Microbundles and Bundles

I. Elementary Theory

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1. Definitions, Statement of Results

In this part I we work in two categories namely the category of topological spaces and continuous maps, and the category of piecewise linear (PL) spaces and PL-maps. A PL-space is a topological space with a complete class of locally finite triangulations, any two of which have a common subdivision. A PL-map \( f: X \to Y \) is a continuous map between PL-spaces \( X \) and \( Y \), which for some triangulation of \( X \) and \( Y \) maps every simplex of \( X \) linearly into a simplex of \( Y \). We consider fibre bundles in the sense of Steenrod with a fixed cross-section, often called the zero section,

\[
\xi: \quad F \xrightarrow{i} E \xrightarrow{p} X; \quad P_s = i^{-1} s(X) \in F. \tag{1}
\]

The base space \( X \) is a locally finite simplicial complex; \( E \) is the total space; \( p \) is the projection which in the PL-category of course has to be a PL-map; \( s \) is the zero section. The fibre \( F \) will be \( n \)-dimensional numberspace \( \mathbb{R}^n \) (the open \( n \)-ball), or the \( n \)-ball (with boundary included)

\[ B^n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \Sigma_i x_i^2 \leq 1 \} \quad \text{with} \quad P_s = (0, \ldots, 0), \]

or the \( n \)-sphere

\[ S^n = \{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \Sigma_i x_i^2 = (4/\pi)^2 \} \]

with

\[ (-4/\pi, 0, \ldots, 0) = P_s \text{(''0'') and } (4/\pi, 0, \ldots, 0) = P_n \text{(''\infty'')} \]

The group will be the group of all homeomorphisms of \((F; P_s)\) or \((F; P_s, P_n)\) resp. onto itself.

Two fibred bundles \( \xi_1 \) and \( \xi_2 \) with the same base space \( X \), are called micro-identical if: the zero sections coincide, \( s_1 X = s_2 X \); the total

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1 See section 2 for a description of these spaces in the PL category.
spaces $E_1$ and $E_2$ have in common some open set $U$ containing this zerosection; and if moreover the restrictions of the projections $p_1$ and $p_2$ to $U$ coincide, $p_1|U = p_2|U$. See Fig. 1.

Let $V_1$ and $V_2$ be subspaces of $X$. Two fibrebundles $\xi_1$ over $V_1$ and $\xi_2$ over $V_2$ are said to micro agree, in case their restrictions to $V = V_1 \cap V_2$ are micro identical fibrebundles over $V$.

A premicrobundle over $X$ is a set of fibrebundles $\xi_\alpha$ with cross-section, one over each open set $V_\alpha$, of a covering of $X = \bigcup_\alpha V_\alpha$, such that any two of them microagree.

Two premicrobundles $\{\xi_\alpha\}$ and $\{\xi_\beta\}$ over $X$ are called strongly equivalent if their union $\{\xi_\alpha, \xi_\beta\}$ is also a premicrobundle.

**Definition.** A microbundle is a strong equivalence class of premicrobundles.

Every premicrobundle determines the unique microbundle (=equivalence class) of which it is an element.

**Example 1.** Every fibrebundle $\xi$ (with fibre $F$ as above), is an example of a premicrobundle. Consequently it determines a unique microbundle $\mu(\xi)^2$. ($\mu(\xi)$ can be considered as a germ of neighborhood of the zerosection of the bundle of the total space, together with what remains of the projection.) The converse is to a certain extent true:

**Theorem 1.** In the topological and in the PL-category every micro-n-bundle over $X$ contains a $R^n$-bundle and a $S^n$-bundle with zero-crosssection, and these bundles are unique up to equivalence.

The special cases of this theorem will be denoted by $\text{Top} - S^n$, $\text{PL} - R^n$ etc.

For the topological case this is a theorem of KISTER [3] and B. Mazur. No proof for the PL-case seems to be published so far. We

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2 One can define "microsets" in a microbundle. They form a partially ordered system with analogy to the system of subsets of a set. It may be interesting to study "microsettheory".
give a proof for the cases top-$R^n$, top-$S^n$, PL-$R^n$ in § 2–4. A second proof for PL-$R^n$ and a proof for PL-$S^n$ is given in § 5. The consideration of the $S^n$-case simplifies the $R^n$-proofs considerably. Observe that BROWDER [1] proved that not every micro $n$-bundle contains a $B^n$-bundle.

Example 2. It was MILNOR [5], who introduced microbundles. He defined them as equivalence classes of certain diagrams.

A Milnor diagram

$$X \xrightarrow{i} Y \xrightarrow{p} X$$

consists of a base space $X$, a total space $Y$, maps $i$ and $p$ with composition $pi=\text{identity}$, such that for every point $x \in X$ there exists a neighborhood $V$ of $i(x)$ in $Y$ and a surjective homeomorphism $h$ which makes the main square of the following diagram commutative

$$
\begin{array}{ccc}
Y & \xrightarrow{h} & p(V) \times R^n \\
\downarrow{p} & & \downarrow{\pi_1} \\
X & \xrightarrow{\text{id} \times 0} & p(V) \times 0 \\
\end{array}
$$

with

$$\pi_1(x,u)=(x,0), \quad i_1(x,0)=(x,0).$$

This special diagram defines a $R^n$-bundle over $p(V) \subset X$ with total space $V$. The bundles so obtained from a Milnor diagram define a premicrobundle, hence a microbundle according to our definition.

Example 3. A special case is the tangent microbundle of a topological or PL manifold $M$ defined by the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p} M$$

with $\Delta: x \mapsto (x, x)$ the diagonal map and $p: (x, y) \mapsto p(x, y) = x$ the projection in the first factor.

If $x \in V_1 \subset V_1 \subset V_2 \subset M$, and $g: V_2 \to R^n$ is a surjective homeomorphism, then $h$ of example 2) is defined by

$$V = V_1 \times V_2 \xrightarrow{h} V_1 \times R^n, \quad (x, y) \mapsto (x, g(y) - g(x)).$$

In stead of the fibrebundles (1), we also consider fibre bundles with fibre $F = R^{p+n} = \{(x_1, \ldots, x_{p+n})\}$
or

$$S^{p+n} = \{(x_0, \ldots, x_{p+n}) \in R^{p+n+1} \mid \Sigma_{0}^{p+n} x_i^2 = 1\}$$
or

$$B^{p+n} = \{(x_1, \ldots, x_{p+n}) \in R^{p+n} \mid \Sigma_{1}^{p+n} x_i^2 \leq 1\}$$
and with group the group of those homeomorphisms that leave invariant the subspace with equation $x_{p+1} = \cdots = x_{p+n} = 0$.

In the case $F = S^{p+n}$ we obtain in this manner a $S^{p+n}$-bundle containing fibrewise a $S^p$-bundle, with a common 0- and $\infty$-cross-section. We call this object a fibrebundle of type $(S^{p+n}, S^p)$ and analogously for the other kinds of fibre. The notions microidentical, premicro-(p+n, p)-bundle and micro-(p+n, p)-bundle are now defined in analogy with the former case. We obtain in § 7:

**Theorem 2.** In the topological and in the PL-category every micro-(p+n, p)-bundle over $X$ contains a bundle of type $(R^{p+n}, R^p)$ and a bundle of type $(S^{p+n}, S^p)$ and they are unique up to equivalence.

**Example 4.** (Normal microbundles.) Let $X$ be a $p$-dimensional locally flat submanifold of a $p+n$-dimensional manifold $Y$, and let the pair be locally homeomorphic to the standard imbedding of $R^p$ in $R^{p+n}$. If $\tau_x$ is the tangentmicrobundle of $X$, $\tau_Y|X$ the restriction to $X$, of the tangent microbundle of $Y$, then these two microbundles form a micro-(p+n, p)-bundle. In particular $X$ may be the zero-section of an $R^n$-bundle $\xi$ over $X$ with total space $Y$. But in the latter case $\tau_Y|X$ can be identified with the Whitney sum (MILNOR [5]) $\tau_x \oplus \mu(\xi)$, with $\mu(\xi)$ the microbundle of $\xi$, $\mu(\xi)$ is in this case a normal microbundle for $X$ in $Y$. The group of the corresponding bundle of type $(R^{p+n}, R^p)$ can then be reduced to the group of homeomorphisms

$$f: R^{p+n} \to R^{p+n} \text{ which split:}$$

$$f_i(x_1, \ldots, x_{n+p}) = \begin{cases} f_i(x_1, \ldots, x_p, 0, \ldots, 0) & \text{for } 1 \leq i \leq p \\ f_i(0, \ldots, 0, x_{p+1}, \ldots, x_{p+n}) & \text{for } p+1 \leq i \leq p+n. \end{cases}$$

It is not yet known whether such a reduction always exists or not. For vectorbundles it is wellknown to exist. In the PL-case for $n>p+1$ it exists and is unique by a theorem of HAEFLIGER and WALL [2].

If it exists in the case of the submanifold $X \subset Y$, hence if $\nu_X$, a submicrobundle of $\tau_Y|X$ exists, and $\tau_Y|X = \tau_x \oplus \nu_X$ then $\nu_X$ is called a normal microbundle of $X$ in $Y$.

Consider a premicro-$n$-bundle over $X$ consisting of bundles $\xi_a$ over open sets $V_a \subset X$ of a locally finite covering.

Let $W_a$ be compact, $W_a \subset V_a$ and $\bigcup_a W_a = X$. Finally let $\xi$ be a $R^n$-bundle which microagrees with all bundles $\xi_a$.

Now we consider an open set $U$ in the total space of $\xi$, that contains the zero-section $s X$ and such that $[U \cap p^{-1}(W_a)] \subset p_a^{-1}(W_a)$. So $U$ is
completely covered by the bundles spaces of the premicro-bundle. It would be nice to have a bundle $\xi'$ with total space contained in $U$, which also microagrees with $\xi_x$ for all $x$. Then $\xi'$ would not require more points then those already offered in the given premicrobundle and no identifications would be needed under restriction to $\{p^{-1}(W_s)\}$. In the case of the tangent microbundle $\xi'$ would have its total space imbedded in $M \times M$ (see example 3). This aim can be reached in view of

**Theorem 3.** If

$$\xi: \mathbb{R}^n \rightarrow E \rightarrow X$$

is a $\mathbb{R}^n$-bundle over $X$, $U$ an open set in $E$ containing $s \times X$, then there exists a bundle $\xi'$ microidentical with $\xi$, with total space contained in $U$.

This will be proved in §6. An analogous theorem holds for bundles of type $(\mathbb{R}^{n+p}, \mathbb{R}^p)$.

The following sequence gives a survey of some related problems

$$[\text{Vector } \mathbb{R}^n] \xrightarrow{b} [\mathbb{B}^n] \xrightarrow{s} [S^n] \xrightarrow{r} [\mathbb{R}^n] \xrightarrow{\mu} \text{micro-n} \xrightarrow{v} [S^{n+p}, S^p] \xrightarrow{r'} [\mathbb{R}^{n+p}, \mathbb{R}^p] \xrightarrow{\mu'} \text{micro}(n+p, p) \quad (2)$$

It concerns bundles with zero-section and microbundles over a $p$-dimensional manifold $X$, and each symbol represents a set of equivalence classes. The arrows represent natural maps, and the problems are injectivity and surjectivity of these maps. $\mathbb{R}^n$ can be compactified by an $n-1$-sphere to get $\mathbb{B}^n$ with the linear group acting well defined on $\mathbb{B}^n$. This defines the map $b$. For any $\mathbb{B}^n$-bundle we can take the bundle twice and identify fibrewise along the $\partial \mathbb{B}^n$-bundle to get an $S^n$-bundle. This defines the map $s$. Given a zero-section in a $S^n$-bundle, there exists a disjoint $\infty$-section; delete it and get a $\mathbb{R}^n$-bundle. This defines $r \cdot \mu$ was defined earlier. $v$ assigns to any bundle over the manifold $X$ the pair consisting of the tangentbundle of $s \times X$ and the restriction of the tangentbundle of the total space to $s \times X$. In the PL case the maps have to be defined with more care.

As a matter of fact $b$ has not been properly defined so far we believe in case PL, but perhaps it can be done along the lines of the work of LASHOF and ROTHENBERG [4].

The following is now known. For some $X$ and some $n$: $b$ is stably neither injective nor surjective (MILNOR [5]); $s$ is not surjective (BROWDER [1]). For every $X$ and $n$, $r$ and $\mu$ as well as $r'$ and $\mu'$ are bijective (KISTER [3], and our theorem 1); $s$ is stably bijective (BROWDER [1]).
2. Tools

We first describe some standard representations of spheres and balls and other tools in the topological category. On the n-sphere

\[ S^n = \{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = (4/\pi)^2 \} \]

in euclidean n+1-space, we distinguish two points \((-4/\pi, 0, \ldots, 0) = P\) also called the south pole or sometimes "0" and \((-4/\pi, 0, \ldots, 0) = P_n\) also called the north pole or sometimes "\(\infty\)".

\(r\) is the distance measured in \(S^n\) from any point to \(P\), \(\omega\) is the shortest geodesic from any point to \(P\). \((r, \omega)\) are southpolar coordinates on \(S^n\).

The ball \(B(a) = \{ x \in S^n \mid r(x) \leq a \}\) with centre \(P\), has interior \(\hat{B}(a) = \{ x \in S^n \mid r(x) < a \}\). The complement of \(\hat{B}(a)\) is \(B'(a) = S^n \setminus B_0(a)\), a ball with centre \(P\). The interior of \(B'(a)\) is denoted by \(\hat{B}'(a)\).

In particular: \(B(0) = P\), \(B(4) = S^n\), \(B(2)\) is called the south hemisphere, \(B'(4) = P\); \(\hat{B}'(4)\) will often be identified with \(\mathbb{R}^n\). Another representation is the ball

\[ D(a) = \{ (x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i^2 \leq a^2 \} \subset \mathbb{R}^n. \]

Lemma 2.1. For every \(0 < a < b < c < d < 4\) there exists a concentric homeomorphism \(\rho(a, b, c, d)\) of \(S^n\) onto \(S^n\) which maps \(B(b)\) onto \(B(c)\) and leaves \(B(a)\) and \(B'(d)\) pointwise fixed.

In southpolar coordinates it is defined by

\[ \rho(a, b, c, d)(r, \omega) = (\varphi(r), \omega). \]

The real function \(\varphi\) is represented in Fig. 2a.

![Fig. 2a](image)

It has the required values for \(r=0, a, b, c, d\) and 4, and is linear in the connecting intervals.

Observe that all hypersurfaces of \(S^n\) like \(x_i = 0\) or \(x_i = x_j\) for \(i, j = 1, \ldots, n\) are invariant under \(\rho\).
Lemma 2.2. For every $0 < a < b < 4$ there exists a continuous map $\lambda(a, b)$, called pinch, of $S^n$ onto $S^n$ which maps $B(a)$ onto $B(0) = P_s$, restricts to a homeomorphism of $\hat{B}'(a)$ onto $\hat{B}'(0)$, and which leaves $B'(b)$ pointwise fixed.

In south polar coordinates it is defined by

$$\lambda(a, b)(r, \omega) = (\psi(r), \omega).$$

The real function $\psi$ is represented in Fig. 2b. It has the required values for $r = 0, a, b, 4$, and is linear in the connecting intervals.

For the piecewise linear category we have to modify these tools. The modified tools are more complicated, but they can be used for the topological category as well.

In $\mathbb{R}^{n+1}$ we consider the Banach norm

$$\| x \| = \max_i |x_i|, \quad x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}.$$

On the $n$-sphere

$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \| x \| = 1 \}$$

we distinguish the south pole $P_s = (-1, 0, \ldots, 0)$ and the north pole $P_n = (1, 0, \ldots, 0)$. $r$ is the Banach distance measured in $S^n$ from any point to $P_s$. $\omega$ is the shortest geodesic from any point to $P_s$. $(r, \omega)$ are south polar coordinates in $S^n$.

The ball $B(a) = \{ x \in S^n \mid r(x) \leq a \}$ with centre $P_s$, has interior $\hat{B}(a) = \{ x \in S^n \mid r(x) < a \}$. $B'(a) = S^n \setminus \hat{B}(a)$, a ball with centre $P_n$, is the complement of $\hat{B}(a)$ in $S^n$.

In particular: $B(0) = P_s$, $B(4) = S^n$, $B(2)$ is the south hemisphere, $B'(4) = P_n$. Observe that $B(1)$ and $B'(3)$ lie in hyperplanes in $\mathbb{R}^{n+1}$.

Another representation is the ball

$$D(a) = \{ (x_1, \ldots, x_n) \mid \max_i |x_i| \leq a \} \subset \mathbb{R}^n.$$
In all these spaces the PL-structure is taken from the natural PL-structure of \( \mathbb{R}^{n+1} \) or \( \mathbb{R}^n \). Every triangulation of \( S^n \) or a part of \( S^n \) or \( \mathbb{R}^n \) to be considered will be a subdivision of the division obtained from the hyperplanes \( x_i = 0 \) and \( x_i = \pm x_j \) for \( i, j = 1, \ldots, n \) in \( \mathbb{R}^{n+1} \) or \( \mathbb{R}^n \).

In order to define a **concentric homeomorphism** \( \rho \) as required in the lemma, also for the PL-category, we first introduce a PL-homeomorphism

\[
\kappa: \hat{B}(4) \rightarrow \hat{D}(4).
\]

For that we take a triangulation of \( \hat{B}(4) \) such that the spheres

\[
\partial B(1), \quad \partial B(2), \quad \partial B\left(\frac{4}{k}\right) \quad \text{for} \quad k = 1, 2, 3, \ldots
\]

are triangulated subspaces, and such that no vertices of the triangulation of \( \hat{B}(4) \) except \( P_s \) are outside these spheres. The restriction of \( \kappa \) to these spheres will be defined in terms of south polar coordinates \((r, \omega)\) for \( \hat{B}(4) \) and polar coordinates \((r', \omega')\) for \( \hat{D}(4) \) by the equations \( r' = r \) and \( \omega' = \omega \). Then \( \kappa \) is completely determined by the condition of linearity on the simplices of the triangulation. **Observe that the equations** \( x_i = 0 \) and \( x_i = \pm x_j \) for \( i, j = 1, \ldots, n \) **are invariant under** \( \kappa \).

**Observe also that the radial Banach distance** \( r \) **is invariant under this map** \( \kappa \).

We will define a concentric homeomorphism (PL) in \( \hat{B}(4) \), which we then can carry over by \( \kappa \) to \( \hat{B}(4) \). Consider a triangulation of \( \hat{D}(4) \) with no vertices in the annuli \( \hat{D}(b) \setminus D(a) \) and \( \hat{D}(d) \setminus D(b) \). In \( \hat{D}(4) \) we apply the formula given above for \( \rho \) \( (a, b, c, d) \) to define a concentric homeomorphism (topological version), but we use it only to define a map for the vertices of the triangulation. After that we extend the map linearly over the simplices, and we call the resulting PL-homeomorphism \( \tau: \hat{D}(4) \rightarrow \hat{D}(4) \). Then \( \kappa^{-1} \tau \kappa: \hat{B}(4) \rightarrow \hat{B}(4) \) is the required PL-version of the concentric homeomorphism. We denote it again as \( \rho \) \( (a, b, c, d) \). It carries "spheres" with centre \( P_s \) onto such spheres. A straight line through \( P_s \) however is in general not mapped onto another such line see Fig. 4). We also define \( \rho \) \( (a, b, c, d) \) to carry the north pole \( P_n \) onto itself.

We leave it to the reader to establish that \( \tau \) is isotopic to the identity map. As a consequence we have (to be used for the proof of theorem 3):

**Remark 2.3.** \( \rho(a, b, c, d) \) **is isotopic with the identity map** (in the PL-category and in the topological category).

The **pinch** \( \lambda(a, b) \) will also be defined using the chart \( \kappa \). We use the above formulas for all points \( x \in \hat{D}(4) \) with \( 0 \leq r \leq a \), \( r = a + (b - a)/k \) for \( k = 1, 2, 3, \ldots \) and \( b \leq r < 4 \). We take a triangulation of \( \hat{D}(4) - D(a) \)
with all vertices in this set of points, and we extend linearly. The continuous map \( \lambda(a, b) \) so obtained is piecewise linear in the complement of the sphere \( \partial D(a) \). There is no open set containing \( \partial D(a) \) in which the pinch \( \lambda(a, b) \) is piecewise linear. Observe that the hypersurfaces \( x_i = 0 \) and \( x_i = x_j \) for \( i, j = 1, \ldots, n \) are invariant under \( \lambda \) as well as under \( \rho \).

Remark 2.4. If \( K \) is a simplicial complex, for example one simplex, then one has the homeomorphism

\[
\text{identity} \times \rho(a, b, c, d): \quad K \times S^n \rightarrow K \times S^n
\]

and the pinch

\[
\text{identity} \times \lambda(a, b): \quad K \times S^n \rightarrow K \times S^n.
\]

In the PL-case these are PL, and PL outside \( K \times \partial B(a) \), respectively. They will be denoted also by \( \rho(a, b, c, d) \) and \( \lambda(a, b) \) respectively. Both maps commute with projection onto the first factor \( K \). That is, they are fibre preserving.

Definition 2.5. If \( f: \Delta_p \times B(b) \rightarrow \Delta_p \times S^n \) is a fibrewise inbedding, \( \Delta_p \) the standard \( p \)-dimensional simplex, \( \lambda = \lambda(a, b) \) the pinch defined above, then the transform \( f(\lambda) \) of the pinch \( \lambda \) is the fibrewise map

\[
f(\lambda): \quad \Delta_p \times S^n \rightarrow \Delta_p \times S^n
\]

given by

\[
f(\lambda)(x, y) = \begin{cases} 
(x, y) & \text{if } (x, y) \notin f(\Delta_p \times B(b)) \\
(x, 0) & \text{if } (x, y) \in f(\Delta_p \times B(a)) \\
\lambda f^{-1}((x, y)) & \text{elsewhere}
\end{cases}
\]

In the PL-case \( f(\lambda) \) is a PL-homeomorphism in the complement of \( f(\Delta_p \times B(a)) \).
3. Reduction of Theorem 1 to Lemma 3

We will first concentrate on the topological category and in particular on the existence proof for the case Top\(-\mathbb{S}^n\) of theorem 1. The other cases are then taken care of by some simple additional remarks as we will see.

Let the micro-n-bundle \(x\) be represented by the premicro-n-bundle \(\{\xi_\sigma, p_\sigma, V_\sigma\}\) where \(\xi_\sigma\) is a bundle over \(V_\sigma\), over the simplicial complex \(X=\bigcup_\alpha V_\alpha\). Take a triangulation \(T\) of \(X\) such that each simplex \(\sigma\) is covered by at least one \(V_\sigma\), of the open sets \(\{V_\alpha\}\) and consider \(\tilde{\xi}_\sigma = \xi_\sigma|_\sigma\), the restriction of the bundle \(\xi_\sigma\) to \(\sigma \subset V_\alpha\). The bundles \(\tilde{\xi}_\sigma\) for \(\sigma \in T\) micro-agree with each other. We may as well restrict, and we will do so, to one (trivial) \(\mathbb{S}^n\)-bundle over each simplex \(\sigma \in T\) that is not on the boundary of some higher dimensional simplex. Such a set of microagreeing bundles already determines the microbundle completely.

If \(A\) is a simplex in the intersection of the simplices \(\sigma_1\) and \(\sigma_2 = \sigma_1\), then \(\xi_{\sigma_1}\left|_A\right.\) and \(\xi_{\sigma_2}\left|_A\right.\) are micro-identical. We assume inductively that \(\xi_{\sigma_1}\left|_A\right.\) and \(\xi_{\sigma_2}\left|_A\right.\) are identical for all triples \((A, \sigma_1, \sigma_2)\) with \(A\) of dimension \(<k\) \((k \geq 0)\). This means that over the \(k-1\)-skeleton of \(T\) we have an \(\mathbb{S}^n\)-bundle already. Now let \(A\) be a \(k\)-simplex in the intersection of \(\sigma_1\) and \(\sigma_2\). The bundles (of the premicrobundle) over \(\sigma_1\) and \(\sigma_2\) are trivial. So are their restrictions to \(A\):

\[
\begin{array}{c}
E_1^{k_1} \xrightarrow{\kappa_1} A \times \mathbb{S}^n \\
p_1 \downarrow \quad \downarrow \\
A \\
\end{array}
\quad \text{and} \quad \begin{array}{c}
E_2^{k_2} \xrightarrow{\kappa_2} A \times \mathbb{S}^n \\
p_2 \downarrow \downarrow \\
A \\
\end{array}
\]

Both can be represented (charts \(\kappa_1\) and \(\kappa_2\)) by the trivial \(\mathbb{S}^n\)-bundle with standard 0-section and \(\infty\)-section. By the inductive assumption the restrictions of the two bundles to \(\partial A\) are identical.

Hence

\[
f_1 = (\kappa_2 \kappa_1^{-1} \mid \partial A \times \mathbb{S}^n): \quad \partial A \times \mathbb{S}^n \xrightarrow{\sim} \partial A \times \mathbb{S}^n
\]

is a well defined bijection.

**Definition.** The word *bijection* will be reserved for a fibrewise surjective homeomorphism of bundles, also in the PL-category. (The diagram is commutative; the nonhorizontal arrows are projections in the first factor.)

The bundles over \(A\) are microidentical. This implies that \(b\), with \(0 < b \leq 3\), exists such that this microidentity is represented by a fibrewise homeomorphism

\[
f_{II} = (\kappa_2 \kappa_1^{-1} \mid A \times B(b)): \quad A \times B(b) \rightarrow A \times \tilde{B}(4)
\]
which agrees with $f_1$ on their common domain. If we replace once and for all $\kappa_1$ by its composition with a suitable concentric homeomorphism $\rho(a, b, 3, 3\frac{1}{2})$. $\kappa_1$ (lemma 2.1), then we get the new value $b = 3$. We assume $b = 3$, and we combine $f_1$ and $f_{11}$ to a fibrewise inbedding

$$f: (\partial A \times S^n) \cup (A \times B(3)) \rightarrow A \times S^n.$$  

Now we need

**Lemma 3.** For every fibrewise inbedding

$$f: A \times B(3) \rightarrow A \times \hat{B}(4)$$

preserving the 0-section, there exists a bijection:

$$g: A \times S^n \rightarrow A \times S^n$$

preserving the 0-section and the $\infty$-section, such that

$$f | A \times B(2) = g | A \times B(2).$$

This lemma will be proved in § 4 and 5. We apply it in our situation and obtain a fibrewise map

$$h_1 = g^{-1} f$$

which is the identity on $A \times B(2)$ and which is also defined on the boundary $\partial (A \times B'(2))$ of $A \times B'(2)$.

We represent the north pole ball $B'(2)$ by the convex ball $D(2) \subset \mathbb{R}^n$, and $A$ by a simplex in $\mathbb{R}^k$ with centroid $0 \in \mathbb{R}^k$.

Next we extend the homeomorphism $h_1$ by defining

$$h(t x, t y) = t h_1(x, y) \in \mathbb{R}^{k+n} = \mathbb{R}^k \times \mathbb{R}^n$$

for $(x, y) \in \partial (A \times D(2))$ and $0 \leq t \leq 1$. Because $h_1$ maps points of the fibre of $t x \in A$ onto points of the fibre of $t x$ so does $h$. Hence $h: A \times S^n \rightarrow A \times S^n$ is a bijection.

Finally $f = g h$ is used to identify the trivial bundles over $\sigma_1$ and $\sigma_2$ after a preliminary separation. As a result, above $\partial A$ nothing is changed. Above $A$ nothing is microchanged.

The $S^n$-bundle is now also defined above $A$. Repeating the process we obtain by induction the existence of the required $S^n$-bundle in the topological category.

If two $S^n$-bundles over $X$ are microidentical then we obtain an isomorphism of the restrictions of the $S^n$-bundles to the $k$-skeleton from the same over the $k-1$-skeleton, $k$ simplexwise with the same method.
Hence uniqueness up to equivalence is also established. Leave out the ∞-cross-sections and one obtains the case Top-$R^n$ of theorem 1.

In the PL-category everything can be done in the same way if one has the pure PL-version of Lemma 3 (Lemma 3 PL). This will be given in § 5. In § 4 however we give a common proof of a) lemma 3 topological and b) the following restricted PL-version of lemma 3.

**Lemma 3' (PL).** For every fibrewise PL-inbedding

$$f: \Delta \times B(3) \to \Delta \times \hat{B}(4)$$

there exists a topological bijection:

$$g: \Delta \times S^n \to \Delta \times S^n$$

such that

$$g | \Delta \times B(2) = f | \Delta \times B(2)$$

and such that

$$g | \Delta \times \hat{B}(4)$$

is a PL-homeomorphism.

Applying lemma 3' instead of lemma 3 we obtain in the PL-case a $S^n$-bundle which may be PL-bad (!) at the ∞-section. If we delete the ∞-section, we obtain an $R^n$-bundle in the PL-category. Hence the common proof of lemma 3 Top and lemma 3' (PL) in § 4 leads to a common proof for the cases Top-$R^n$, PL-$R^n$ and Top-$S^n$ of theorem 1.

**4. Proof of Lemmas 3 (Top) and 3' (PL)**

Let $f_1 = f$ be the fibrewise inbedding assumed in lemma 3 or 3'(PL). We define the fibrewise inbeddings

$$f_k: \Delta \times B(3) \to \Delta \times \hat{B}(4), \quad k=2, 3, \ldots$$

inductively by

$$f_{k+1}(x, y) = \begin{cases} f_k(x, y) & \text{for } y \in B_k = B \left( 3 - \frac{1}{k} \right) \\ \left[ f_k(\lambda_k) \right]^{-1} \rho_k \left[ f_k(\lambda_k) \right] f_k(x, y) & \text{for } y \notin B_k \end{cases}$$

with

$$\lambda_k = \lambda \left( 3 - \frac{1}{k}, 3 - \frac{1}{k+1} \right)$$

(pinch),

$$\rho_k = \rho \left( \varepsilon, 2\varepsilon, 4 - \frac{1}{k}, 4 - \frac{1}{k+1} \right)$$

and $\varepsilon > 0$, so small that

$$\Delta \times B(2\varepsilon) \subseteq f(\Delta \times B(2)).$$
Observe that \([f_k(\lambda_k)]f_k = f_k \lambda_k\), and analyse in particular what happens with \(A \times B_{k+1} \setminus A \times B_k\) in the four steps (composition of maps) of the formula for \(f_{k+1}(x, y)\). Observe also that

\[ f(A \times B(2)) = f_k(A \times B_{k+1}) = f_k \lambda_k(A \times B_{k+1}). \]

Compare definition 2.5 for \(f_k(\lambda_k)\). The pinch is needed in order to be able to leave \(f_k\) unchanged in the pinched part. \(\rho_k\) is used in order to make \(f_{k+1}(A \times B(3))\) very large: it contains

\[ B \left( 4 - \frac{1}{k} \right). \]

For the PL-case it should be remarked that \(f_k\) and \(f_{k+1}\) are identical in some open set containing \(A \times B_k\). Therefore the piecewise linearity is not hurt by the change from \(f_k\) to \(f_{k+1}\) although a pinch occurs twice in the formula for this change.

In the topological as well as in the PL-category we obtain a limit which is a bijection

\[ f_\infty: A \times \hat{B}(3) \rightarrow A \times \hat{B}(4) \]

with \(f_\infty|A \times B(2) = f|A \times B(2)\) as we see.

Next let \(\tau\) be the reflection of \(S^n\) with respect to the equator

\[ \tau(x_0, x_1, \ldots, x_n) = (-x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}. \]

Then

\[ \tau \lambda(1, 2) \tau \]

is a mapping which pinches \(B'(3)\) into the North pole \(P_n\), leaves \(B(2)\) pointwise fixed, and defines a bijection of

\[ A \times \hat{B}(3) \quad \text{onto} \quad A \times \hat{B}(4). \]

The topological bijection \(g\), required in lemma 3 (Top) and in lemma 3' (PL), is then

\[ g = \begin{cases} f_\infty \cdot [\tau \lambda(1, 2) \tau]^{-1}: A \times \hat{B}(4) \rightarrow A \times \hat{B}(4) \\ \text{identity}: A \times P_n \rightarrow A \times P_n. \end{cases} \]

We recall that now in the cases \(\text{Top} - \mathbb{R}^n\), \(\text{Top} - S^n\) and \(\text{PL} - \mathbb{R}^n\), existence and uniqueness of theorem 1 are proved completely.

5. Proof of Lemma 3 (PL)

In this § we work in the PL-category.

Referring to lemma 3, we first consider the special case where \(A\) is one point (dimension zero).
We write $B(3)$ instead of $\Delta \times B(3)$ for this case. As we have no need to consider the part between $B(3)$ and $B(2)$ we let $f$ denote the (given) inbedding:

$$f: B(2) \rightarrow B^0(4) \subset S^n.$$ 

Newman (Theorem 3 in [6]) proved that any two inbedded combinatorial $n$-balls in an $n$-manifold are "similarly situated". Applying this to $B(2) \subset B^0(4)$ and $f(B(2)) \subset B^0(4)$, this means the existence of a bijection which can be assumed pointwise fixed for points in $B'(4-\varepsilon)$ for some $\varepsilon > 0$,

$$h_1: S^n \rightarrow S^n$$

such that $h_1 f (B(2)) = B(2)$.

Of course it can and will also be assumed that $f (P_3) = P_3$. Let the bijection $h_2: S^n \rightarrow S^n$ be defined by

$$h_2(y) = \begin{cases} h_1 f (y) & \text{for } y \in B(2) \\ \tau h_1 f (\tau (y)) & \text{for } y \in B'(2). \end{cases}$$

$\tau$ is the reflection of $S^n$ in the equator $\partial B(2)$ discussed earlier.

Then the bijection

$$g_0 = h_1^{-1} h_2: S^n \rightarrow S^n$$

is the extension to $S^n$, required in the lemma, for the case that $\Delta$ is one point:

$$g_0 \mid B(2) = f \mid B(2),$$

$$\begin{array}{ccc}
B(2) & \xrightarrow{f} & S^n \\
\downarrow_{\cap} & & \downarrow_{h_1} \\
S^n & \xrightarrow{h_2} & S^n.
\end{array}$$

Next consider the general case. Let $x_0$ be a vertex of $\Delta$. Let $f (\Delta \times B(2))$ be contained in the interior of $\Delta \times B(4-\varepsilon) \subset \Delta \times S^n$.

Denote by

$$f_0: \Delta \times B(2) \rightarrow \Delta \times S^n$$

the inbedding which equals $f$ on the fibre $x_0 \times B(2)$ and is constant in the variabele $x \in \Delta$:

$$f_0(x, y) = (x, \psi (y)); \quad f_0(x_0, y) = f(x, y).$$

According to Hudson (Zeeman, Remark, page 74 in [7]) there exists a bijection (a higher dimensional PL-isotopy with the variable $x$ running in $\Delta$ instead of in the 1-simplex $\Delta_1$):

$$h: \Delta \times S^n \rightarrow \Delta \times S^n.$$
with

\[(h | A \times B' (4 - \varepsilon)) = \text{identity}\]

such that

\[f = h f_0.\]

By the first part of this §, \(f_0\) is the restriction of some bijection (the same in each fibre): \(g_0 : A \times S^n \to A \times S^n\), with the property:

\[g_0 | A \times B(2) = f_0.\]

Then \(g = h g_0\) is the bijection required in lemma 3:

\[g | A \times B(2) = h g_0 | A \times B(2) = h f_0 = f.\]

Hence lemma 3 (PL) and theorem 1 (PL) are also proved.

A simpler obvious proof can be obtained with a deeper theorem to the effect that if \(f\) is as above, and orientation preserving, then \(f | A \times B(2)\) is ambient isotopic to the identity map.

6. Proof of Theorem 2

Theorem 2 is analogous to theorem 1 with \(\mathbb{R}^n\)-bundles replaced by bundles of type \((\mathbb{R}^{p+n}, \mathbb{R}^p)\) and \(S^n\)-bundles replaced by bundles of type \((S^{p+n}, S^p)\). The proof is obtained by following the proof of theorem 1 very closely and making small modifications.

First we follow § 2 the tools. Here the standard fibre \(S^n\) is replaced by the pair consisting of

\[S^{p+n} = \{(x_0, \ldots, x_{p+n}) \mid \Sigma_0^{p+n} x_i^2 = (4/\pi)^2\}\]

and

\[S^p = S^{p+n} \cap \{(x_0, \ldots, x_{p+n}) \mid x_{p+1} = x_{p+2} = \cdots = x_{p+n} = 0\}\]

and analogous for \(\mathbb{R}^n\) and for the PL case.

Because the concentric homeomorphism \(\rho\) as well as the pinch \(\lambda\) leave \(S^p \subset S^{p+n}\) and \(\mathbb{R}^p \subset \mathbb{R}^{p+n}\) invariant, all operations used in § 2 and § 3 can be repeated in the new situation and are seen to preserve these pairs for each fibre. Of course it is necessary for example in § 3 to choose the charts \(\kappa_1\) and \(\kappa_2\) such that they are charts of type \((S^{p+n}, S^p)\) instead of type \(S^n\). Also in lemma 3 the given map, \(f\) must be replaced by an inbedding of bundle pairs:

\[f : (\Delta \times B^{p+n}(3), \Delta \times B^p(3)) \to (\Delta \times \hat{B}^{p+n}(4), \Delta \times \hat{B}^p(4)).\]
This being assumed the resulting bijection \( g \) of the lemma is automatically a bijection of bundles of type \((S^{p+n}, S^p)\).

In § 4 the proof of lemma 3 (Top) and 3 (PL - \( \mathbb{R}^n \)) can be repeated with the same formulas with \( n \) replaced by \( p+n \), to obtain the lemma required for theorem 2.

This being so, we have proved theorem 2 for the topological category and the case of bundles of type \((\mathbb{R}^{p+n}, \mathbb{R}^p)\) in the PL-category.

Zeeman has informed us that from a forthcoming paper of Haefliger and Zeeman, it follows that any simplex of ball-bundle-pair imbeddings, like \( f \) above, orientation preserving in each fibre-pair, is ambient isotopic, preserving fibrepairs, to the identity map. The analogue of lemma 3 (PL) then follows immediately, and hence theorem 2 both PL-cases.

7. Proof of Theorem 3

\( U \) is an open set containing the zero cross-section in the \( \mathbb{R}^p \)-bundle

\[
\xi: \mathbb{R}^n \to E \cong X
\]

over the locally finite simplicial complex \( X: s X \subseteq U \subseteq E^s \). We construct as follows a locally finite covering \( \{ V_\sigma \} \) of \( X \).

For each simplex \( \sigma \), which is not on the boundary of another simplex of \( X \), let \( \varphi_\sigma \) be a continuous function, which is linear on each simplex of \( X \) and for which

\[
\varphi_\sigma(x) = \begin{cases} 
-1 & \text{for } x \in \sigma \\
2 & \text{for } x \in L(\sigma)
\end{cases}
\]

where \( L(\sigma) \) is the union of all simplices that have no point in common with \( \sigma \). Let

\[
V_\sigma = \varphi_\sigma^{-1}([-1, 1]), \quad V'_\sigma = \varphi_\sigma^{-1}([-1, 0]), \quad W_\sigma = \varphi_\sigma^{-1}(0).
\]

Then we can identify \( \varphi_\sigma^{-1}(0, 1]) = W_\sigma \times I \), and we have

\[
V_\sigma = V'_\sigma \cup (W \times I), \quad V'_\sigma \cap (W_\sigma \times I) = W_\sigma \times 0 = W_\sigma.
\]

\( V_\sigma \) is contractible and so \( \xi \) is trivial over \( V \). We use a chart of the kind

\[
\kappa_\sigma: p^{-1}(V_\sigma) \xrightarrow{\cong} V_\sigma \times \mathbb{B}(2)
\]

to represent the bundle \((\xi | V_\sigma)\).

Now let \( b_\sigma > 0 \) be so small that

\[
\kappa_\sigma(U \cap p^{-1}(\sigma)) \supset \sigma \times B(b_\sigma).
\]
We map $E$ into $E$ with the fibrewise homeomorphism $\varphi_\sigma$ defined by:

$$\varphi_\sigma(z) = k_\sigma^{-1}[\rho(\frac{1}{2} b_\sigma, b_\sigma, 2, 3)]^{-1} k_\sigma(z) \quad \text{for} \quad z \in p^{-1}(V')$$

$$\varphi_\sigma(z) = z \quad \text{(identity)} \quad \text{for} \quad z \notin p^{-1}(V).$$

The remaining part of the bundle, with total space $p^{-1}(W \times I)$, is equivalent to

$$W_\sigma \times I \times \hat{B}(2) \to W_\sigma \times I.$$

$\varphi_\sigma$ is already defined for the parts corresponding to the endpoints 0 and 1 of $I$:

$$\partial V_\sigma \times 0 \times \hat{B}(2): \text{a concentric homeomorphism, the same in each fibre}$$
and

$$\partial V_\sigma \times 1 \times \hat{B}(2): \text{identity}.$$

We connect these two by the isotopy described in remark 2.3, in order to complete the definition of $\varphi_\sigma$.

The image $\varphi_\sigma(E) \subset E$ is the total space of a bundle with projection $p_\sigma = (p|\varphi_\sigma(E))$ such that $(p_\sigma)^{-1}(\sigma) \subset U \subset E$. By repeating this process, for each simplex $\sigma$ once, we obtain the required bundle $\xi'$ with total space $E' \subset U$. Observe that by local finiteness every point of $E$ is involved in at most a finite number of moves.

References


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