ANNALS OF MATHEMATICS
Vol. 75, No. 2, March, 1962
Printed in Japan

LOCALLY FLAT IMBEDDINGS OF TOPOLOGICAL MANIFOLDS

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(Received August 14, 1961)

I. Introduction

Let us say that a topological $(n-1)$-sphere Σ^{n-1} imbedded in the n-sphere S^n is *flat* if there is a homeomorphism of S^n upon itself which carries Σ^{n-1} onto the equator S^{n-1} of S^n. The classical result of Schoenflies states that every Σ^1 in S^2 is flat. Antoine [1] and Alexander [3] exhibited imbeddings of Σ^2 in S^3 which are not flat. These examples can be modified to produce non-flat imbeddings of Σ^{n-1} in S^n for $n \geqq 3$. However, Alexander [2] proved that a sufficient condition for Σ^2 to be flat in S^3 is that it be a polyhedron (i.e., the union of a finite collection of convex cells).

In view of Alexander's theorem, let us define a compact subset X of S^n to be *tame* if there is a homeomorphism of S^n upon itself carrying X onto a polyhedron, *semi-locally tame* if there is a homeomorphism of a neighborhood of X into S^n carrying X onto a polyhedron, and *locally tame* if for each point $x \in X$ there is a neighborhood N_x of x in S^n and a homeomorphism of $\bar{N}_x$ into S^n such that the image of $\bar{N}_x \cap X$ is a polyhedron. Moise [8] proved that a Σ^2 in S^3 is tame (and hence flat) if it is semi-locally tame. Bing [4] (and independently Moise [9]) proved that Σ^2 is semi-locally tame in S^3 if it is locally tame.

For $n > 4$ it is still unknown whether a tame Σ^{n-1} in S^n must be flat. (See however Newman [11] or Theorem 7 of this paper.) Recently, attempts to circumvent this barrier have been successful. Let us define a Σ^{n-1} in S^n to be *bi-collared*[0] if there is a homeomorphism of some neighborhood of Σ^{n-1} into S^n carrying Σ^{n-1} onto the equator S^{n-1} of S^n and *locally flat* if for each point x of Σ^{n-1} there is a neighborhood N_x of x in S^n and a homeomorphism of N_x into S^n such that the image of $N_x \cap \Sigma^{n-1}$ lies in S^{n-1}. In 1959 Mazur [6] proved that a bi-collared Σ^{n-1} is flat if the defining homeomorphism is piecewise linear on some non-empty open set. An important consequence of Mazur's theorem is that a differentiably imbedded Σ^{n-1} in S^n is flat. Of equally great importance was the indication (in view of Mazur's elegant proof) that some important theorems in higher dimensions might be accessible by elementary techniques.

In 1960 Brown [5] proved that every bi-collared Σ^{n-1} in S^n is flat. Shortly

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[0] Mazur calls this "collared". It is also referred to as the "shell hypothesis". We prefer to reserve the term collar for the one sided case.

afterward Morse [10] succeeded in removing the hypothesis of piecewise linearity from Mazur's argument. One of the consequences of the results in this paper is that a locally flat Σ^{n-1} in S^n is bi-collared. The following diagram indicates the present status of affairs for Σ^{n-1} in S^n.

FLAT	→	TAME
↓ ↑		↓
BI-COLLARED	→	SEMI-LOCALLY TAME
↓ ↑		↓
LOCALLY FLAT	→	LOCALLY TAME

Statement of results. The main theorem of this paper is that *a manifold with boundary has collared boundary* (see §§ II; IV for definitions). From this we derive the result that a *two-sided* $(n-1)$-*manifold imbedded* in a locally *flat fashion in an n-manifold is bi-collared.* We also give a new proof (originally due to Newman [11]) that a combinatorial $(n-1)$-sphere imbedded as a subcomplex of a combinatorial subdivision of the n-sphere is flat.

The author is indebted to E. A. Michael for numerous helpful discussions. In fact, many aspects of the formulation and proof of Theorem 1 in its present generality are the result of joint work.

II. Collared subsets

Let X be a topological space and B a subset of X. Then B is *collared* in X if there is a homeomorphism h carrying $B \times I'$[1] onto a neighborhood[2] of B such that $h(b, 0) = b$ for all $b \in B$[3]. If B can be covered by a collection of open subsets (relative to B) each of which is collared in X, then B is *locally collared* in X. (The most important example of a locally collared subset is the case where B is the boundary of a manifold with boundary.) If there is a homeomorphism h carrying $B \times (-1, 1)$ onto a neighborhood of B such that $h(b, 0) = b$ for all $b \in B$, then B is *bi-collared* in X.[3] Similarly, B is *locally bi-collared* in X if B can be covered by a collection of bi-collared open subsets. (The unit $(n-1)$-sphere of E^n is bi-collared in E^n. On the other hand the central circle of a Möbius band is locally bi-collared but not bi-collared.)

LEMMA 0. *Let B be a subset of a topological space X. A necessary and sufficient condition for B to be collared in X is that every homeomorphism $g: B \to\to B$[4] can be "extended" to homeomorphism $\bar{g}$ of $B \times I'$ onto*

[1] I' denotes the sect [01).
[2] All neighborhoods will be open.
[3] The emply set will be considered to be both collared and bicollared.
[4] "$\to\to$" means "onto".

a neighborhood of B in X such that $\bar{g}(b, 0) = g(b)$ for each $b \in B$.[5]

PROOF. The proof of sufficiency is trivial, for we may take g to be the identity map. Suppose, on the other hand, that B is collared in X and $g: B \to\to B$ is a homeomorphism. By hypothesis there is a homeomorphism h of $B \times I'$ onto a neighborhood of B such that $h(b, 0) = b$, $b \in B$. Let $g^*: B \times I' \to\to B \times I'$ be the homeomorphism defined by $g^*(b, t) = (g(b), t)$, and let $\bar{g}: B \times I' \to X$ be defined by $\bar{g} = hg^*$. Then $\bar{g}$ is a homeomorphism of $B \times I'$ onto a neighborhood of B (i.e., $h(B \times I')$) and

$$\bar{g}(b, 0) = hg^*(b, 0) = h(g(b), 0) = g(b)$$

In the next lemma we show that if a homeomorphism of $B \times 0$ into X can be extended to neighborbood of $B \times 0$ in $B \times I'$, then it can be extended to $B \times I'$.

LEMMA 1. *Let B, X be metric spaces, and N a neighborhood of $B \times 0$ in $B \times I'$. Suppose $h: N \to X$ is a homeomorphism of N onto a neighborhood of $h(B \times 0)$ in X. Then there is a homeomorphism $h': B \times I' \to X$ such that $h' | B \times 0 = h | B \times 0$ and $h'(B \times I')$ is a neighborhood of $h(B \times 0)$.*

PROOF. Let d be a metric for B such that under d the diameter of B is less than 1. Let D be the metric for $B \times I'$ defined by $D((x, t), (x', t')) = \max(d(x, x'), |t - t'|)$. For $b \in B$ let $g(b) = D(b, (B \times I') - N)$. Then g is a continuous positive real valued function on B, and for all $b \in B$ we have $g(b) < 1$. Let $\Gamma: B \times I' \to N$ be the homeomorphism defined by $\Gamma(b, t) = (b, tg(b))$, and let $h' = h\Gamma$.

III. Spindle neighborhoods

Suppose $Z = B \times I'$ where B is a metric space. Let U be an open subset of B and $\lambda: \bar{U} \to [0, 1]$ a map[6] such that $\lambda(x) = 0$ if and only if $x \in \bar{U} - U$. We define the *spindle neighborhood* $S(U, \lambda)$ by:

$$S(U, \lambda) = \{(x, t) \in B \times I' \,|\, (x, 0) \in U, t < \lambda(x)\} .$$

It is easily seen that $S(U, \lambda)$ is a neighborbood of $U \times 0$ in $B \times I'$, and that the spindle neighborhoods form a neighborhood basis for $U \times 0$ in $B \times I'$. For, suppose O is an open subset of $B \times I'$ containing $U \times 0$. Let D be the metric for $B \times I'$ defined in the proof of Lemma 1. For $x \in U$ let $\lambda(x) = \min[D(x, \bar{U} - U), D(x, (B \times I') - O)]$. Then $S(U, \lambda) \subset O$.

The map $\pi_{S(U, \lambda)}$. Suppose $S(U, \lambda)$ is defined as above. We define a map $\pi_{S(U, \lambda)}: B \times I' \to\to B \times I'$ by

[5] A Similar argument proves the corresponding theorem for the bi-collared case.

[6] A "map" is a continuous function.

$$\pi(x, t) = \begin{cases} (x, t), & (x, t) \notin S(U, \lambda) \\ (x, 0), & (x, t) \in S\left(U, \frac{\lambda}{2}\right)^{7} \\ (x, 2t - \lambda(x)), & (x, t) \in S(U, \lambda) - S\left(U, \frac{\lambda}{2}\right). \end{cases}$$

See Figure 1. In words, π is the identity on the complement of $S(U, \lambda)$, collapses $S(U, (\lambda/2))$ "vertically" onto $U\times 0$, and carries the interval $[(x, (\lambda/2)(x)), (x, \lambda(x))]$ linearly onto $[(x, 0), (x, \lambda(x))]$ for each x in U. Note that π maps $B \times I' - S(U, (\lambda/2))$ *homeomorphically* onto $B \times I'$.

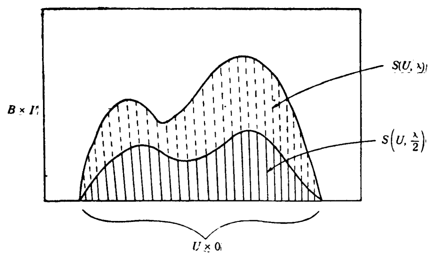

Figure 1

LEMMA 2. *Let U be an open subset of the metric space B, N a neighborhood of $U\times 0$ in $B\times I'$, and f a homeomorphism of $\bar{N}$ onto the closure of a neighborhood of $U\times 0$ such that $f\,|\,\bar{U}\times 0 = 1$. Then there is a homeomorphism f': $\bar{N} \to B\times I'$ and a neighborhood V of $U\times 0$ in N such that*

(2.1) $f'\,|\,(\bar{N} - N) = f\,|\,(\bar{N} - N)$,

(2.2) $f'(\bar{N}) = f(\bar{N})$,

(2.3) $f'\,|\,V = 1$.

PROOF. (See Figure 2). Let $S(U, \lambda)$ be a spindle neighborhood of $U\times 0$ such that $S(U, \lambda) \subset N \cap f(N)$. Let π be the associated mapping $\pi_{S(U, \lambda)}$ and let f': $\bar{N} \to B\times I'$ be defined by

$$f'(x) = \begin{cases} x, & x \in \overline{S\left(U, \frac{\lambda}{2}\right)}, \\ \pi^{-1} f \pi(x), & x \in \bar{N} - S\left(U, \frac{\lambda}{2}\right). \end{cases}$$

(2.4) *$\pi^{-1}f\pi$ is well defined on $\bar{N} - S(U, (\lambda/2))$ and carries it homeomorphically onto $f(\bar{N}) - S(U, (\lambda/2))$.* First notice that π carries $\bar{N} - S(U, (\lambda/2))$ homeomorphically onto $\bar{N}$. In turn, f carries $\bar{N}$ homeomorphically onto $f(\bar{N})$. Now π^{-1} carries $B \times I'$ homeomorphically onto $B\times I' - S(U, (\lambda/2))$ and is the identity on the complement of $S(U, \lambda)$.

[7] $(\lambda/2)$ is defined by $(\lambda/2)(x) = (1/2)\lambda(x)$.

Since $S(U, \lambda) \subset f(\bar{N})$, π^{-1} carries $f(\bar{N})$ homeomorphically onto

$$f(\bar{N}) - S(U, (\lambda/2)) .$$

(2.5) *f' is a well defined map.* Suppose $y \in \overline{S(U,(\lambda/2))} \cap (\bar{N} - S(U,(\lambda/2)) = (\bar{U} - U) \cup \{(x, t) \mid x \in U, t = (\lambda/2)(x)\}$. If $y \in \bar{U} - U$, $\pi^{-1}f\pi(y) = y$ since π and f are the identity on $\bar{U} - U$. Suppose $y = (x, (\lambda/2)(x))$, $x \in U$. Then

$$\pi^{-1}f\pi(y) = \pi^{-1}f\pi\left(x, \frac{\lambda}{2}(x)\right) = \pi^{-1}f(x, 0) = \pi^{-1}(x, 0) = \left(x, \frac{\lambda}{2}(x)\right) = y .$$

(2.6) *f' is a homeomorphism.* It is evident from (2.4) and the definition of f' that f' is $1 - 1$. On the other hand $f'(\bar{N} - S(U, (\lambda/2))$ and $f'(\overline{S(U, (\lambda/2))}$ are closed subsets of $f'(\bar{N})$. Finally, f' is a homeomorphism on each of its domains of definition.

Evidently (2.2) is satisfied. Choosing $V = S(U, (\lambda/2))$ we see that (2.3) is satisfied. Finally, let $y \in \bar{N} - N$. Since $S(U, \lambda) \subset N \cap f(N)$, neither y nor $f(y)$ is in $S(U, \lambda)$. Furthermore π is the identity on the complement of $S(U, \lambda)$. Hence

$$f'(y) = \pi^{-1}f\pi(y) = \pi^{-1}f(y) = f(y) .$$

This completes the proof of Lemma 2.

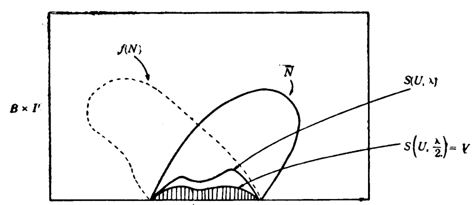

Figure 2

LEMMA 3. *Let X, B be metric spaces and h: $B \to X$ a homeomorphism. Suppose U_1, U_2 are open subsets of B, K is a closed (subset relative to B) of $U_1 \cap U_2$, and $U_1 \cup U_2 = B$. Suppose also that for $i = 1, 2$, $h \mid U_i$ can be extended to a homeomorphism h_i of $U_i \times I'$ onto a neighborhood of $h(U_i)$ in X such that $h_i \mid U_i \times 0 = h \mid U_i$. Then there is a homeomorphism h_2': $U_2 \times I' \to\to h_2(U_2 \times I')$ such that $h_2' \mid U_2 \times 0 = h \mid U_2$ and $h_2' \mid V = h_1 \mid V$ for some neighborhood V of $K \times 0$ in $(U_1 \cap U_2) \times I'$.* (*See Figure* 3).

PROOF. Let U be an open subset of $U_1 \cap U_2$ such that $K \subset U \subset \bar{U} \subset U_1 \cap U_2$. Then there is a spindle neighborhood N of $U \times 0$ in $B \times I'$ such that $\bar{N} \subset h_2^{-1}(h_1(U_1 \times I') \cap h_2(U_2 \times I'))$. Hence the map f: $\bar{N} \to B \times I'$ defined by $f(y) = h_1^{-1}h_2(y)$ is a well defined homeomorphism, $f \mid \bar{U} \times 0 = 1$ and $f(N)$ is open in $B \times I'$. Applying Lemma 2 we obtain a homeomorphism f': $\bar{N} \to B \times I'$ and a neighborhood V of $U \times 0$[8] such that:

[8] V can be chosen as a subset of $(U_1 \cap U_2) \times I'$.

(3.1) $f' | (\bar{N} - N) = f | (\bar{N} - N)$,
(3.2) $f'(\bar{N}) = f(\bar{N})$,
(3.3) $f' | V = 1$.
Define h_2': $U_2 \times I' \rightarrow X$ by

$$(3.4) \qquad h_2'(x) = \begin{cases} h_1 f'(x) , & x \in \bar{N} \cap (U_2 \times I') , \\ h_2(x) , & x \in (U_2 \times I') - N . \end{cases}$$

Observe that h_2' is a homeomorphism on each of the domains of definition and that the domains are closed in $U_2 \times I'$.

(3.5) *h_2' is well defined.* Suppose $x \in [\bar{N} \cap (U_2 \times I')] \cap [(U_2 \times I') - N] = (\bar{N} - N) \cap (U_2 \times I')$. Then since $x \in \bar{N} - N$, $h_1 f'(x) = h_1 f(x) = h_1 h_1^{-1} h_2(x) = h_2(x)$.

$$h_2'(\bar{N} \cap (U_2 \times I')) = h_2(\bar{N} \cap (U_2 \times I')) .$$

$$(3.6) \qquad \begin{aligned} h_2'(\bar{N} \cap (U_2 \times I')) &= h_1 f'(\bar{N} \cap (U_2 \times I')) , \\ &= h_1 f(\bar{N} \cap (U_2 \times I')) , \\ &= h_1 h_1^{-1} h_2(\bar{N} \cap (U_2 \times I')) , \\ &= h_2(\bar{N} \cap (U_2 \times I')) . \end{aligned}$$

(3.7) *h_2' is a homeomorphism.* If follows from (3.6) and (3.4) that

$$h_2'(\bar{N} \cap (U_2 \times I')) \cap h_2'((U_2 \times I') - N) = h_2(\bar{N} \cap (U_2 \times I') \cap h_2((U_2 \times I') - N) = 0 .$$

Hence h_2' is $1 - 1$. On the other hand the image of each domain is closed in $h_2'(U_2 \times I')$(again by (3.6) and (3.4) and the fact that h_2' is a homeomorphism on each domain.

Suppose $x \in V$. Then since $f' | V = 1$, $h_2'(x) = h_1 f'(x) = h_1(x)$. Finally, suppose $x \in U_2$. If $(x, 0) \notin N$ then $h_2'(x, 0) = h_2(x, 0) = h(x)$. If $(x, 0) \in N$ then, since N is a spindle neighborhood of $U \times 0$, $(x, 0) \in V$. Hence $h_2'(x, 0) = h_1 f'(x, 0) = h_1(x, 0) = h(x)$.

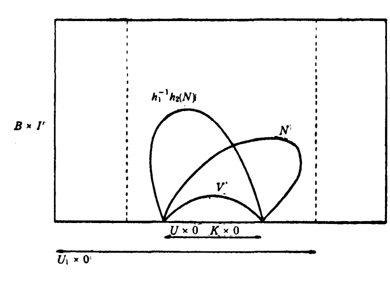

Figure 3

LEMMA 4. *Let B be a subset of a metric space X. Suppose $B = U_1 \cup U_2$ where U_1, U_2 are open in B and $U_1 \cap U_2 \neq 0$. If both of U_1, U_2 are col-*

lared in X then B is collared in X.

PROOF. Since B is a normal space there are open subsets O_1, O_2 of B such that $\bar{O}_1 \subset U_1, \bar{O}_2 \subset U_2$ and $B = O_1 \cup O_2$. Let $K = \bar{O}_1 \cap \bar{O}_2$. Then K is a closed subset rel B of $U_1 \cap U_2$. By the hypothesis there exist homeomorphisms $h_i (i = 1, 2)$ of $U_i \times I'$ onto a neighborhood of U_i in X such that $h_i(b, 0) = b, b \in U_i$. Applying Lemma 3 (with h the identity map) we get a homeomorphism $h_2'\colon U_2 \times I' \to \to h_2(U_2 \times I')$ and a neighborhood V of $K \times 0$ in $(U_1 \cap U_2) \times I'$ such that $h_2' | V = h_1 | V$ and $h_2' | U_2 \times 0 = h_2 | U_2 \times 0$.

Obviously $(O_1 - O_2) \cap \overline{O_2 - O_1} = \overline{O_1 - O_2} \cap (O_2 - O_1) = 0$, i.e., $O_1 - O_2$ and $O_2 - O_1$ are *completely separated* in X. Since X is a metric space there exist disjoint open subsets W_1, W_2 of X such that

$$O_1 - O_2 \subset W_1 \subset h_1(U_1 \times I')$$
$$O_2 - O_1 \subset W_2 \subset h_2'(U_2 \times I') .$$

Let V_1, V_2 be spindle neighborhoods of $(O_1 - \bar{O}_2) \times 0, (O_2 - \bar{O}_1) \times 0$ respectively such that $h_1(V_1) \subset W_1, h_2'(V_2) \subset W_2$. Then V_i is open in $B \times I'$, $h(V_1) \cap h_2'(V_2) = 0$, and $B \times 0 \subset V_1 \cup V_2 \cup V$. Let $f\colon V_1 \cup V_2 \cup V \to X$ be defined by

$$f(x) = \begin{cases} h_1(x) , & x \in V_1 , \\ h_2'(x) , & x \in V_2 , \\ h_1(x) = h_2'(x) , & x \in V . \end{cases}$$

Clearly f is a well defined homeomorphism and $f(b, 0) = b, b \in B$. Note that $V_1 \cup V_2 \cup V$ is a neighborhood of $B \times 0$ in $B \times I'$. For $V_1 \supset (O_1 - \bar{O}_2) \times 0$, $V_2 \supset (O_2 - \bar{O}_1) \times 0$ and $V \supset (\bar{O}_1 \cap \bar{O}_2) \times 0$. In view of Lemma 1 the proof is complete.

We are now in a position to prove the main result of this section.

THEOREM 1. *A locally collared subset of a metric space is collared.*

PROOF. Suppose B is a locally collared subset of the metric space X. Let us say that an open subset of B has *property* C if it is collared in X.

(i) C *is hereditary*, i.e., *if U has property* C *and V is an open subset of U then V has property* C.

If V is empty it has property C by definition. Suppose $V \neq 0$. Then $U \neq 0$, and there is a homeomorphism h_u of $U \times I'$ onto a neighborhood of U in X such that $h_u(x, 0) = x, x \in U$. Let $h_v = h_u | V \times I'$.

(ii) C *is closed under disjoint union*, i.e., *if $\{U_\alpha\}_{\alpha \in A}$ is a pairwise disjoint collection of open subsets of B each having property* C, *then $\bigcup_{\alpha \in A}\{U_\alpha\}$ has property* C.

Suppose h_α is the homeomorphism of $U_\alpha \times I'$ onto a neighborhood of U_α in X such that $h_\alpha(x, 0) = x, x \in U_\alpha$. Since X is a metric space there is a pairwise disjoint collection $\{W_\alpha\}_{\alpha\in A}$ of open subsets of X such that $U_\alpha \subset W_\alpha \subset h_\alpha(U_\alpha \times I')$, $\alpha \in A$.[9] Let $O = \bigcup_{\alpha\in A} h_\alpha^{-1}(W_\alpha)$. Then O is an open subset of $B \times I'$ and $O \supset \bigcup_{\alpha\in A}\{U_\alpha \times 0\}$. Let $h\colon O \to X$ be the homeomorphism defined by $h \mid (U_\alpha \times I') \cap O = h_\alpha \mid (U_\alpha \times I') \cap O$. In view of Lemma 1, $\bigcup_{\alpha\in A}\{U_\alpha\}$ is collared.

(iii) *Suppose U_1, U_2 are open subsets of B each having property* C. *Then $U_1 \cup U_2$ has property* C.

If $U_1 \cap U_2 = 0$, (iii) is a consequence of (ii).

If $U_1 \cap U_2 \neq 0$, (iii) is a consequence of Lemma 4.

In a metric space, a property of open sets satisfying conditions (i)–(iii), and which is satisfied locally, is possessed by all open subsets [7]. In particular, B itself has property C. This completes the proof of Theorem 1.

The following is a restatement of Theorem 1 into a theorem about extensions of homeomorphisms (cf. Lemma 0).

COROLLARY. *Let X, B, $B \times I'$ be metric spaces and $h\colon B \times 0 \to X$ be a homeomorphism. Suppose B can be covered by a collection of open subsets $\{U_\alpha\}_{\alpha\in A}$ such that for each $\alpha \in A$, $h \mid U_\alpha \times 0$ has a homeomorphic extension h_α mapping $U_\alpha \times I'$ onto a neighborhood of $h(U_\alpha \times 0)$. Then h has a homeomorphic extension mapping $B \times I'$ onto a neighborhood of $h(B \times 0)$.*

IV. Applications to manifolds

An *n-manifold with boundary* is a connected metrizable topological space such that each point has a closed neighborhood homeomorphic to an n-cell. As usual the *boundary* consists of the subset of points which do not have (open) neighborhoods homeomorphic to E^n. If the boundary is empty, the manifold with boundary will be called a *manifold*. Suppose X is an n-manifold, and B is a subset of X which is an r-manifold under the relative topology. Then B is an *r-submanifold* of X. Suppose, in particular, that $r = n - 1$. Then B is *two-sided* in X if there is a connected neighborhood N of B which is separated by B.[10] Finally B is *locally flat* in X if for each point $b \in B$ there is a neighborhood N_b of b in X and a homeomorphism $h_b\colon N_b \to E^n$ such that $h_b(N_b \cap B) \subset E^{n-1} \subset E^n$.

REMARK. In the definition of locally flat there is no loss of generality in requiring that $h_b(N_b) = E^n$ and $h_b(N_b \cap B) = E^{n-1}$. The definition is equivalent to that given in § I. The following two lemmas are easily established, and we state them without proof.

LEMMA 5. *The boundary of an n-manifold with boundary is locally*

[9] Let $W_\alpha = h_\alpha(U_\alpha \times I') \cap \{x \in X \mid D(x, U_\alpha) < D(x, \bigcup_{\beta\neq\alpha} U_\beta)\}$.

[10] In this case $N - B$ has two components.

collared.

LEMMA 6. *A submanifold B^{n-1} of a manifold X^n is locally flat in X^n if and only if it is locally bi-collared in X^n.*

THEOREM 2. *The boundary of an n-manifold with boundary is collared.* This follows directly from Theorem 1 and Lemma 5.

THEOREM 3. *Let B^{n-1} be a locally flat two-sided $(n-1)$-submanifold of a manifold X^n. Then B^{n-1} is bi-collared in X^n.*

PROOF. Let N be a connected neighborhood of B in X which is separated by B, and let Q, R be the components of $N - B$.[10] Since B is locally flat in N, $Q \cup B$ and $R \cup B$ are manifolds with boundary B. It follows from Theorem 2 that B is collared in each. Hence B is bi-collared in X.

REMARK. The case of a one sided manifold will be treated in a forthcoming paper by E.A.Michael.

THEOREM 4. *Let Σ^{n-1} be locally flat in S^n. Then Σ^{n-1} is flat in S^n.*

PROOF. This follows from Theorem 3 above and Theorem 5 of [5].

V. Applications to polyhedral manifolds

DEFINITIONS.[11] A *0-star sphere* Σ^0 is a pair of points. A *0-star cell* $\mathcal{I}^0$ is a single point. For $n > 0$ an *n-star sphere* Σ^n(*n-star cell* $\mathcal{I}^n$) is a finite complex homeomorphic to the n-sphere S^n(n-cell I^n) and such that the link[12] of each vertex is a Σ^{n-1}(Σ^{n-1} or $\mathcal{I}^{n-1}$). An *n-star manifold* M^n (manifold with boundary N^n) is a locally finite complex such that the link of each vertex is a Σ^{n-1}(Σ^{n-1} or $\mathcal{I}^{n-1}$). A 0-star manifold (manifold with boundary) is an even (odd) numbered set of points.

A *combinatorial n-cell* I^n(n-sphere S^n) is a finite complex which has a linear subdivision isomorphic to some linear subdivision of an n-simplex (the boundary of an $(n+1)$-simplex). A *combinatorial n-manifold* (n-manifold with boundary) is a locally finite complex such that the link of each vertex is an S^{n-1}(S^{n-1} or I^{n-1}).

REMARK. The reader is referred to [11] for a more complete discussion of star manifolds. Combinatorial manifolds are special cases of star manifolds. If every combinatorial manifold homeomorphic to an n-sphere is a *combinatorial n-sphere* (and this has been proved for $n \neq 4, 5, 7$ by Smale [12]), then all n-star spheres are combinatorial n-spheres). Unfortunately, the only proof we know of this implication requires induction on n; hence even with Smale's result, n-star spheres are known to be combinatorial

[11] These definitions are due to Newman [11].

[12] The link of a vertex v in a complex K consists of the union of the closed simplexes σ of K not containing v but such that the join of v and σ is a simplex of K. We denote it by $\operatorname{lk}(v, K)$. $\operatorname{St}(v, K) \equiv$ star of v in K is the join of v with $\operatorname{lk}(v, K)$.

spheres only for $n \geqq 3$ (and combinatorial manifolds for $n \geqq 4$).

THEOREM 5. *Let M^{n-1} be an $(n-1)$-star manifold imbedded as a subcomplex of an n-star manifold M^n. Then M^{n-1} is locally flat in M^n.*

PROOF. The theorem is evidently true for $n = 1$. Inductively, suppose we have proven the theorem for $n = k$. Let M^k be a k-star manifold imbedded as a subcomplex of the $(k+1)$-star manifold M^{k+1}. Let v be a vertex of M^k. Then $\text{lk}(v, M^k)$ is a Σ^{k-1} imbedded as a subcomplex of $\text{lk}(v, M^{k+1})$ which is a Σ^k. By the induction hypothesis $\text{lk}(v, M^k)$ is locally flat in $\text{lk}(v, M^{k+1})$. Applying Theorem 4 we obtain a homeomorphism $h: \text{lk}(v, M^{k+1}) \to \to S^k$ such that $h(\text{lk}(v, M^k))$ is the equator S^{k-1} of S^k. We may think of S^k as the unit sphere of E^{k+1} with S^{k-1} in the hyperplane E^k. Since $\text{St}(v, M^{k+1})$[12] is the join of v and $\text{lk}(v, M^{k+1})$ and, since the unit ball B^{k+1} is the join of the origin and S^k, h can be extended in the obvious way to a homeomorphism $\bar{h}: \text{St}(v, M^{k+1}) \to \to B^{k+1}$. Furthermore, $\text{St}(v, M^k)$ is the join of v with $\text{lk}(v, M^k)$. Hence $\bar{h}(\text{St}(v, M^k)) \subset E^k$. Since each point of M^k lies in the interior of the star of some vertex of M^k we have established that M^k is locally flat in M^{k+1}. The following theorem is an immediate consequence of Theorem 5 and Theorem 3.

THROREM 6. *Let M^{n-1} be an $(n-1)$-star manifold imbedded as a 2-sided subcomplex of an n-star manifold M^n. Then M^{n-1} is bi-collared in M^n.*

THEOREM 7. (Newman). Let Σ^{n-1} be an $(n-1)$-star sphere imbedded as a subcomplex of an n-star triangulation of the n-sphere S^n. Then Σ^{n-1} is flat in S^n.

QUESTION. Suppose K is bi-collared $(n-1)$-polyhedron in E^n. Is K a manifold? The answer is affirmative if and only if the link of every vertex in a triangulated n-manifold is an $(n-1)$-manifold. A negative answer would give a counter example to a very weak form of the Hauptvermutung for spheres.

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