LINEAR ALGEBRA I

1 December 2022

Some remarks on Proofs

This document sets out a few notes about *Proofs*, why they are important, how to understand them, and how to learn them.

Like most mathematics modules, the proofs of the results we have covered form an integral and important part of the subject. Proofs are there not because we have had a loss of confidence that the results are actually true, but because they explain *why* the results are true. Hence they are really the fundamental aspect of the subject, explaining the ideas and perspectives that make the concepts work. Without an appreciation of that level of the subject, it will remain no more than a collection of abstract definitions and templates to solve a fixed set of problems: it will not help you to apply the techniques to other, different problems, which is the basic skill a mathematics module ought to achieve.

Mathematics could, perhaps crudely, be said to consist of *Definitions*, *Theorems* and *Proofs*; that is at least the order they appear in the unfolding story of a module. But a good way to think of these three interrelated objects is that the Theorems give clean statements that make order out of otherwise seeming chaos, the Proofs are the ideas and perspectives that allow you to see the clear view of what it is the Theorem states, and the Definitions can be thought of as the names and rules you set up that make the Proofs work[1].

That a proof is really an explanation of something may seem hard to believe at first, as the first time one reads any but the simplest proof it probably seems confusingly complicated. However, that is because any proof needs a certain amount of effort to understand what it is doing, and how it is working. There is a curious historical issue that arose some decades ago. You may have heard about the *Four Colour Theorem* (if not, look it up: it is simple to state, clearly true in any example you might imagine, but hard to see why it is true in general). It was conjectured to be true in the mid 1800's but remained unproven until a computer proof was announced in 1976. This led to debate, even assuming there was not an error in the computer programming, in what way was the result *proved*? Just having the computer tell you it was true (no one had doubted it anyway) gives no insight as to *why* it is true, something that most mathematicians see as the essence and one of the basic worths of a proof.

So, proofs are important as they explain why the subject actually works, and to a depth that allows the mathematician to adapt the ideas to learn other things, or solve new problems. But, when presented with a proof, how should you set about understanding it, and eventually learning it?

[1]Definitions are themselves interesting conceptual structures. It is easy to see that when a definition is made it is somehow 'creating' the object defined, or at least giving it a name by which it can be discussed. That is true, but it can also be informative to ask yourself when a definition is made what it is that it *excludes*: defining something is often about saying we *don't* want to include this or that, as much as saying what we do want. To take an example, we made the Defintion of a *Linear Map* in Chapter 7; with it we could prove results such as (part of 7.3.5) 'If $T\colon V \to W$ is a linear map then it is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$. This result would simply not be true if we allowed other, more general functions between vector spaces. The definition is what made the proof work.

How do you analyse a Proof?

There are several basic and necessary steps to take in trying to understand a proof; they may seem obvious, but are not always followed, which then leads to more confusion that is necessary. I would recommend the following checklist.

1. Do you know what the result you are trying to prove actually is? I mean *really* know what it means? By that I mean can you make an accurate statement of the result, and importantly, can you accurately state what each of the main terms used mean? If not, go back to where those terms are defined and read again the definition. In the example in the footnote on page 1, you won't stand a chance of understanding the proof if you do not know what it means for a function to be a linear map.

2. What is the structure of the proof? The structures of simple proofs are usually one of

    (a) **Direct Proof.** This is a proof of a statement of the form 'If A is true then B is true', and proceeds straightforwardly, beginning 'Suppose A is true', and then by a process of argument concludes that B is true. *An easy example would be a proof of the statement 'If $2x = 2$ then $x = 1$': you would start by assuming $2x = 2$ and then you would divide both sides by 2 to reach your conclusion.* A variation on this method of proof is to note that 'If A is true then B is true' is equivalent to 'If B is false then A is false', and then proceeding with assuming B false and deducing A false.

    (b) **Proof by Contradiction.** Here one proves 'If A is true then B is true' by starting with assuming that 'A is true and B is false', and then creating an argument to show that this cannot be possible. *A simple example might be a proof of the statement 'If the integer x is odd then it is not divisible by 2 with integer answer': one might start with assuming x is odd and it is divisible by 2, then one has to know what these terms mean (see 1. above), which I take as an integer x is* odd *if it is of the form $x = 2y + 1$ for some integer y, and an integer x is* divisible by 2 *if it can be written as $x = 2z$ for some integer z. Then your assumptions mean there are integers y and z with $x = 2y + 1 = 2z$, from which you do a bit of algebra and conclude that $\frac{1}{2} = z - y \in \mathbb{Z}$, clearly not true.*

    (c) **Proof by Induction.** This would typically apply to a statement that involved something being true for each value of $n = 1, 2, 3 \ldots$, or something similar. We have seen a few of these, and I'll not comment more on them here.

    More complicated proofs may break down into sub-proofs, each of which might fit into one or other of these structures.

3. Finally, once you have got to grips with the precise meanings of the terms involved, and the underlying structure, you should look to see if there is a particular *idea* or trick that is used, some way of starting, or a particular lemma that is appealed to, or something of that nature. For simple proofs there often isn't, it is simply a case of understanding clearly enough what it is that the Theorem is asserting, and then it just follows from the definitions of the terms.

4. (Optional, but can be very powerful.) Try to wreck the theorem: try to construct a counter-example to prove the theorem is not true. Hopefully you will fail, but in the process you will learn a lot about how the proof works.

How do you learn a Proof?

So, you have analysed your proof. That in itself may well have embedded it pretty well in your mind, but maybe not fully yet. What do you do now? My recommendation is that you should

1. Close your notes. Now try to write down precisely what the statement is you are trying to prove. If you can't, go back to your notes and look it up again. Note where you got confused: can you understand the point the actual statement is making that you overlooked before? Now close your notes again.
2. Do you know what all the terms in the statement mean, precisely? (If not, you are unlikely to be successful in understanding the proof, let alone being able to write it down.) If you don't, look them up.
3. Do you know what the structure of the proof is, in particular how it begins? If you don't, go back and look it up again.
4. Are there any tricks or special perspectives that you need to remember?
5. If you get this far, you should be in a good position to try to write down the proof in full detail. Try to do it. If you get stuck, note the particular place you get stuck and look back at your notes: what is it that you had forgotten at that point, was it a key aspect of a definition, or a particular perspective that you need to hold to get the next line, or what?
6. Ideally, as you write, imagine you were writing so as to explain it to a friend who didn't understand the proof. That will keep you clear and honest. Even better, try to actually explain it to a friend: you learn more about mathematics by teaching it than by reading it.

Finally, if you find all this takes effort, don't be discouraged. Learning higher level maths does take effort and everyone, even the most senior professors take time and have to put effort into understanding proofs they have not seen before. However, the more proofs you have seen, the easier it is to 'read' the next ones you come across, so the effort is not wasted.