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Microbundles and Bundles

I. Elementary Theory

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1. Definitions, Statement of Results

In this part I we work in two categories namely the category of topological spaces and continuous maps, and the category of piecewise linear (PL) spaces and PL-maps. A PL-space is a topological space with a complete class of locally finite triangulations, any two of which have a common subdivision. A PL-map $f\colon X \to Y$ is a continuous map between PL-spaces X and Y, which for some triangulation of X and Y maps every simplex of X linearly into a simplex of Y. We consider fibrebundles in the sense of STEENROD with a fixed cross-section, often called the zerosection,

$$\xi\colon\quad F \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} X; \qquad P_s = i^{-1} s(X) \in F. \tag{1}$$

The base space X is a locally finite simplicial complex; E is the total space; p is the projection which in the PL-category of course has to be a PL-map; s is the zero section. The fibre F will be n-dimensional numberspace $\boldsymbol{R}^n$ (the open n-ball), or the n-ball (with boundary included)

$$B^n = \{(x_1, \dots, x_n) \in \boldsymbol{R}^n \mid \Sigma_1^n x_i^2 \leq 1\} \quad \text{with} \quad P_s = (0, \dots, 0),$$

or the n-sphere

$$S^n = \{(x_0, \dots, x_n) \in \boldsymbol{R}^{n+1} \mid \Sigma_0^n x_i^2 = (4/\pi)^2\}$$

with

$$(-4/\pi, 0, \dots, 0) = P_s(\text{“}0\text{”}) \quad \text{and} \quad (4/\pi, 0, \dots, 0) = P_n(\text{“}\infty\text{”})\,^1.$$

The group will be the group of all homeomorphisms of $(F; P_s)$ or $(F; P_s, P_n)$ resp. onto itself.

Two fibrebundles ξ_1 and ξ_2 with the same base space X, are called *micro-identical* if: the zerosections coincide, $s_1 X = s_2 X$; the total

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[1] See section 2 for a description of these spaces in the PL category.

spaces E_1 and E_2 have in common some open set U containing this zerosection; and if moreover the restrictions of the projections p_1 and p_2 to U coincide, $p_1|U = p_2|U$. See Fig. 1.

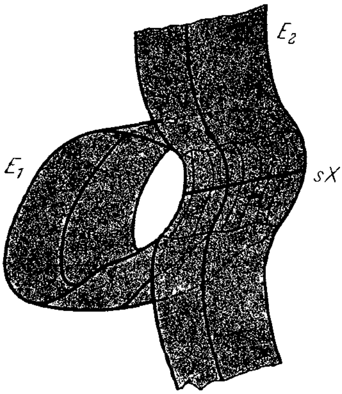

Fig. 1

$$\begin{tikzcd} & sX \arrow[d, phantom, "\cap"] & \\ E_1 \arrow[dr, "p_1"'] & U \arrow[l, phantom, "\supset"] \arrow[r, phantom, "\subset"] \arrow[d] & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Let V_1 and V_2 be subspaces of X. Two fibrebundles ξ_1 over V_1 and ξ_2 over V_2 are said to *micro agree*, in case their restrictions to $V = V_1 \cap V_2$ are micro identical fibrebundles over V.

A *premicrobundle* over X is a set of fibrebundles ξ_α with cross-section, one over each open set V_α, of a covering of $X = \bigcup_\alpha V_\alpha$, such that any two of them microagree.

Two premicrobundles $\{\xi_\alpha\}$ and $\{\xi_\beta\}$ over X are called *strongly equivalent* if their union $\{\xi_\alpha, \xi_\beta\}$ is also a premicrobundle.

Definition. *A microbundle is a strong equivalence class of premicrobundles.*

Every premicrobundle determines the unique microbundle (=equivalence class) of which it is an element.

Example 1. Every fibrebundle ξ (with fibre F as above), is an example of a premicrobundle. Consequently it determines a unique microbundle $\mu(\xi)$[2]. ($\mu(\xi)$ can be considered as a germ of neighborhood of the zerosection of the bundle of the total space, together with what remains of the projection.) The converse is to a certain extent true:

Theorem 1. *In the topological and in the PL-category every micro-n-bundle over X contains a* $\boldsymbol{R}^n$*-bundle and a* S^n*-bundle with zero-cross-section, and these bundles are unique up to equivalence.*

The special cases of this theorem will be denoted by Top$-S^n$, PL$-\boldsymbol{R}^n$ etc.

For the topological case this is a theorem of KISTER [*3*] and B. Mazur. No proof for the PL-case seems to be published so far. We

[2] One can define "microsets" in a microbundle. They form a partially ordered system with analogy to the system of subsets of a set. It may be interesting to study "microsettheory".

give a proof for the cases top $-\boldsymbol{R}^n$, top $-S^n$, PL $-\boldsymbol{R}^n$ in § 2–4. A second proof for PL $-\boldsymbol{R}^n$ and a proof for PL $-S^n$ is given in § 5. The consideration of the S^n-case simplifies the $\boldsymbol{R}^n$-proofs considerably. Observe that BROWDER [1] proved that not every micro n-bundle contains a B^n-bundle.

Example 2. It was MILNOR [5], who introduced microbundles. He defined them as equivalence classes of certain diagrams.

A *Milnor diagram*

$$X \xrightarrow{i} Y \xrightarrow{p} X$$

consists of a base space X, a total space Y, maps i and p with composition $p\,i$=identity, such that for every point $x \in X$ there exists a neighborhood V of $i(x)$ in Y and a surjective homeomorphism h which makes the main square of the following diagram commutative

$$\begin{array}{ccc} Y \supset V & \xrightarrow{\quad h \quad} & p(V) \times \boldsymbol{R}^n \\ {\scriptstyle i}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle i_1}\uparrow\downarrow{\scriptstyle \pi_1} \\ X \supset p(V) & \xrightarrow{\ id \times 0\ } & p(V) \times 0 \end{array}$$

with

$$\pi_1(x, u) = (x, 0), \qquad i_1(x, 0) = (x, 0).$$

This special diagram defines a $\boldsymbol{R}^n$-bundle over $p(V) \subset X$ with total space V. The bundles so obtained from a Milnor diagram define a premicrobundle, hence a microbundle according to our definition.

Example 3. A special case is the *tangent microbundle* of a topological or PL manifold M defined by the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p} M$$

with $\Delta: x \rightsquigarrow (x, x)$ the diagonal map and $p: (x, y) \rightsquigarrow p(x, y) = x$ the projection in the first factor.

If $x \in V_1 \subset \overline{V}_1 \subset V_2 \subset M$, and $g: V_2 \to \boldsymbol{R}^n$ is a surjective homeomorphism, then h of example 2) is defined by

$$V = V_1 \times V_2 \xrightarrow{h} V_1 \times \boldsymbol{R}^n, \qquad (x, y) \rightsquigarrow \big(x, g(y) - g(x)\big).$$

In stead of the fibrebundles (1), we also consider fibre bundles with fibre $F=$

$$\boldsymbol{R}^{p+n} = \{(x_1, \ldots, x_{p+n})\}$$

or

$$S^{p+n} = \{(x_0, \ldots, x_{p+n}) \in \boldsymbol{R}^{p+n+1} \mid \Sigma_0^{p+n} x_i^2 = 1\}$$

or

$$B^{p+n} = \{(x_1, \ldots, x_{p+n}) \in \boldsymbol{R}^{p+n} \mid \Sigma_1^{p+n} x_i^2 \leqq 1\}$$

and with group the group of those homeomorphisms that leave invariant the subspace with equation $x_{p+1}=\cdots=x_{p+n}=0$.

In the case $F=S^{p+n}$ we obtain in this manner a S^{p+n}-bundle containing fibrewise a S^p-bundle, with a common 0- and ∞-cros-section. We call this object a fibrebundle of type (S^{p+n}, S^p) and analogously for the other kinds of fibre. The notions microidentical, premicro-$(p+n, p)$-bundle and micro-$(p+n, p)$-bundle are now defined in analogy with the former case. We obtain in § 7:

Theorem 2. *In the topological and in the* PL-*category every micro-$(p+n,p)$-bundle over X contains a bundle of type $(\boldsymbol{R}^{p+n}, \boldsymbol{R}^p)$ and a bundle of type (S^{p+n}, S^p) and they are unique up to equivalence.*

Example 4. (Normal microbundles.) Let X be a p-dimensional locally flat submanifold of a $p+n$-dimensional manifold Y, and let the pair be locally homeomorphic to the standard imbedding of $\boldsymbol{R}^p$ in $\boldsymbol{R}^{p+n}$. If τ_x is the tangentmicrobundle of X, $\tau_Y|X$ the restriction to X, of the tangent microbundle of Y, then these two microbundles form a micro-$(p+n,p)$-bundle. In particular X may be the zero-section of an $\boldsymbol{R}^n$-bundle ξ over X with total space Y. But in the latter case $\tau_Y|X$ can be identified with the Whitneysum (MILNOR [5]) $\tau_X\oplus\mu(\xi)$, with $\mu(\xi)$ the microbundle of ξ. $\mu(\xi)$ is in this case a normal microbundle for X in Y. The group of the corresponding bundle of type $(\boldsymbol{R}^{p+n}, \boldsymbol{R}^p)$ can then be reduced to the group of homeomorphisms

f: $\boldsymbol{R}^{p+n}\to\boldsymbol{R}^{p+n}$ which split:

$$f_i(x_1,\ldots,x_{n+p})=\begin{cases} f_i(x_1,\ldots,x_p,0,\ldots,0) & \text{for } 1\leqq i\leqq p\\ f_i(0,\ldots,0,x_{p+1},\ldots,x_{p+n}) & \text{for } p+1\leqq i\leqq p+n\,. \end{cases}$$

It is not yet known whether such a reduction always exists or not. For vectorbundles it is wellknown to exist. In the PL-case for $n>p+1$ it exists and is unique by a theorem of HAEFLIGER and WALL [2].

If it exists in the case of the submanifold $X\subset Y$, hence if ν_X, a submicrobundle of $\tau_Y|X$ exists, and

$$\tau_Y\,|\,X=\tau_X\oplus\nu_X$$

then ν_X is called a *normal microbundle* of X in Y.

Consider a premicro-n-bundle over X consisting of bundles ξ_α over open sets $V_\alpha\subset X$ of a locally finite covering.

Let W_α be compact, $W_\alpha\subset V_\alpha$ and $\bigcup_\alpha W_\alpha=X$. Finally let ξ be a $\boldsymbol{R}^n$-bundle which microagrees with all bundles ξ_α.

Now we consider an open set U in the total space of ξ, that contains the zero-section sX and such that $[U\cap p^{-1}(W_\alpha)]\subset p_\alpha^{-1}(W_\alpha)$. So U is completely covered by the bundles spaces of the premicro-bundle. It would be nice to have a bundle ξ' with total space contained in U, which also microagrees with ξ_α for all α. Then ξ' would not require more points then those allready offered in the given premicrobundle and no identifications would be needed under restriction to $\{p^{-1}(W_\alpha)\}$. In the case of the tangent microbundle ξ' would have its total space imbedded in $M \times M$ (see example 3). This aim can be reached in view of

Theorem 3. *If*

$$\xi: \quad \boldsymbol{R}^n \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} X$$

is a $\boldsymbol{R}^n$-bundle over X, U an open set in E containing $s\,X$, then there exists a bundle ξ' microidentical with ξ, with total space contained in U.

This will be proved in § 6. An analogous theorem holds for bundles of type $(\boldsymbol{R}^{p+n}, \boldsymbol{R}^p)$.

The following sequence gives a survey of some related problems

$$\begin{array}{ccr} [\text{Vector } \boldsymbol{R}^n] \xrightarrow{b} [B^n] \xrightarrow{s} [S^n] \xrightarrow{r} [\boldsymbol{R}^n] \xrightarrow{\mu} \text{micro-}n & & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow{\scriptstyle v} & & (2) \\ [S^{n+p}, S^p] \xrightarrow{r'} [\boldsymbol{R}^{n+p}, \boldsymbol{R}^p] \xrightarrow{\mu'} \text{micro}(n+p, p)\,. & & \end{array}$$

It concerns bundles with zero-section and microbundles over a p-dimensional manifold X, and each symbol represents a set of equivalence classes. The arrows represent natural maps, and the problems are injectivity and surjectivity of these maps. $\boldsymbol{R}^n$ can be compactified by an $n-1$-sphere to get B^n with the linear group acting well defined on B^n. This defines the map b. For any B^n-bundle we can take the bundle twice and identify fibrewise along the ∂B^n-bundle to get an S^n-bundle. This defines the map s. Given a zero-section in a S^n-bundle, there exists a disjoint ∞-section; delete it and get a $\boldsymbol{R}^n$-bundle. This defines $r \cdot \mu$ was defined earlier. v assigns to any bundle over the manifold X the pair consisting of the tangentbundle of $s\,X$ and the restriction of the tangent-bundle of the total space to $s\,X$. In the PL case the maps have to be defined with more care.

As a mather of fact b has not been properly defined so far we believe in case PL, but perhaps it can be done along the lines of the work of Lashof and Rothenberg [*4*].

The following is now known. For some X and some n: b is stably neither injective nor surjective (Milnor [*5*]); s is not surjective (Browder [*1*]). For every X and n, r and μ as well as r' and μ' are bijective (Kister [*3*], and our theorem 1); s is stably bijective (Browder [*1*]).

2. Tools

We first describe some standard representations of spheres and balls and other tools in the *topological category*. On the n-sphere

$$S^n = \{(x_0, \ldots, x_n) \in \boldsymbol{R}^{n+1} \mid \Sigma_0^n x_i^2 = (4/\pi)^2\}$$

in euclidean $n+1$-space, we distinguish two points $(-4/\pi, 0, \ldots 0) = P_s$ also called the *south pole* or sometimes "0" and $(-4/\pi, 0, \ldots 0) = P_n$ also called the north pole or sometimes "∞".

r is the distance measured in S^n from any point to P_s. ω is the shortest geodesic from any point to P_s. (r, ω) are southpolar coordinates on S^n.

The *ball* $B(a) = \{x \in S^n \mid r(x) \leqq a\}$ with centre P_s, has *interior* $\mathring{B}(a) = \{x \in S^n \mid r(x) < a\}$. The *complement* of $\mathring{B}(a)$ is $B'(a) = S^n \setminus B_0(a)$, a ball with centre P_n. The interior of $B'(a)$ is denoted by $\mathring{B}'(a)$.

In particular: $B(0) = P_s$, $B(4) = S^n$, $B(2)$ is called the *south hemisphere*, $B'(4) = P_n$; $\mathring{B}(4)$ will often be identified with $\boldsymbol{R}^n$. Another representation is the *ball*

$$D(a) = \{(x_1, \ldots, x_n) \mid \Sigma_1^n x_i^2 \leqq a^2\} \subset \boldsymbol{R}^n.$$

Lemma 2.1. For every $0 < a < b < c < d < 4$ there exists a *concentric homeomorphism* $\rho(a, b, c, d)$ of S^n onto S^n which maps $B(b)$ onto $B(c)$ and leaves $B(a)$ and $B'(d)$ pointwise fixed.

In southpolar coordinates it is defined by

$$\rho(a, b, c, d)(r, \omega) = (\varphi(r), \omega).$$

The real function φ is represented in Fig. 2a.

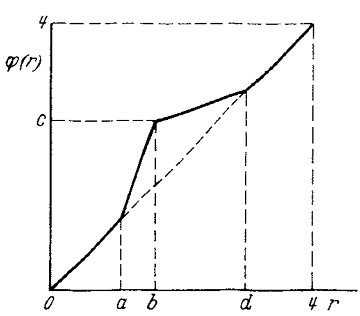

Fig. 2a

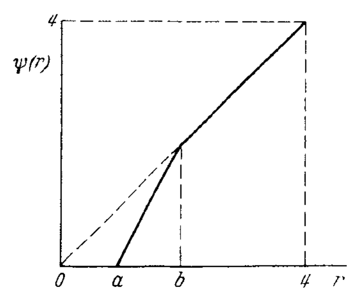

Fig. 2b

It has the required values for $r = 0, a, b, c, d$ and 4, and is linear in the connecting intervals.

Observe that all hypersurfaces of S^n like $x_i = 0$ or $x_i = x_j$ for $i, j = 1, \ldots n$ are invariant under ρ.

Lemma 2.2. For every $0<a<b<4$ there exists a continuous map $\lambda(a,b)$, called *pinch*, of S^n onto S^n which maps $B(a)$ onto $B(0)=P_s$, restricts to a homeomorphism of $\mathring{B}'(a)$ onto $\mathring{B}'(0)$, and which leaves $B'(b)$ pointwise fixed.

In south polar coordinates it is defined by

$$\lambda(a,b)(r,\omega)=(\psi(r),\omega).$$

The real function ψ is represented in Fig. 2b. It has the required values for $r=0$, a, b, 4, and is linear in the connecting intervals.

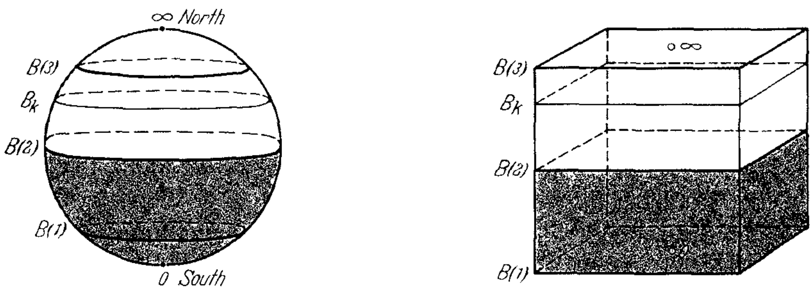

Fig. 3

For the *piecewise linear category* we have to modify these tools. The modified tools are more complicated, but they can be used for the topological category as well.

In $\boldsymbol{R}^{n+1}$ we consider the Banach norm

$$\|x\|=\max_i|x_i|, \qquad x=(x_0,\dots,x_n)\in\boldsymbol{R}^{n+1}.$$

On the *n-sphere*

$$S^n=\{x\in\boldsymbol{R}^{n+1}\,|\,\|x\|=1\}$$

we distinguish the *south pole* $P_s=(-1,0,\dots 0)$ and the north pole $P_n=(1,0,\dots 0)$. r is the Banach distance measured in S^n from any point to P_s. ω is the shortest geodesic from any point to P_s. (r,ω) are *south polar coordinates* in S^n.

The *ball* $B(a)=\{x\in S^n\,|\,r(x)\leq a\}$ with centre P_s, has *interior* $\mathring{B}(a)=\{x\in S^n\,|\,r(x)<a\}$. $B'(a)=S^n\setminus\mathring{B}(a)$, a ball with centre P_n, is the *complement* of $\mathring{B}(a)$ in S^4.

In particular: $B(0)=P_s$, $B(4)=S^n$, $B(2)$ is the *south hemisphere*, $B'(4)=P_n$. Observe that $B(1)$ and $B'(3)$ lie in hyperplanes in $\boldsymbol{R}^{n+1}$.

Another representation is the ball

$$D(a)=\{(x_1,\dots,x_n)\,|\,\max_i|x_i|\leq a\}\subset\boldsymbol{R}^n.$$

In all these spaces the PL-structure is taken from the natural PL-structure of $\boldsymbol{R}^{n+1}$ or $\boldsymbol{R}^n$. Every triangulation of S^n or a part of S^n or $\boldsymbol{R}^n$ to be considered will be a subdivision of the division obtained from the hyperplanes $x_i=0$ and $x_i=\pm x_j$ for $i,j=1,\ldots n$ in $\boldsymbol{R}^{n+1}$ or $\boldsymbol{R}^n$.

In order to define a *concentric homeomorphism* ρ as required in the lemma, also for the PL-category, we first introduce a PL-homeomorphism

$$\kappa\colon\ \mathring{B}(4)\to\mathring{D}(4).$$

For that we take a triangulation of $\mathring{B}(4)$ such that the spheres

$$\partial B(1), \quad \partial B(2), \quad \partial B\left(4-\frac{1}{k}\right) \quad \text{for} \quad k=1,2,3,\ldots$$

are triangulated subspaces, and such that no vertices of the triangulation of $\mathring{B}(4)$ except P_s are outside these spheres. The restriction of κ to these spheres will be defined in terms of south polar coordinates (r,ω) for $\mathring{B}(4)$ and polar coordinates (r',ω') for $\mathring{D}(4)$ by the equations $r'=r$ and $\omega'=\omega$. Then κ is completely determined by the condition of linearity on the simplices of the triangulation. *Observe that the equations $x_i=0$ and $x_i=x_j$ for $i,j=1,\ldots,n$ are invariant under κ.*

Observe also that the radial Banach distance r is invariant under this map κ.

We will define a concentric homeomorphism (PL) in $\mathring{D}(4)$, which we then can carry over by κ to $\mathring{B}(4)$. Consider a triangulation of $\mathring{D}(4)$ with no vertices in the annuli $\mathring{D}(b)\setminus D(a)$ and $\mathring{D}(d)\setminus D(b)$. In $\mathring{D}(4)$ we apply the formula given above for ρ (a,b,c,d) to define a concentric homeomorphism (topological version), but we use it only to define a map for the vertices of the triangulation. After that we extend the map linearly over the simplices, and we call the resulting PL-homeomorphism $\tau\colon\mathring{D}(4)\to\mathring{D}(4)$. Then $\kappa^{-1}\,\tau\,\kappa\colon\mathring{B}(4)\to\mathring{B}(4)$ is the required PL-version of the concentric homeomorphism. We denote it again as ρ (a,b,c,d). *It carries "spheres" with centre P_s onto such spheres. A straight line through P_s however is in general not mapped onto another such line* see Fig. 4). We also define ρ (a,b,c,d) to carry the north pole P_n onto itself.

We leave it to the reader to establish that τ is isotopic to the identity map. As a consequence we have (to be used for the proof of theorem 3):

Remark 2.3. $\rho(a,b,c,d)$ is isotopic with the identity map (in the PL-category and in the topological category).

The *pinch* $\lambda(a,b)$ will also be defined using the chart κ. We use the above formulas for all points $x\in\mathring{D}(4)$ with $0\leqq r\leqq a$, $r=a+(b-a)/k$ for $k=1,2,3,\ldots$ and $b\leqq r<4$. We take a triangulation of $\mathring{D}(4)-D(a)$

with all vertices in this set of points, and we extend linearly. *The continuous map $\lambda(a, b)$ so obtained is piecewise linear in the complement of the sphere $\partial D(a)$. There is no open set containing $\partial D(a)$ in which the pinch $\lambda(a, b)$ is piecewise linear.* Observe that the hypersurfaces $x_i=0$ and $x_i=x_j$ for $i, j=1, \dots n$ are invariant under λ as well as under ρ.

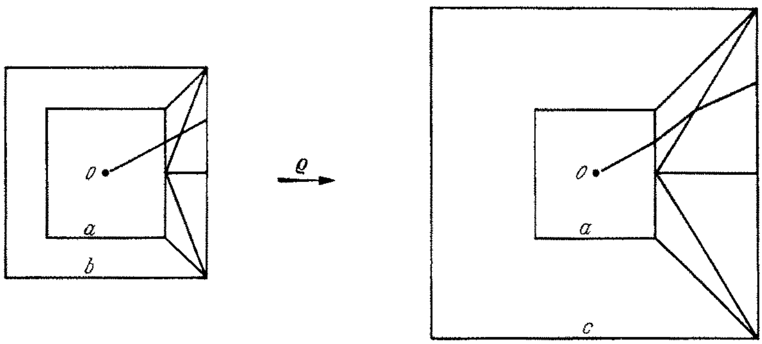

Fig. 4

Remark 2.4. If K is a simplicial complex, for example one simplex, then one has the homeomorphism

$$\text{identity} \times \rho(a, b, c, d)\colon \quad K \times S^n \to K \times S^n$$

and the pinch

$$\text{identity} \times \lambda(a, b)\colon \qquad K \times S^n \to K \times S^n .$$

In the PL-case these are PL, and PL outside $K \times \partial B(a)$, respectively. They will be denoted also by $\rho(a, b, c, d)$ and $\lambda(a, b)$ respectively. Both maps commute with projection onto the first factor K. That is, they are fibre preserving.

Definition 2.5. If $f\colon \Delta_p \times B(b) \to \Delta_p \times S^n$ is a fibrewise inbedding, Δ_p the standard p-dimensional simplex, $\lambda = \lambda(a, b)$ the pinch defined above, then the *transform $f(\lambda)$ of the pinch* λ is the fibrewise map

$$f(\lambda)\colon \quad \Delta_p \times S^n \to \Delta_p \times S^n$$

given by

$$f(\lambda)(x, y) = \begin{cases} (x, y) & \text{if} \quad (x, y) \notin f(\Delta_p \times B(b)) \\ (x, 0) & \text{if} \quad (x, y) \in f(\Delta_p \times B(a)) \\ f \lambda f^{-1}((x, y)) & \qquad \text{elsewhere} \end{cases}$$

In the PL-case $f(\lambda)$ is a PL-homeomorphism in the complement of $f(\Delta_p \times B(a))$.

3. Reduction of Theorem 1 to Lemma 3

We will first concentrate on the topological category and in particular on the existence proof for the case Top $-S^n$ of theorem 1. The other cases are then taken care of by some simple additional remarks as we will see.

Let the micro-n-bundle $\underline{\underline{x}}$ be represented by the premicro-n-bundle $\{\xi_\alpha, p_\alpha, V_\alpha\}$ where ξ_α is a bundle over V_α, over the simplicial complex $X = \bigcup_\alpha V_\alpha$. Take a triangulation T of X such that each simplex σ is covered by at least one V_α, of the open sets $\{V_\alpha\}$ and consider $\xi_\sigma = \xi_\alpha | \sigma$, the restriction of the bundle ξ_α to $\sigma \subset V_\alpha$. The bundles ξ_σ for $\sigma \in T$ microagree with each other. We may as well restrict, and we will do so, to one (trivial) S^n-bundle over each simplex $\sigma \in T$ that is not on the boundary of some higher dimensional simplex. Such a set of microagreeing bundles allready determines the microbundle completely.

If Δ is a simplex in the intersection of the simplices σ_1 and $\sigma_2 \neq \sigma_1$, then $\xi_{\sigma_1} | \Delta$ and $\xi_{\sigma_2} | \Delta$ are micro-identical. We assume inductively that $\xi_{\sigma_1} | \Delta$ and $\xi_{\sigma_2} | \Delta$ are identical for all triples $(\Delta, \sigma_1, \sigma_2 \,|\, \Delta \subset \sigma_1 \cap \sigma_2)$ with Δ of dimension $<k$ $(k \geqq 0)$. This means that over the $k-1$-skeleton of T we have an S^n-bundle allready. Now let Δ be a k-simplex in the intersection of σ_1 and σ_2. The bundles (of the premicrobundle) over σ_1 and σ_2 are trivial. So are their restrictions to Δ:

$$\begin{array}{ccc} E_1 & \xrightarrow{\kappa_1} & \Delta \times S^n \\ \downarrow{\scriptstyle p_1} & & \downarrow \\ \Delta & & \Delta \end{array} \quad \text{and} \quad \begin{array}{ccc} E_2 & \xrightarrow{\kappa_2} & \Delta \times S^n \\ \downarrow{\scriptstyle p_2} & & \downarrow \\ \Delta & & \Delta \end{array}$$

Both can be represented (charts κ_1 and κ_2) by the trivial S^n-bundle with standard 0-section and ∞-section. By the inductive assumption the restrictions of the two bundles to $\partial \Delta$ are identical.

Hence

$$f_{\mathrm{I}} = (\kappa_2 \kappa_1^{-1} \,|\, \partial\Delta \times S^n): \quad \begin{tikzcd} \partial\Delta \times S^n \arrow[rr, "\cong"] \arrow[dr] & & \partial\Delta \times S^n \arrow[dl] \\ & \partial\Delta & \end{tikzcd}$$

is a well defined bijection.

Definition. The word *bijection* will be reserved for a fibrewise surjective homeomorphism of bundles, also in the PL-category. (The diagram is commutative; the nonhorizontal arrows are projections in the first factor.)

The bundles over Δ are microidentical. This implies that b, with $0 < b \leqq 3$, exists such that this microidentity is *represented by* a fibrewise homeomorphism

$$f_{\mathrm{II}} = (\kappa_2 \kappa_1^{-1} \,|\, \Delta \times B(b)): \quad \Delta \times B(b) \to \Delta \times \mathring{B}(4)$$

which agrees with f_{I} on their common domain. If we replace once and for all κ_1 by its composition with a suitable concentric homeomorphism $\rho(a, b, 3, 3\frac{1}{2})$. κ_1 (lemma 2.1), then we get the new value $b=3$. We assume $b=3$, and we combine f_{I} and f_{II} to a fibrewise inbedding

$$f\colon\ (\partial\Delta\times S^n)\cup(\Delta\times B(3))\to\Delta\times S^n.$$

Now we need

Lemma 3. *For every fibrewise inbedding*

$$\begin{tikzcd} \Delta\times B(3) \arrow[rr] \arrow[dr] & & \Delta\times\mathring{B}(4) \arrow[dl] \\ & \Delta & \end{tikzcd}$$

preserving the 0*-section, there exists a bijection*:

$$\begin{tikzcd} \Delta\times S^n \arrow[rr] \arrow[dr] & & \Delta\times S^n \arrow[dl] \\ & \Delta & \end{tikzcd}$$

preserving the 0*-section and the* ∞*-section, such that*

$$f\,|\,\Delta\times B(2)=g\,|\,\Delta\times B(2).$$

This lemma will be proved in § 4 and 5. We apply it in our situation and obtain a fibrewise map

$$h_{\mathrm{I}}=g^{-1}f$$

which is the identity on $\Delta\times B(2)$ and which is also defined on the boundary $\partial(\Delta\times B'(2))$ of $\Delta\times B'(2)$.

We represent the north pole ball $B'(2)$ by the convex ball $D(2)\subset\boldsymbol{R}^n$, and Δ by a simplex in $\boldsymbol{R}^k$ with centroid $0\in\boldsymbol{R}^k$.

Next we extend the homeomorphism h_{I} by defining

$$h(t\,x, t\,y)=t\,h_{\mathrm{I}}(x, y)\in\boldsymbol{R}^{k+n}=\boldsymbol{R}^k\times\boldsymbol{R}^n$$

for $(x, y)\in\partial(\Delta\times D(2))$ and $0\leqq t\leqq 1$. Because h_{I} maps points of the fibre of $t\,x\in\Delta$ onto points of the fibre of $t\,x$ so does h. Hence $h\colon\Delta\times S^n\to\Delta\times S^n$ is a bijection.

Finally $\hat{f}=g\,h$ is used to identify the trivial bundles over σ_1 and σ_2 after a preliminary separation. As a result, above $\partial\Delta$ nothing is changed. Above Δ nothing is microchanged.

The S^n-bundle is now also defined above Δ. Repeating the process we obtain by induction the existence of the required S^n-bundle in the topological category.

If two S^n-bundles over X are microidentical then we obtain an isomorphism of the restrictions of the S^n-bundles to the k-skeleton from the same over the $k-1$-skeleton, k simplexwise with the same method.

Hence uniqueness up to equivalence is also established. Leave out the ∞-cross-sections and one obtains the case Top-$\boldsymbol{R}^n$ of theorem 1.

In the PL-category everything can be done in the same way if one has the pure PL-version of Lemma 3 (Lemma 3 PL). This will be given in § 5. In § 4 however we give a *common proof* of a) lemma 3 topological and b) the following restricted PL-version of lemma 3.

Lemma 3′(PL). For every fibrewise PL-inbedding

$$f:\quad \Delta\times B(3)\to\Delta\times\mathring{B}(4)$$

there exists a topological bijection:

$$g:\quad \Delta\times S^n\to\Delta\times S^n$$

such that

$$g\,|\,\Delta\times B(2)=f\,|\,\Delta\times B(2)$$

and such that

$$g\,|\,\Delta\times\mathring{B}(4)\quad\text{is a PL-homeomorphism}.$$

Applying lemma 3′ instead of lemma 3 we obtain in the PL-case a S^n-bundle which may be PL-bad (!) at the ∞-section. If we delete the ∞-section, we obtain an $\boldsymbol{R}^n$-bundle in the PL-category. Hence the common proof of lemma 3 Top and lemma 3′ (PL) in § 4 leads to a common proof for the cases Top$-\boldsymbol{R}^n$, PL$-\boldsymbol{R}^n$ and Top$-S^n$ of theorem 1.

4. *Proof of Lemmas 3* (Top) *and* 3′(PL)

Let $f_1=f$ be the fibrewise inbedding assumed in lemma 3 or 3′(PL). We define the fibrewise inbeddings

$$\begin{tikzcd} f_k:\ \Delta\times B(3) \arrow[rr] \arrow[dr] & & \Delta\times\mathring{B}(4) \arrow[dl] \\ & \Delta & \end{tikzcd}\qquad k=2,3,\dots$$

inductively by

$$f_{k+1}(x,y)=\begin{cases} f_k(x,y) & \text{for}\quad y\in B_k=B\left(3-\dfrac{1}{k}\right)\\ [f_k(\lambda_k)]^{-1}\,\rho_k\,[f_k(\lambda_k)]\,f_k(x,y) & \text{for}\quad y\notin B_k\end{cases}$$

with

$$\lambda_k=\lambda\left(3-\frac{1}{k},\,3-\frac{1}{k+1}\right)\qquad\text{(pinch)},$$

$$\rho_k=\rho\left(\varepsilon,\,2\varepsilon,\,4-\frac{1}{k},\,4-\frac{1}{k+1}\right)$$

and $\varepsilon>0$, so small that

$$\Delta\times B(2\varepsilon)\subset f(\Delta\times B(2)).$$

Observe that $[f_k(\lambda_k)]f_k = f_k\,\lambda_k$, and analyse in particular what happens with $\Delta\times B_{k+1}\setminus\Delta\times B_k$ in the four steps (composition of maps) of the formula for $f_{k+1}(x,y)$. Observe also that

$$f(\Delta\times B(2))\subset f_k(\Delta\times B_{k+1}) = f_k\,\lambda_k(\Delta\times B_{k+1}).$$

Compare definition 2.5 for $f_k(\lambda_k)$. The pinch is needed in order to be able to leave f_k unchanged in the pinched part. ρ_k is used in order to make $f_{k+1}(\Delta\times B(3))$ very large: it contains

$$B\left(4-\frac{1}{k}\right).$$

For the PL-case it should be remarked that f_k and f_{k+1} are identical in some open set containing $\Delta\times B_k$. *Therefore the piecewise linearity is not hurt by the change from f_k to f_{k+1} although a pinch occurs twice in the formula for this change.*

In the topological as well as in the PL-category we obtain a limit which is a *bijection*

$$f_\infty\colon\ \Delta\times\mathring{B}(3)\xrightarrow{\ \simeq\ }\Delta\times\mathring{B}(4)$$

with $f_\infty\,|\,\Delta\times B(2) = f\,|\,\Delta\times B(2)$ as we see.

Next let τ be the reflection of S^n with respect to the equator

$$\tau(x_0,x_1,\dots,x_n) = (-x_0,x_1,\dots,x_n)\in\boldsymbol{R}^{n+1}.$$

Then

$$\tau\,\lambda(1,2)\,\tau$$

is a mapping which pinches $B'(3)$ into the North pole P_n, leaves $B(2)$ pointwise fixed, and defines a bijection of

$$\Delta\times\mathring{B}(3)\quad\text{onto}\quad\Delta\times\mathring{B}(4).$$

The topological bijection g, required in lemma 3 (Top) and in lemma 3′ (PL), is then

$$g=\begin{cases} f_\infty\cdot[\tau\,\lambda(1,2)\,\tau]^{-1}\colon & \Delta\times\mathring{B}(4)\to\Delta\times\mathring{B}(4)\\ \text{identity:} & \Delta\times P_n\to\Delta\times P_n.\end{cases}$$

We recall that now in the cases Top$-\boldsymbol{R}^n$, Top$-S^n$ and PL$-\boldsymbol{R}^n$, existence and uniqueness of theorem 1 are proved completely.

5. *Proof of Lemma* 3 (PL)

In this § we work in the PL-*category*.

Referring to lemma 3, we first consider the special case where Δ *is one point* (dimension zero).

We write $B(3)$ instead of $\Delta \times B(3)$ for this case. As we have no need to consider the part between $B(3)$ and $B(2)$ we let f denote the (given) inbedding:

$$f\colon\ B(2) \to B^0(4) \subset S^n.$$

NEWMAN (Theorem 3 in [6]) proved that any two inbedded combinatorial n-balls in an n-manifold are "similarly situated". Applying this to $B(2) \subset B^0(4)$ and $f(B(2)) \subset B^0(4)$, this means the existence of a bijection which can be assumed pointwise fixed for points in $B'(4-\varepsilon)$ for some $\varepsilon > 0$,

$$h_1\colon\ S^n \to S^n$$

such that $h_1 f(B(2)) = B(2)$.

Of course it can and will also be assumed that $f(P_s) = P_s$. Let the bijection $h_2\colon S^n \to S^n$ be defined by

$$h_2(y) = \begin{cases} h_1 f(y) & \text{for } y \in B(2) \\ \tau h_1 f(\tau(y)) & \text{for } y \in B'(2). \end{cases}$$

τ is the reflection of S^n in the equator $\partial B(2)$ discussed earlier.

Then the bijection

$$g_0 = h_1^{-1} h_2\colon\ S^n \to S^n$$

is the extension to S^n, required in the lemma, for the case that Δ is one point:

$$g_0 \mid B(2) = f \mid B(2),$$

$$\begin{tikzcd} B(2) \arrow[r, "f"] \arrow[d, "\cap"] & S^n \arrow[d, "h_1"] \\ S^n \arrow[r, "h_2"] \arrow[ur, "g_0"] & S^n. \end{tikzcd}$$

Next consider the general case. Let x_0 be a vertex of Δ. Let $f(\Delta \times B(2))$ be contained in the interior of $\Delta \times B(4-\varepsilon) \subset \Delta \times S^n$.

Denote by

$$f_0\colon\ \Delta \times B(2) \to \Delta \times S^n$$

the inbedding which equals f on the fibre $x_0 \times B(2)$ and is constant in the variabele $x \in \Delta$:

$$f_0(x, y) = (x, \psi(y)); \qquad f_0(x_0, y) = f(x, y).$$

According to HUDSON (ZEEMAN, Remark, page 74 in [7]) there exists a bijection (a higher dimensional PL-isotopy with the variable x running in Δ instead of in the 1-simplex Δ_1):

$$h\colon\ \Delta \times S^n \to \Delta \times S^n$$

with

$$(h \mid \Delta \times B'(4-\varepsilon)) = \text{identity}$$

such that

$$f = h\, f_0 .$$

By the first part of this §, f_0 is the restriction of some bijection (the same in each fibre): $g_0 \colon \Delta \times S^n \to \Delta \times S^n$, with the property:

$$g_0 \mid \Delta \times B(2) = f_0 .$$

Then $g = h\, g_0$ is the bijection required in lemma 3:

$$g \mid \Delta \times B(2) = h\, g_0 \mid \Delta \times B(2) = h\, f_0 = f .$$

Hence lemma 3 (PL) and theorem 1 (PL) are also proved.

A simpler obvious proof can be obtained with a deeper theorem to the effect that if f is as above, and orientation preserving, then $f \mid \Delta \times B(2)$ is ambient isotopic to the identity map.

6. *Proof of Theorem 2*

Theorem 2 is analogous to theorem 1 with $\boldsymbol{R}^n$-bundles replaced by bundles of type $(\boldsymbol{R}^{p+n}, \boldsymbol{R}^p)$ and S^n-bundles replaced by bundles of type (S^{p+n}, S^p). The proof is obtained by following the proof of theorem 1 very closely and making small modifications.

First we follow § 2 the tools. Here the standard fibre S^n is replaced by the pair consisting of

$$S^{p+n} = \{(x_0, \dots, x_{p+n}) \mid \Sigma_0^{p+n} x_i^2 = (4/\pi)^2\}$$

and

$$S^p = S^{p+n} \cap \{(x_0, \dots, x_{p+n}) \mid x_{p+1} = x_{p+2} \cdots = x_{p+n} = 0\}$$

and analogous for $\boldsymbol{R}^n$ and for the PL case.

Because the concentric homeomorphism ρ as well as the pinch λ leave $S^p \subset S^{p+n}$ and $\boldsymbol{R}^p \subset \boldsymbol{R}^{p+n}$ invariant, all operations used in § 2 and § 3 can be repreated in the new situation and are seen to preserve these pairs for each fibre. Of course it is necessary for example in § 3 to choose the charts κ_1 and κ_2 such that they are charts of type (S^{p+n}, S^p) instead of type S^n. Also in lemma 3 the given map, f must be replaced by an inbedding of bundle pairs:

$$\begin{tikzcd}[column sep=tiny] f\colon (\Delta \times B^{p+n}(3), \Delta \times B^p(3)) \arrow[rr] \arrow[dr] & & (\Delta \times \mathring{B}^{p+n}(4), \Delta \times \mathring{B}^p(4)) . \arrow[dl] \\ & \Delta & \end{tikzcd}$$

This being assumed the resulting bijection g of the lemma is automatically a bijection of bundles of type (S^{p+n}, S^p).

In § 4 the proof of lemma 3 (Top) and 3 (PL$-\boldsymbol{R}^n$) can be repeated with the same formulas with n replaced by $p+n$, to obtain the lemma required for theorem 2.

This being so, we have proved theorem 2 for the topological category and the case of bundles of type $(\boldsymbol{R}^{p+n}, \boldsymbol{R}^p)$ in the PL-category.

ZEEMAN has informed us that from a forthcoming paper of HAEFLIGER and ZEEMAN, it follows that any simplex of ball-bundle-pair imbeddings, like f above, orientation preserving in each fibre-pair, is ambient isotopic, preserving fibrepairs, to the identity map. The analogue of lemma 3 (PL) then follows immediately, and hence theorem 2 both PL-cases.

7. *Proof of Theorem 3*

U is an open set containing the zero cross-section in the $\boldsymbol{R}^n$-bundle

$$\xi:\ \boldsymbol{R}^n \to E \overset{p}{\leftrightarrows} X$$

over the locally finite simplicial complex X: $s\,X \subset U \subset E^s$. We construct as follows a locally finite covering $\{V_\sigma\}$ of X.

For each simplex σ, which is not on the boundary of another simplex of X, let φ_σ be a continuous function, which is linear on each simplex of X and for which

$$\varphi_\sigma(x) = \begin{cases} -1 & \text{for } x \in \sigma \\ 2 & \text{for } x \in L(\sigma) \end{cases}$$

where $L(\sigma)$ is the union of all simplices that have no point in common with σ. Let

$$V_\sigma = \varphi_\sigma^{-1}([-1,1]), \qquad V'_\sigma = \varphi_\sigma^{-1}([-1,0]), \qquad W_\sigma = \varphi_\sigma^{-1}(0).$$

Then we can identify $\varphi_\sigma^{-1}([0,1]) = W_\sigma \times I$, and we have

$$V_\sigma = V'_\sigma \cup (W \times I), \qquad V'_\sigma \cap (W_\sigma \times I) = W_\sigma \times 0 = W_\sigma.$$

V_σ is contractible and so ξ is trivial over V. We use a chart of the kind

$$\begin{tikzcd} \kappa_\sigma:\ p^{-1}(V_\sigma) \arrow[rr, "\simeq"] \arrow[dr] & & V_\sigma \times \mathring{B}(2) \arrow[dl] \\ & V_\sigma & \end{tikzcd}$$

to represent the bundle $(\xi \mid V_\sigma)$.

Now let $b_\sigma > 0$ be so small that

$$\kappa_\sigma\big(U \cap p^{-1}(\sigma)\big) \supset \sigma \times B(b_\sigma).$$

We map E into E with the fibrewise homeomorphism φ_σ defined by:

$$\varphi_\sigma(z)=\kappa_\sigma^{-1}[\rho(\tfrac{1}{2}b_\sigma, b_\sigma, 2, 3)]^{-1}\kappa_\sigma(z) \quad \text{for} \quad z\in p^{-1}(V'_\sigma)$$

$\varphi_\sigma(z)=z$ (identity) for $z\notin p^{-1}(V_\sigma)$. The remaining part of the bundle, with total space $p^{-1}(W\times I)$, is equivalent to

$$W_\sigma\times I\times \mathring{B}(2)\to W_\sigma\times I.$$

φ_σ is already defined for the parts corresponding to the endpoints 0 and 1 of I:

$\partial V_\sigma\times 0\times \mathring{B}(2)$: a concentric homeomorphism, the same in each fibre

and

$\partial V_\sigma\times 1\times \mathring{B}(2)$: identity.

We connect these two by the isotopy described in remark 2.3, in order to complete the definition of φ_σ.

The image $\varphi_\sigma(E)\subset E$ is the total space of a bundle with projection $p_\sigma=(p\,|\,\varphi_\sigma(E))$ such that $(p_\sigma)^{-1}(\sigma)\subset U\subset E$. By repeating this process, for each simplex σ once, we obtain the required bundle ξ' with total space $E'\subset U$. Observe that by local finiteness every point of E is involved in at most a finite number of moves.

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