K and M ($= M \times 1$ or $M \times 0$). Let $p: M \times I \to M$ be projection, inducing retractions $\bar{A}, \bar{B}$ to M, and a map $g: K \to M$. Let $a: K \to A$, $b: K \to B$, $\alpha: A \to M \times R$, $\beta: B \to M \times R$ be inclusions and the same letters with bars ($\bar{a}, \bar{b}, \bar{\alpha}, \bar{\beta}$) the corresponding inclusions with $\bar{A}, \bar{B}, M \times I$, replacing A, B, $M \times R$.

LEMMA 2·1. *$g_*: H_*(K) \to H_*(M)$ is of degree* 1.

Proof. Let μ_K, μ_M be the generators of $H_n(K)$, $H_n(M)$ respectively given by the orientation of $M \times R$. Then $\partial \bar{A} = K \cup M$ and if $\nu \in H_{n+1}(\bar{A}, \partial \bar{A})$ is the orientation generator, $\partial \nu = \mu_M - \mu_K$, so that μ_M and μ_K have the same image μ in $H_n(\bar{A})$. But $p_* \alpha_* \mu = \mu_M$, so $p_* \alpha_* a_* \mu_K = g_* \mu_K = \mu_M$.

LEMMA 2·2. *There are maps $\gamma: H_q(M) \to H_q(K)$, $\gamma^*: H^q(K) \to H^q(M)$ such that $g_* \gamma = 1$, $\gamma^* g^* = 1$, and* $(\text{kernel } g_*)_q \cong (\text{kernel } \gamma^*)^{n-q}$.

Proof. Define γ, γ^* as follows:

If $x \in H_q(M)$, $x = y \cap \mu_M$, then $\gamma x = g^* y \cap \mu_K$. If $x' \in H^{n-q}(K)$, $\gamma^* x = z$, where $z \cap \mu_M = g_*(x' \cap \mu_K)$. Then it is easy to check using properties of cap product, that $g_* \gamma = 1$ and $\gamma^* g^* = 1$. If $x' \in (\text{kernel } \gamma^*)^{n-q}$ then $g_*(x' \cap \mu_K) = 0$ so

$$\cap \mu_K : (\text{kernel } \gamma^*)^{n-q} \to (\text{kernel } g_*)_q.$$

It only remains to show it is onto. But if $z \in (\text{kernel } g_*)_q$, $z = w \cap \mu_K$ and $\gamma^* w = 0$, so that

$$(\text{kernel } \gamma^*)^{n-q} \xrightarrow[\cong]{\cap \mu_K} (\text{kernel } g_*)_q.$$

LEMMA 2·3. (I) $\text{kernel } g_* = \text{kernel } a_* + \text{kernel } b_*$.

(II) $a_* | \text{kernel } b_*$ *and* $b_* | \text{kernel } a_*$ *are mono.*

(III) $a_*(\text{kernel } b_*) = \text{kernel } \alpha_*$ *and*

$b_*(\text{kernel } a_*) = \text{kernel } \beta_*$.

Proof. Since M is a retract of A (or B), α_* (or β_*) is onto, so that we get from the Mayer–Viettoris sequence for $M \times R = A \cup B$,

$$0 \to H_q(K) \xrightarrow{a_* + b_*} H_q(A) + H_q(B) \xrightarrow{\alpha_* - \beta_*} H_q(M \times R) \to 0.$$

Now $g_* = \alpha_* a_* = \beta_* b_*$ (using the isomorphism of $H_*(M)$ and $H_*(M \times R)$). If $x \in \text{kernel } g_*$ then $\alpha_* a_* x = 0$, so that $(\alpha_* - \beta_*)(a_* x) = 0$ and $a_* x = (a_* + b_*)(z)$, $z \in H_*(K)$, and $z \in \text{kernel } b_*$. Similarly $b_* x = (a_* + b_*)(z')$, $z' \in \text{kernel } a_*$. Then $(a_* + b_*)(x) = (a_* + b_*)(z + z')$ and since $a_* + b_*$ is mono, $x = z + z'$. But

$$\text{kernel } a_* \cap \text{kernel } b_* \subset \text{kernel } (a_* + b_*) = 0$$

so they are disjoint, which proves (II), and thus we have a direct sum

$$\text{kernel } g_* = \text{kernel } a_* + \text{kernel } b_*,$$

and (I) is proved.

To prove (III), we note that if $x \in \text{kernel } b_*$, then

$$\alpha_* a_* x = (\alpha_* - \beta_*)(a_* x) = (\alpha_* - \beta_*)(a_* + b_*)(x) = 0$$

so $a_* x \in \text{kernel } \alpha_*$. But if $y \in \text{kernel } \alpha_* \subset H_*(A)$, then $(\alpha_* - \beta_*)(y) = 0$ so

$$y = (a_* + b_*)(z) \in H_*(A), \quad \text{so} \quad b_* z = 0 \quad \text{and} \quad y = a_* z.$$

LEMMA 2·4. *Let $j_A: M \to \bar{A}$. Then $\cap \mu_K$ induces an isomorphism of $\bar{a}^*(\text{kernel } j_A^*)^q$ with $(\text{kernel } \bar{a}_*)_{n-q}$ and similarly with A in place of $\bar{A}$.*

Proof. Since $A \subset \bar{A}$ are the same homotopy type, it suffices to prove it for $\bar{A}$.

Consider the commutative diagram with exact rows (see (2))

$$\begin{array}{ccccc}
\dots \to H^q(\bar{A}) & \xrightarrow{i^*} & H^q(\partial \bar{A}) & \xrightarrow{\delta} & H^{q+1}(\bar{A}, \partial \bar{A}) \to \dots \\
\downarrow \cap \nu & & \downarrow \cap \bar{\mu} & & \downarrow \cap \nu \\
\dots \to H_{n+1-q}(\bar{A}, \partial \bar{A}) & \xrightarrow{\partial} & H_{n-q}(\partial \bar{A}) & \xrightarrow{i_*} & H_{n-q}(\bar{A}) \to \dots
\end{array}$$

where $\bar{\mu} = \partial \nu \in H_n(\partial \bar{A})$ is the fundamental class. It follows that $\cap \bar{\mu}$ induces an isomorphism of $(\text{image } i^*)^q$ with $(\text{kernel } i_*)_{n-q}$. Now $\partial \bar{A} = K \cup M$ (disjoint union), so that $i_* = \bar{a}_* + j_*$, $j: M \to \bar{A}$ is inclusion, and $i^* = \bar{a}^* + j^*$. Now for the fundamental class $\bar{\mu} \in H_n(\partial \bar{A}) = H_n(K) + H_n(M)$, $\bar{\mu} = \mu_M - \mu_K$, and $\cap \mu_K = 0$ on $H^*(M)$ and $\cap \mu_M = 0$ on $H^*(K)$. Hence $\cap \bar{\mu}$ sends $H^*(K)$ into $H_*(K)$ and $H^*(M)$ into $H_*(M)$. It follows that $\cap\bar{\mu}|H^*(K) = \cap\mu_K$ induces an isomorphism between $(\text{image}\, i^*)^q \cap H^*(K)$ and $(\text{kernel}\, i_*) \cap H_*(K)$. But

$$(\text{image}\, i^*) \cap H^*(K) = \bar{a}^*(\text{kernel}\, j_A^*) \quad \text{and} \quad (\text{kernel}\, i_*) \cap H_*(K) = \text{kernel}\, \bar{a}_*,$$

which proves the lemma.

LEMMA 2·5. *The sequences of the pair (A,K) are the sum of short exact sequences*:

$$0 \to H_{k+1}(A,K) \xrightarrow{\partial_1} H_k(K) \xrightarrow{a_*} H_k(A) \to 0$$
$$0 \leftarrow H^{k+1}(A,K) \xleftarrow{\delta_1} H^k(K) \xleftarrow{a^*} H^k(A) \leftarrow 0.$$

Proof. Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
\to H_{k+1}(A,K) \arrow[r, "\partial_1"] \arrow[d, "k_*"] & H_k(K) \arrow[r, "a_*"] \arrow[d, "b_*"] & H_k(A) \to \arrow[d] \\
\to H_{k+1}(M\times R, B) \arrow[r, "\partial_2"] & H_k(B) \arrow[r, "\beta_*"] & H_k(M\times R) \to
\end{tikzcd}$$

Now k is an excision so k_* is an isomorphism. Now β_* is onto, since $B \supset M \times t$ for some t (hence β_* splits). Hence ∂_2 is mono. Then $b_*\partial_1 = \partial_2 k_*$ is mono, so ∂_1 is mono and the upper sequence breaks into short exact sequences. The case of cohomology is treated similarly.

LEMMA 2·6. kernel $j_A^* \cong$ coker α^*, *and* kernel $\alpha_* \cong$ coker j_{A*}.

Proof.
$$M \xrightarrow{j_A} A \xrightarrow{\alpha} M \times R$$
and composition αj_A is a homotopy equivalence. The lemma follows.

PROPOSITION 2·7. $(\text{kernel}\, a_*)_q \cong (\text{coker}\, \alpha^*)^{n-q}$, *and similarly for* b, β.

Proof. By (2·4) $(\text{kernel}\, a_*)_q \cong a^*(\text{kernel}\, j_A^*)^{n-q}$ and a^* is a monomorphism by (2·5). Hence
$$(\text{kernel}\, a_*)_q \cong (\text{kernel}\, j_A^*)^{n-q} \cong (\text{coker}\, \alpha^*)^{n-q}$$
by (2·6).

3. *Finding discs, handles, etc.* In this section we prove various lemmas about embedding spheres and discs which we will need. Also we show how to kill the fundamental group.

LEMMA 3·1. *Let K^n be a manifold embedded in W^{n+1}, with $\pi_1(W) = 0$ and $\pi_1(K)$ finitely generated, with $W = A \cup B$, $A \cap B = K$, $n \geqslant 4$. Then by exchanging 2-discs between A and B we may get $K' \subset W$, $A' \cup B' = W$, $A' \cap B' = K'$ such that*

$$\pi_1(K') = \pi_1(A') = \pi_1(B') = 0.$$

Proof. Choose a set of generators $g_1, \ldots, g_k$ for $\pi_1(K)$, and disjoint 2-discs $d_1, \ldots, d_k$ in general position with respect to K, such that $\partial d_i \subset K$ is in the homotopy class g_i. Then $d_1 \cap K$ is a collection of disjoint simple closed curves in d_1, and we choose an innermost one q, which bounds a disc δ, with $\delta \subset A$ or $\delta \subset B$. Adding a neighbourhood N of a 2-disc δ to A (or B) is adding a 2-dimensional handle to A (or B), and the difference $V = (\widetilde{A \cup N}) - A$ is a cobordism of K with $K' = \partial(\widetilde{A \cup N})$, where $\widetilde{A \cup N}$ is a regular neighbourhood of $A \cup N$, containing $A \cup N$ in its interior. Now $V \cong K \cup \delta$, so $\pi_1(V) = \pi_1(K)/(\partial\delta)$. But $V \cong K' \cup \delta'$, where δ' is an $(n-1)$-disc, and since $n \geqslant 4$, $\pi_1(K') = \pi_1(V)$. Hence exchanging a 2-disc reduces $\pi_1(K)$, so that in K' $(\partial d_1), \ldots, (\partial d_k)$ are still a set of generators for $\pi_1(K')$ (parentheses indicate homotopy classes). Hence we may continue exchanging the subdiscs which bound innermost curves of intersection of $K \cap d_1$ *until* finally we exchange d_1 and we have killed a generator g_1 of $\pi_1(K)$. We continue this way until all g_i's are killed so that the resulting K' is simply connected, and then it follows from van Kampen's theorem that A' and B' are simply connected.

Thus we may assume that we have $M \times R = A \cup B$, $A \cap B = K$. A, B are combinatorial manifolds with boundary K, and $\pi_1(K) = \pi_1(A) = \pi_1(B) = 0$, and K is closed.

Now let us assume that the inclusion $i: K \to M \times R$ is such that $i_*: H_j(K) \to H_j(M \times R)$ is an isomorphism for $j < k$, onto in all dimensions by Lemma 2.2.

LEMMA 3·2. *Let $h: \pi_k(K) \to H_k(K)$ be the Hurewicz homorphism. Then*

$$\text{kernel } i_* = h(\text{kernel } i_\#)$$

in dimension k, where $i_\#$ is the map of homotopy groups.

This follows easily from the exact sequence of the map i and the relative Hurewicz theorem.

LEMMA 3·3. kernel $a_* = h(\text{kernel } a_\#)$, kernel $b_* = h(\text{kernel } b_\#)$.

The proof is the same as the previous lemma.

LEMMA 3·4. *If $k \leqslant \frac{1}{2}n$, $n > 4$, then a map $f: S^k \to K^n$ is homotopic to a piecewise linear (smooth) embedding.*

Proof. If $k < \frac{1}{2}n$ this follows easily from a general position argument. If $k = \frac{1}{2}n$ (n even), then an argument following ((10)) (or (7) in smooth case) yields the result.

LEMMA 3·5. *Let $2k+1 \leqslant n$, $S^k \subset K^n$, S^k homotopic to zero in A. Then there is a disc $D^{k+1} \subset A$, $D^{k+1} \cap K = \partial D^{k+1} = S^k$ which meets K transversally.*

Proof. Since S^k is homotopic to zero in A, there is a map $f: D^{k+1} \to A$ which has all the properties required, except that it is not an embedding in the interior of D^{k+1}. Since $\dim A \geqslant 2k+2$ the arguments of Lemma 3·4 apply to get an embedding homotopic to f modulo S^k.

LEMMA 3·6. *Let $S^k \subset K^{2k}$ $(k > 2)$, S^k homotopic to zero in A, all smoothly. Then S^k bounds an immersed disc $D^{k+1} \subset A^{2k+1}$, so that its normal bundle is trivial.*

This follows from the argument of Whitney in (17) that the map of $D^{k+1} \to A^{2k+1}$ which is an embedding in a neighbourhood of ∂D^{k+1} into ∂A, is homotopic (modulo ∂D^{k+1}) to an immersion.

LEMMA 3·7. *Let $S^k \subset K^{2k}$ $(k > 2)$, S^k homotopic to a constant in A. Then a homotopic sphere $\bar{S}^k$ has a product neighbourhood $\bar{S}^k \times D^k \subset K^{2k}$, $\bar{S}^k$ arbitrarily close to S^k.*

Proof. In the smooth case, this follows at once from Lemma 3·6. In the combinatorial case we map a disc D^{k+1} into A in general position with $\partial D^{k+1} = S^k$. Then the singularities of the map are at most 1-dimensional so that the singular disk Δ = image of D^{k+1}, has no homology above dimension 2. Let U be a regular neighbourhood of Δ, which therefore has the same homology as Δ. Then the obstructions to smoothing U are in $H^q(U; \Gamma_{q-1})$ and since $\Gamma_i = 0$ for $i < 4$, U is smoothable, i.e. U has a differentiable structure compatible with the triangulation (see (4) and (5)). Then the lemma follows from the differentiable case.

4. *Exchanging handles to kill homology.* Now we are in a position to kill homology by exchanging handles. This is sufficient to prove the Theorem when n is odd, but for even n, we will use another step to get the result.

Recall that we have $M \times R = A \cup B$, $A \cap B = K$, K, A, B, M all 1-connected, by Lemma 3·1.

PROPOSITION 4·1. *By exchanging handles between A and B we may make $M \times R = A' \cup B'$, $A' \cap B' = K'$ with $\pi_j(M \times R, K') = 0$ for $0 \leq j < k+1$, for $k < \frac{1}{2}n$.*

Proof. By induction, we suppose $\pi_j(M \times R, K') = 0$ for $j < k$. By the relative Hurewicz theorem, $\pi_j(M \times R, K) = H_j(M \times R, K)$ for $j \leq k$, and since

$$i_* : H_i(K) \rightarrow H_i(M \times R)$$

is onto by Lemma 2·1 ($i_* = g_*$) we have

$$0 \rightarrow H_k(M \times R, K) \rightarrow H_{k-1}(K) \xrightarrow{g_*} H_{k-1}(M) \rightarrow 0.$$

By (2·2) and (2·3), kernel g_* is a direct summand of $H_{k-1}(K)$ and kernel g_* = kernel a_* + kernel b_*. By Lemmas 3·2–3·5, if $x \in (\text{kernel } a_*)_{k-1}$ there is a disc $D^k \subset A$ meeting K transversally in $S^{k-1} = \partial D^k \subset K$ and the homology class of S^{k-1} represents x. If we exchange a neighbourhood $\bar{D}$ of D^k (a handle) from A to B, then in $B' = B \cup \bar{D}$, $b_* x$ is killed, i.e. $H_{k-1}B' = H_{k-1}B/(b_* x)$, while $H_j B' = H_j B$ for $j < k-1$. Now $A' = A$ − interior $\bar{D}$ is homotopy equivalent to $A - D^k$, and since co-dimension D^k in A is $n+1-k > k$, since $k < \frac{1}{2}n$, and it follows that $H_j A' = H_j A$ for $j \leq k-1$. Hence we have reduced kernel a_*, since by (2·3) b'_* is mono on kernel a'_* and kept kernel b'_* = kernel b_*, and thus we may continue until $(\text{kernel } a'_*)_{k-1}$ and $(\text{kernel } b'_*)_{k-1}$ are both 0 so that by Lemma 2·3 $(\text{kernel } g'_*)_{k-1} = 0$ and hence $H_k(M \times R, K') = 0$ for this new K'.

PROPOSITION 4·2. *Let $n = 2k+1$. Then we may exchange handles to make $K' \subset M \times R$ so that K' is homotopy equivalent to $M \times R$.*

Proof. By Proposition 4·1 we may assume $g_* : H_j(K) \rightarrow H_j(M)$ is an isomorphism for $j < k$. If we can make it an isomorphism on H_k, then it follows since kernel g_* satisfies Poincaré duality (Lemma 2·2) that g_* is mono, and epi by (2·2), so that g_* is an isomorphism and hence g is a homotopy equivalence.

We may proceed as in (4·1) to transfer neighbourhoods of $(k+1)$ discs from B to A to make $(\text{kernel } b_*)_k = 0$.

Then we will show that $(\text{kernel } a_*)_k$ is free. First $(\text{kernel } a_*)_k \cong (\text{coker } a^*)^{k+1}$ by (2·7). Now $(\text{kernel } b_*)_j = 0$ for $j \leq k$ implies $\alpha_* : H_j(A) \rightarrow H_j(M \times R)$ is an isomorphism for $j \leq k$ by (2·3, III) and onto for all j, so $H_{k+1}(M \times R, A) = 0$. But

$$(\text{coker } \alpha^*)^{k+1} = H^{k+2}(M \times R,\, A) = \text{Hom}\,(H_{k+2}(M \times R,\, A),\, Z) + \text{Ext}\,(H_{k+1}(M \times R, A) Z)$$

by the Universal Coefficient Theorem. Since $H_{k+1}(M \times R, A) = 0$ and $\mathrm{Hom}(G, Z)$ is free for any G, it follows that $(\mathrm{coker}\,\alpha^*)^{k+1}$ is free and hence $(\mathrm{kernel}\,a_*)_k$ is free.

Since $(\mathrm{kernel}\,a_*)_k$ is free when we add handles as before to B (transferring them from A) if we add the handles to kill exactly a basis of $(\mathrm{kernel}\,a_*)_k$, then if $B' = B \cup$ (handles), $H_k B' = H_k B / b_*(\mathrm{kernel}\,a_*)_k$ and $H_j B' = H_j B$ for $j \neq k$, and $H^j B' = H^j B$ for $j \neq k$.

Now by (2·7), $(\mathrm{kernel}\,b_*)_k \cong (\mathrm{coker}\,\beta^*)^{k+1}$ and $(\mathrm{kernel}\,b_*)_k = 0$, hence

$$(\mathrm{coker}\,\beta^*)^{k+1} = 0.$$

Since $H^j B' = H^j B$ for $j \neq k$,

$$(\mathrm{coker}\,\beta^*)^{k+1} = (\mathrm{coker}\,\beta'^*)^{k+1} = 0, \quad \text{and} \quad H^{k+2}(M \times R, B') = 0.$$

By excision $$H^{k+2}(M \times R, B') \cong H^{k+2}(A', K') = 0.$$

Hence $H_{k+2}(A', K')$ is a torsion group $\cong$ torsion subgroup of $H^{k+3}(A', K')$. But

$$H^{k+3}(A', K') \cong (\mathrm{coker}\,\beta'^*)^{k+2} \cong (\mathrm{kernel}\,b'_*)_{k-1} = 0$$

so that $H_{k+2}(A', K') = 0$. Hence $a'_*: H_i(K') \to H_i(A')$ is an isomorphism for $i \leqslant k+1$. Since kernel $g'_* =$ kernel $a'_* +$ kernel b'_* and $(\mathrm{kernel}\,g'_*)_i = 0$ for $i > k+1$, and a'_* is onto by (2·5), it follows that a'_* is an isomorphism. Hence it follows that g'_* is an isomorphism and the proposition is proved.

COROLLARY 4·3. *K' is h-cobordant to M.*

For $\bar{A}'$ above is an h-cobordism, since $\bar{A}' \cong K'$ so $H_*(\bar{A}', K') = 0 = H^*(\bar{A}', M)$ by the Poincaré duality theorem (see (2)).

PROPOSITION 4·4. *Let $n = 2k > 4$, and assume $M \times R = A \cup B$, $A \cap B = K$ as above, with $g_*: H_i(K) \to H_i(M)$ isomorphism for $i \neq k$. Then we may add handles to $\bar{A}$ along $K \subset \partial \bar{A}$ to get an h-cobordism $\bar{A}'$ between M and a new manifold K', $\partial \bar{A}' = M \cup K'$, K' a combinatorial (or smooth) manifold.*

Proof. Since $(\mathrm{kernel}\,g_*)$ satisfies Poincaré duality and g_* is an isomorphism for $i \neq k$, it follows that $(\mathrm{kernel}\,g_*)_k$ is free, so that $(\mathrm{kernel}\,a_*)_k$ and $(\mathrm{kernel}\,b_*)_k$ are free.

By (3·4) we may represent a basis for $(\mathrm{kernel}\,b_*)_k$ by embedded k-spheres, S_i^k. By (2·4), the Poincaré duals of elements in $(\mathrm{kernel}\,b_*)$ are in image b^*, so it follows that if $x, y \in \mathrm{kernel}\,b_*$, their intersection number $x \,.\, y = 0$. Hence using Whitney's theorem ((18) or (7)) the spheres $\{S_i^k\}$ representing a basis for $(\mathrm{kernel}\,b_*)_k$ may be embedded disjointly. By (3·7), the spheres S_i^k may be chosen to have disjoint product neighbourhood $S_i^k \times D^k \subset K$. Then we may attach disjoint handles $D_i^{k+1} \times D^k$ to $\bar{A}$ along K by identifying $\partial D_i^{k+1} \times D^k$ with $S_i^k \times D^k$. This gives us a new manifold

$$\bar{A}' = \bar{A} \cup \bigcup_i D_i^{k+1} \times D^k,$$

with $\partial \bar{A}' = M \cup K'$, $\bar{A}' - M$ a combinatorial (or smooth) manifold with boundary K'.

Then $\bar{A}' \cong \bar{A} \cup \bigcup_i D_i^{k+1}$, and since the homology classes of S_i^k are a basis for $(\mathrm{kernel}\,b_*)$ which is free, and $\mathrm{kernel}\,b_* \cong \mathrm{coker}\,j_{A*}$ by (2·3) and (2·6) it follows that

$$H_j(\bar{A}') \cong H_j(\bar{A}) \quad \text{for} \quad j \neq k \quad \text{and} \quad H_k(\bar{A}') \cong (\mathrm{image}\,j_{A*})_k \subset H^k(\bar{A}).$$

Hence the inclusion $j': M \to \bar{A}'$ is such that $j'_*: H_* M \to H_* \bar{A}'$ is an isomorphism. It follows that $\bar{A}'$ is an h-cobordism between M and K'.

Then the proof of the theorem is completed by using the following theorem due to Stallings ((15)).

THEOREM. *Let W^n be an h-cobordism between M and N, M, N, W compact, and suppose $W - M$ is a combinatorial manifold, $n \geqslant 5$. Then $W - N$ is homeomorphic to $M \times [0, 1)$ and $W - M$ is combinatorially equivalent to $N \times [0, 1)$, so that $W - \partial W$ is combinatorially equivalent to $N \times (0, 1)$ and homeomorphic to $M \times (0, 1)$.*

We indicate an outline of the proof:

Take collars $M \times [0, a_K) \subset W$ around M, and using the engulfing theorem ((14)) in $W - \partial W$ and the fact that W is an h-cobordism we may find a piecewise linear equivalence f_k of $W - \partial W$ fixed on $f_{k-1}(M \times [0, a_{k-1}])$ a neighbourhood of M which contains $M \times [0, a_{k-1})$, which takes $M \times [0, a_k)$ onto the complement of a small collar C_k around N, where $\cap C_k = N$, $a_i < a_{i+1}$, $\lim_{n \to \infty} a_n = 1$. Define $f = \lim f_k$, which is only a finite iterate at any point, and hence piecewise linear. Then f is the desired map. A similar argument gives the other part.

It follows that, if A^n is an h-cobordism between M and a combinatorial manifold K, such that $A - M$ is combinatorial, $n \geqslant 5$, $A - \partial A$ is piecewise linearly equivalent to $K \times R$. But $M \times R$ is contained as a combinatorial open submanifold of

$$A - \partial A = K \times R,$$

as a deformation retract. Applying the engulfing theorem again, we get $K \times R$ piecewise linearly equivalent (combinatorially equivalent) to $M \times R$.

In the differentiable case, we apply the theory of Munkres ((8)) to show that $K \times R$ and $M \times R$ are diffeomorphic.

To show the uniqueness of K, we note that $K \times R$ equivalent to $K' \times R$ implies they are h-cobordant (as smooth or combinatorial manifolds) and hence K is equivalent to K' by the h-cobordism theorem (see (13)), which has been shown to hold under the same hypotheses for piecewise linear manifolds.

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