Stable homeomorphisms and the annulus conjecture*

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A homeomorphism h of R^n to R^n is stable if it can be written as a finite composition of homeomorphisms, each of which is somewhere the identity, that is, $h = h_1 h_2 \cdots h_r$ and $h_i \mid U_i =$ identity for each i where U_i is open in R^n.

Stable Homeomorphism Conjecture, SHC_n : All orientation preserving homeomorphisms of R^n are stable.

Stable homeomorphisms are particularly interesting because (see [3]) $\mathrm{SHC}_n \Rightarrow \mathrm{AC}_n$, and AC_k for all $k \leqq n \Rightarrow \mathrm{SHC}_n$ where AC_n is the

Annulus Conjecture, AC_n : Let $f, g : S^{n-1} \rightarrow R^n$ be disjoint, locally flat imbeddings with $f(S^{n-1})$ inside the bounded component of $R^n - g(S^{n-1})$. Then the closed region A bounded by $f(S^{n-1})$ and $g(S^{n-1})$ is homeomorphic to $S^{n-1} \times [0, 1]$.

Numerous attempts on these conjectures have been made; for example, it is known that an orientation preserving homeomorphism is stable if it is differentiable at one point [10] [12], if it can be approximated by a PL homeomorphism [6], or if it is $(n-2)$-stable [4]. "Stable" versions of AC_n are known; $A \times [0, 1)$ is homeomorphic to $S^{n-1} \times I \times [0, 1)$, $A \times R$ is $S^{n-1} \times I \times R$, and $A \times S^k$ is $S^{n-1} \times I \times S^k$ if k is odd (see [7] and [13]). A counter-example to AC_n would provide a non-triangulable n-manifold [3].

Here we reduce these conjectures to the following problem in PL theory. Let T^n be the cartesian product of n circles.

Hauptvermutung for Tori, HT_n: Let T^n and τ^n be homeomorphic PL n-manifolds. Then T^n and τ^n are PL homeomorphic.

THEOREM 1. *If* $n \geqq 6$, *then* $\mathrm{HT}_n \Rightarrow \mathrm{SHC}_n$.

(*Added December* 1, 1968. It can now be shown that SHC_n is true for $n \neq 4$. If $n \neq 3$, this is a classical result. Theorem 1 also holds for $n = 5$, since Wall [19, p. 67] has shown that an end which is homeomorphic to $S^4 \times R$ is also PL homeomorphic to $S^4 \times R$.

In the proof of Theorem 1, a homeomorphism $f\colon T^n \rightarrow \tau^n$ is constructed. If $\hat{f}\colon \hat{T}^n \rightarrow \hat{\tau}^n$ is any covering of $f\colon T^n \rightarrow \tau^n$, then clearly f is stable if and only if

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$\hat{f}$ is stable. Using only the fact that f is a simple homotopy equivalence, Wall's non-simply connected surgery techniques [15] provide an "obstruction" in $H^3(T^n; Z_2)$ to finding a PL homeomorphism between T^n and τ^n. It is Siebenmann's idea to investigate the behavior of this obstruction under lifting $f: T^n \to \tau^n$ to a 2^n-fold cover; he suggested that the obstruction would become zero. Wall [16] and Hsiang and Shaneson [17] have proved that this is the case; that is, if $\hat{\tau}^n$ is the 2^n-fold cover of a homotopy torus τ^n, $n \geqq 5$, then $\hat{\tau}^n$ is PL homeomorphic to $T^n (= \hat{T}^n)$. Therefore, following the proof of Theorem 1, $\hat{f}: \hat{T}^n \to \hat{\tau}^n$ is stable, so f is stable, and thus SHC$_n$ holds for $n \neq 4$. Hence the annulus conjecture AC$_n$ holds for $n \neq 4$.)

(*Added April* 15, 1969. Siebenmann has found a beautiful and surprising counter-example which leads to non-existence and non-uniqueness of triangulations of manifolds. In particular HT$_n$ is false for $n \geqq 5$, so it is necessary to take the 2^n-fold covers, as above. One may then use the fact that $\hat{f}: T^n \to \hat{\tau}^n$ is homotopic to a PL homeomorphism to show that $f: T \to \tau^n$ was actually isotopic to a PL homeomorphism. Thus, although there are homeomorphisms between T^n and another PL manifold which are not even homotopic to PL homeomorphisms, they cannot be constructed as in Theorem 1. Details will appear in a forthcoming paper by Siebenmann and the author. See also R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, to appear in Bull. Amer. Math. Soc.)

Let $\mathcal{H}(M^n)$ denote the space (with the compact-open topology) of orientation preserving homeomorphisms of an oriented stable n-manifold M, and let $\mathcal{SH}(M^n)$ denote the subspace of stable homeomorphisms.

THEOREM 2. *$\mathcal{SH}(R^n)$ is both open and closed in $\mathcal{H}(R^n)$.*

Since a stable homeomorphism of R^n is isotopic to the identity, we have the

COROLLARY. *$\mathcal{SH}(R^n)$ is exactly the component of the identity in $\mathcal{H}(R^n)$.*

COROLLARY. *A homeomorphism of R^n is stable if and only if it is isotopic to the identity.*

THEOREM 3. *If M^n is a stable manifold, then $\mathcal{SH}(M^n)$ contains the identity component of $\mathcal{H}(M^n)$.*

In general this does not imply that the identity component is arcwise connected (as it does for $M^n = R^n$ or S^n), but arcwise connectivity does follow from the remarkable result of Cernavskii [5] that $\mathcal{H}(M^n)$ is locally contractible if M^n is compact and closed or $M^n = R^n$. From the techniques in this paper, we have an easy proof of the last case.

THEOREM 4. *$\mathcal{H}(R^n)$ is locally contractible.*

We now give some definitions, then a few elementary propositions, the crucial lemma, and finally the proofs of Theorems 1 − 4 in succession.

The following definitions may be found in Brown and Gluck [3], a good source for material on stable homeomorphisms. A homeomorphism h between open subsets U and V of R^n is called stable if each point $x \in U$ has a neighborhood $W_x \subset U$ such that $h \mid W_x$ extends to a stable homeomorphism of R^n. Then we may define stable manifolds and stable homeomorphisms between stable manifolds in the same way as is usually done in the PL and differential categories. Whenever it makes sense, we assume that a stable structure on a manifold is inherited from the PL or differential structure. Homeomorphisms will always be assumed to preserve orientation.

PROPOSITION 1. *A homeomorphism of R^n is stable if it agrees with a stable homeomorphism on some open set.*

PROPOSITION 2. *Let $h \in \mathcal{H}(R^n)$ and suppose there exists a constant $M > 0$ so that $|h(x) - x| \leqq M$ for all $x \in R^n$. Then h is stable.*

PROOF. This is Lemma 5 of [6].

Letting rB^n be the n-ball of radius r, we may consider $5D^n = i(5B^n)$ as a subset of T^n, *via* some fixed differentiable imbedding $i: 5B^n \to T^n$.

PROPOSITION 3. *There exists an immersion $\alpha: T^n - D^n \to R^n$.*

PROOF. Since $T^n - D^n$ is open and has a trivial tangent bundle, this follows from [8, Th. 4. 7].

PROPOSITION 4. *If A is an $n \times n$ matrix of integers with determinant one, then there exists a diffeomorphism $f: T^n \to T^n$ such that $f_* = A$ where $f_*: \pi_1(T^n, t_0) \to \pi_1(T^n, t_0)$.*

PROOF. A can be written as a product of elementary matrices with integer entries, and these can be represented by diffeomorphisms.

PROPOSITION 5. *A homeomorphism of a connected stable manifold is stable if its restriction to some open set is stable.*

For the proof, see [3, p. 35]

PROPOSITION 6. *Let $f: S^{n-1} \times [-1, 1] \to R^n$ be an imbedding which contains S^{n-1} in its interior. Then $f \mid S^{n-1} \times 0$ extends canonically to an imbedding of B^n in R^n.*

PROOF. This is shown in [9]. However, there is a simple proof; one just re-proves the necessary part of [2] in a canonical way. This sort of canonical construction is done carefully in the proof of Theorem 1 of [11].

The key to the paper is the following observation.

LEMMA. *Every homeomorphism of T^n is stable.*

PROOF. Let $e\colon R^n \longrightarrow T^n$ be the usual covering map defined by

$$e(x_1, \cdots, x_n) = (e^{2\pi i x_1}, \cdots, e^{2\pi i x_n})$$

and let $t_0 = (1, 1, \cdots, 1) = e(0, \cdots, 0)$. e fixes a differential and hence stable structure on T^n.

Let h be a homeomorphism of T^n, and assume at first that $h(t_0) = t_0$ and $h_*\colon \pi_1(T^n, t_0) \longrightarrow \pi_1(T^n, t_0)$ is the identity matrix. h lifts to a homeomorphism $\hat{h}\colon R^n \longrightarrow R^n$ so that the following diagram commutes.

$$\begin{array}{ccc} R^n & \xrightarrow{\hat{h}} & R^n \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ T^n & \xrightarrow{h} & T^n \end{array}$$

Since $I^n = [0, 1] \times \cdots \times [0, 1]$ is compact,

$$M = \sup \{|\hat{h}(x) - x| \mid x \in I^n\}$$

exists. The condition $h_* =$ identity implies that $\hat{h}$ fixes all lattice points with integer coordinates. Thus $\hat{h}$ moves any other unit n-cube with vertices in this lattice in the "same" way it moves I^n; in particular $|\hat{h}(x) - x| < M$ for all $x \in R^n$. By Proposition 2, $\hat{h}$ is stable. e provides the coordinate patches on T^n, so h is stable because $e^{-1}he \mid e^{-1}$ (patch) extends to the stable homeomorphism $\hat{h}$ for all patches.

Given any homeomorphism h of T^n, we may compose with a diffeomorphism g so that $gh(t_0) = t_0$. If $A = (gh)_*^{-1}$, then Proposition 4 provides a diffeomorphism f with $f_* = A = (gh)_*^{-1}$, so $(fgh)_* =$ identity. We proved above that fgh was stable so $h = g^{-1}f^{=1}(fgh)$ is the product of stable homeomorphisms and therefore stable.

PROOF OF THEOREM 1. Let g be a homeomorphism of R^n. $g\alpha$ induces a new differentiable structure on $T^n - D^n$, and we call this differential manifold $\widetilde{T^n - D^n}$. We have the following commutative diagram,

$$\begin{array}{ccc} T^n - D^n & \xrightarrow{\text{id}} & \widetilde{T^n - D^n} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g\alpha} \\ R^n & \xrightarrow{g} & R^n \end{array} .$$

α and $g\alpha$ are differentiable and therefore stable, so g is stable if and only if the identity is stable (use Proposition 1).

Since $\widetilde{T^n - D^n}$ has one end, which is homeomorphic to $S^{n-1} \times R$, and $n \geqq 6$, there is no difficulty in adding a differentiable boundary [1]. Since the boundary is clearly a homotopy $(n-1)$-sphere, we can take a C^1-triangulation and use the PL h-cobordism theorem to see that the boundary is a PL $(n-1)$-sphere. To be precise, there is a proper PL imbedding $\beta\colon S^{n-1} \times [0, 1) \to \widetilde{T^n - D^n}$, and we add the boundary by taking the union $\widetilde{T^n - D^n} \cup_\beta S^{n=1} \times [0, 1]$ over the map β.

Finally we add B^n to this union, *via* the identity map on the boundaries, to obtain a closed PL manifold τ^n.

We can assume that $\partial 2D^n$ lies in $\beta(S^{n-1} \times [0, 1))$. Thus $\partial 2D^n$ lies in an n-ball of τ^n and, since it is locally flat, bounds an n-ball by the topological Schoenflies theorem [2]. Now, we may extend the $id \mid T^n - 2D^n$, by coning on $\partial 2D^n$, to a homeomorphism $f\colon T^n \to \tau^n$.

Using HT_n, we have a PL (hence stable) homeomorphism $h\colon T^n \to \tau^n$. By the Lemma, $h^{-1}f\colon T^n \to T^n$ is stable, so $f = h(h^{-1}f)$ is stable, $f \mid T^n - 2D^n =$ identity is stable, and finally g is stable.

Note that it is only necessary that HT_n gives a stable homeomorphism h.

PROOF OF THEOREM 2. We shall show that a neighborhood of the identity consists of stable homeomorphisms. But then by translation in the topological group $\mathcal{H}(R^n)$, any stable homeomorphism has a neighborhood of stable homeomorphisms, so $\mathcal{SH}(R^n)$ is open. Now it is well known that an open subgroup is also closed (for a coset of $\mathcal{SH}(R^n)$ in $\mathcal{H}(R^n)$ is open, so the union of all cosets of $\mathcal{SH}(R^n)$ is open and is also the complement of $\mathcal{SH}(R^n)$, which is therefore closed).

If C is a compact subset of R^n and $\varepsilon > 0$, then it is easily verified that $N(C, \varepsilon) = \{h \in \mathcal{H}(R^n) \mid |h(x) - x| < \varepsilon \text{ for all } x \in C\}$ is an open set in the co-topology. Let C be a compact set containing $\alpha(T^n - D^n)$. If $\varepsilon > 0$ is chosen small enough, then

$$\overline{h\alpha(T^n - 5D^n)} \subset \alpha(T - 4D^n) \subset \overline{\alpha(T^n - 4D^n)} \subset h\alpha(T^n - 3D^n)$$
$$\subset \overline{h\alpha(T^n - 2D^n)} \subset \alpha(T^n - D^n)$$

for any $h \in N(C, \varepsilon)$. There exists an imbedding $\hat{h}$, which "lifts" h so that the following diagram commutes.

$$\begin{array}{ccc} T^n - 2D^n & \xrightarrow{\hat{h}} & T^n - D^n \\ \downarrow \alpha & & \downarrow \alpha \\ R^n & \xrightarrow{h} & R^n \end{array}$$

To define $\hat{h}$, first we cover C with finitely many open sets $\{U_i\}, i = 1, \cdots, k$, so that α is an imbedding on each component of $\alpha^{-1}(U_i)$, $i = 1, \cdots, k$. Let $\{V_i\}$, $i = 1, \cdots, k$, be a refinement of $\{U_i\}$. If ε was chosen small enough, then $h(V_i) \subset U_i$. Let $W_i = U_i \cap \alpha(T^n - D^n)$ and $X_i = V_i \cap \alpha(T^n - 2D^n)$. Since $h\alpha(T^n - 2D^n) \subset \alpha(T^n - D^n)$, we have $h(X_i) \subset W_i$, $i = 1, \cdots, k$. Let $W_{i,j}, j = 1, \cdots, w_i$ be the components of $\alpha^{-1}(W_i)$, let $X_{i,j} = W_{i,j} \cap T^n - 2D^n$, and let $\alpha_{i,j} = \alpha \mid W_{i,j}$ for all i and j. Now we can define $\hat{h}$ by

$$\hat{h} \mid X_{i,j} = (\alpha_{i,j})^{-1} h\alpha \mid X_{i,j} \qquad \text{for all } i \text{ and } j.$$

Clearly $\hat{h}$ is an imbedding.

$\alpha(T^n - 4D^n) \subset h\alpha(T^n - 3D^n)$ which implies that $\alpha(4D^n - D^n) \supset h\alpha(\partial 3D^n)$, so $\hat{h}(\partial 3D^n) \subset 4D^n$ and hence $\hat{h}(\partial 3D^n)$ bounds an n-ball in $4D^n$. By coning, we extend $\hat{h} \mid T^n - 3D^n$ to a homeomorphism $H: T^n \to T^n$. H is stable by the lemma, so $\hat{h}$ is stable and h is stable. Hence $N(C, \varepsilon)$ is a neighborhood of the identity consisting of stable homeomorphisms, finishing the proof of Theorem 2.

Proof of Theorem 3. As in the proof of Theorem 2, it suffices to show that a neighborhood of the identity consists of stable homeomorphisms; then $\mathcal{SH}(M^n)$ is both open and closed and therefore contains the identity component.

Let $j: R^n \to M$ be a coordinate patch. Let $\varepsilon > 0$ and $r > 0$ be chosen so that $N(rB^n, \varepsilon) \subset \mathcal{H}(R^n)$ consists of stable homeomorphisms. Then there exists a $\delta < 0$ such that if $h \in N(j(rB^n), \delta) \subset \mathcal{H}(M^n)$, then $hj(2rB^n) \subset j(R^n)$ $j^{-1}hj \mid 2rB^n \in N(rB^n, \varepsilon)$. We may isotope $j^{-1}hj \mid 2rB^n$ to a homeomorphism H of R^n with $H = j^{-1}hj$ on rB^n and therefore $H \in N(rB^n, \varepsilon) \subset \mathcal{H}(R^n)$. Thus H is stable and so $j^{-1}hj \mid 2rB^n$ is stable. By Proposition 5, h is stable, and hence $N(j(rB^n), \delta)$ is our required neighborhood of the identity.

Proof of Theorem 4. We will observe that Theorem 2 can be proved in a "canonical" fashion; that is, if h varies continuously in $\mathcal{H}(R^n)$, then H varies continuously in $\mathcal{H}(T^n)$. First note that $\mathcal{H}(R^n)$ may be contracted onto $\mathcal{H}_0(R^n)$, the homeomorphisms fixing the origin. The immersion $\alpha: T^n - D^n \to R^n$ can be chosen so that $\alpha e = \text{id}$ on $1/4B^n$. Pick a compact set C and $\varepsilon > 0$ as in the proof of Theorem 2 and let $h \in N(C, \varepsilon)$. h lifts canonically to $\hat{h}: T^n - 2D^n \to T^n - D^n$. Since $\hat{h}(\text{int}\, 5D^n - 2D^n)$ contains $\partial 4D^n$, it follows from Proposition 6 that $\hat{h}(\partial 3D^n)$ bounds a canonical n-ball in $4D^n$. Then $\hat{h} \mid T^n - 3D^n$ extends by coning to $H: T^n \to T^n$.

Clearly $H(t_0) = t_0$ and $H_* = \text{identity}$ so H lifts uniquely to a homeomorphism $g: R^n \to R^n$, with $|g(x) - x| <$ constant for all $x \in R$, (see the lemma). We have the commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{g} & R^n \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ T^n & \xrightarrow{H} & T^n \\ \cup & & \cup \\ T^n - 3D^n & \xrightarrow{\hat{h}} & T^n - 2D^n \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ R^n & \xrightarrow{h} & R^n \end{array} .$$

Since $e(1/4B^n) \cap 4D^n = \emptyset$ and $\alpha e = \text{id}$ on $1/4B^n$, it follows that $g = h$ on $1/4B^n$. The construction of g being canonical means that the map $\psi: \mathcal{H}_0(R^n) \to \mathcal{H}_0(R^n)$, defined by $\psi(h) = g$, is continuous.

Let $P_t: R^n \to R^n$, $t \in [0, 1]$, be the isotopy with $P_0 = h$ and $P_1 = g$ defined by

$$P_t(x) = g\left\{\frac{1}{1-t}\cdot\left[g^{-1}h((1-t)x)\right]\right\} \qquad \text{if } t < 1, \text{ and } P_1 = g .$$

Let $Q_t: R^n \to R^n$, $t \in [0, 1]$ be the isotopy with $Q_0 = g$ and $Q_1 = \text{identity}$ defined by

$$Q_t(x) = (1-t)\cdot g\left(\frac{1}{1-t}\cdot x\right) \qquad \text{if } t < 1, \text{ and } Q_1 = \text{identity}.$$

Now let $h_t: R^n \to R^n$, $t \in [0, 1]$ be defined by

$$h_t(x) = \begin{cases} P_{2t}(x) & \text{if } 0 \leqq t \leqq 1/2 \\ Q_{2t-1}(x) & \text{if } 1/2 \leqq t \leqq 1 . \end{cases}$$

It can be verified that h_t is an isotopy of h to the identity which varies continuously with respect to h. Then $H_t: N(C, \varepsilon) \to \mathcal{H}_0(R^n)$, $t \in [0, 1]$ defined by $H_t(h) = h_t$ is a contraction of $N(C, \varepsilon)$ to the identity where $H_t(\text{identity}) = \text{identity}$ for all $t \in [0, 1]$.

This proof can be easily modified to show that if a neighborhood V of the identity in $\mathcal{H}_0(R^n)$ is given, then C and ε may be chosen so that $N(C, \varepsilon)$ contracts to the identity and the contraction takes place in V. To see this, pick $r > 0$ and δ so that $N(rB^n, \delta) \subset V$. Then we may re-define α and e so that $\alpha e = \text{identity}$ on rB^n. If $h \in N(rB^n, \delta)$, then $P_t \in N(rB^n, \delta)$, and if ε is chosen small enough (with respect to δ), then $h \in N(rB^n, \varepsilon)$ implies that $Q_t \in N(rB^n, \delta)$. Therefore $N(rB^n, \varepsilon)$ contracts in V.

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