MOCK MODULAR FORMS AND GEOMETRIC THETA FUNCTIONS FOR INDEFINITE QUADRATIC FORMS

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Abstract. Mock modular forms are central objects in the recent discoveries of new instances of Moonshine. In this paper, we discuss the construction of Mock modular forms via integrals of theta series associated to indefinite quadratic forms. As an example, we obtain Zwegers' Mock theta function for hyperbolic space in this geometric setting.

1. Introduction

Theta series are a very important tool for the construction of automorphic forms with many and significant applications ranging from number theory to physics. While positive theta series are well understood, the scope and possibilities of indefinite theta series are still developing. As an example we mention Borcherds' celebrated construction of automorphic products for Hermitian domains arising from a regularized theta lift for an indefinite quadratic space of signature $(p, 2)$ [4]. These ideas also play a central role for the proof of the umbral moonshine conjecture given in [6].

Indefinite theta functions play an important role in the construction of Mock modular forms and their completions. In 2002, Zwegers [35] constructed a Mock theta function of weight $(p+1)/2$ associated to a quadratic space of signature $(p, 1)$ whose non-holomorphic completion involves the error function. In recent months, [2] (signature $(p, 2)$) and then [27] (general signature (p, q), already indicated in [2]) extended Zwegers' construction to arbitrary signature. Now the completion involves generalized error functions. For a related result, see [34]. Of course, Mock modular forms are central objects in the recent discoveries of instances of Moonshine, for example via the elliptic genus of K3 surfaces.

Throughout the 1980's [22, 23, 24], the second author in joint work with J. Millson employed the Weil representation and the theta correspondence to systematically construct *holomorphic* (Siegel) modular forms associated to indefinite quadratic forms. More precisely, they obtain a lift from the (co)homology of the underlying locally symmetric space to the space of modular forms. The first author jointly with Millson [9, 10, 12, 13] has since studied the non-compact situation and also considered local coefficient systems to construct modular forms of higher weight. The second author [21] employed this machinery to recover the results of [2] from a more geometric point of view. For the relationship to cycles on moduli spaces of K3 surfaces, see [19].

In this note, we first outline the representation theoretic background in the construction of theta series stressing the role of the Weil representation. We then give an introduction to the theory developed in [22, 23, 24]. The key object is the indefinite theta function $\theta(\tau, z, \varphi_{KM})$,which as a function of $\tau \in \mathbb{H}$, the upper half plane, is a modular form of weight $(p+q)/2$, while as function of $z \in D$, the symmetric space associated to V, defines a closed differential q-form. For a compact q-chain C in D and η a (not necessarily closed) differential $(p-1)q$-form on D with compact support, we consider the theta integrals

$$I(\tau, C) := \int_C \theta(\tau, z, \varphi_{KM}) \qquad \text{and} \qquad I(\tau, \eta) := \int_D \eta \wedge \theta(\tau, z, \varphi_{KM}),$$

which, by construction, are then (non-holomorphic) modular forms of weight $(p+q)/2$ for a certain level. For those forms we then give a natural splitting into a holomorphic 'Mock theta' part with a geometric interpretation and its non-holomorphic modular completion. We then discuss the hyperbolic case explicitly and recover Zwegers' theta series in this setting.

In a separate paper [8] (following [20]), we will explain in detail how the results of [27] can also be obtained in this setting.

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2. The Weil Representation and Theta Series

In this section, we recall the (Segal-Shale)-Weil or oscillator representation over $\mathbb{R}$ and explain how it provides a representation-theoretic framework for the construction and properties of theta series. A good reference is Shintani's treatment [30].

2.1. **Weil representation.** Let V be a rational vector space over $\mathbb{Q}$ with a non-degenerate bilinear form $(\,,\,)$ of signature (p, q) and dimension $m = p+q$. We pick an ordered orthogonal basis $\{v_i\}$ of $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ such that $(v_\alpha, v_\alpha) = 1$ for $\alpha = 1, \ldots, p$ and $(v_\mu, v_\mu) = -1$ for $\mu = p+1, \ldots, p+q$. We denote the corresponding coordinates for a vector x by x_i.

We let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$ and let $\omega = \omega_{V,\psi}$ be the Weil representation of $\mathrm{Mp}_2(\mathbb{R}) \times \mathrm{O}(V(\mathbb{R}))$ associated to the additive character $\psi(t) = e^{2\pi i t}$, acting on $\mathcal{S}(V(\mathbb{R}))$, the space of Schwartz functions on $V(\mathbb{R})$. For m even, the representation factors through $\mathrm{SL}_2(\mathbb{R})$. The orthogonal group acts linearly, that is, $\omega(g)\varphi(x) = \varphi(g^{-1}x)$. For matrices in $\mathrm{SL}_2(\mathbb{R})$, there are preimages in $\mathrm{Mp}_2(\mathbb{R})$ so that

$$\begin{aligned} \omega\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right)\varphi(x) &= e^{\pi i (x,x) b}\varphi(x), \\ \omega\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right)\varphi(x) &= a^{m/2}\varphi(ax) \quad (a>0), \\ \omega\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\varphi(x) &= \gamma_V\, \hat{\varphi}(x). \end{aligned}$$

Here $\gamma_V = e^{2\pi i \frac{p-q}{8}}$ and $\hat{\varphi}(x) := \int_{V(\mathbb{R})} \varphi(y) e^{-2\pi i (x,y)} dy$ is the Fourier transform.

On the Lie algebra level, the actions of the standard basis elements $H = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $R = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$, and $L = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$ of $\mathfrak{sl}_2(\mathbb{C})$ are given by

$$\begin{aligned}
\omega(H) &= -\pi r^2 + \frac{1}{4\pi}\Delta, \\
\omega(L) &= \frac{\pi}{2} r^2 + \frac{1}{8\pi}\Delta + \frac{1}{2}E + \frac{m}{4}, \\
\omega(R) &= -\frac{\pi}{2} r^2 - \frac{1}{8\pi}\Delta + \frac{1}{2}E + \frac{m}{4}.
\end{aligned}$$

Here $r^2 = \sum_{\alpha=1}^{p} x_\alpha^2 - \sum_{\mu=p+1}^{m} x_\mu^2$ is the metric of V, $\Delta = \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial x_\alpha^2} - \sum_{\mu=p+1}^{m} \frac{\partial^2}{\partial x_\mu^2}$ is the Laplace operator, and $E = \sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i}$ is the Euler operator.

We let K' be the inverse image of $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ in $\mathrm{Mp}_2(\mathbb{R})$ under the covering map. Then K' admits a one-dimensional character $\chi_{1/2}$ whose square descends to the map $\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \mapsto \cos(\theta) + i\sin(\theta) = e^{i\theta}$ on $\mathrm{SO}(2)$. We say $\varphi \in S(V(\mathbb{R}))$ has weight $\frac{1}{2}r \in \frac{1}{2}\mathbb{Z}$ if $\omega(k')\varphi = \chi_{1/2}^{r}(k')\varphi$ for $k' \in K'$. Note that in terms of the Lie algebra action this is equivalent to

$$\omega(-H)\varphi = \frac{r}{2}\,\varphi. \tag{2.1}$$

In particular, φ is an eigenfunction under the Fourier transform. Given $\varphi \in \mathcal{S}(V(\mathbb{R}))$, it will be convenient to set

$$\varphi^0(x) = e^{\pi(x,x)}\varphi(x).$$

Then a quick calculation shows that (2.1) is equivalent to

$$(E - \frac{1}{4\pi}\Delta)\varphi^0 = \left(\frac{r-m}{2}\right)\varphi^0, \tag{2.2}$$

see also [32]. For example, the standard Gaussian φ_0 on $V(\mathbb{R})$ given by

$$\varphi_0(x) = e^{-\pi(x,x)_0}$$

with $(x,x)_0 = \sum_{i=1}^{p+q} x_i^2$ has weight $(p-q)/2$ under the action of K'.

Remark 2.1. Roughly speaking, the local theta correspondence or Howe duality correspondence for $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{O}(V(\mathbb{R}))$ is concerned with the question which irreducible representations $\pi \otimes \pi'$ of $\mathrm{Mp}_2(\mathbb{R}) \times \mathrm{O}(V(\mathbb{R}))$ occur as quotients of the Weil representation, that is, $\mathrm{Hom}_{\mathrm{equiv}}(\omega_{V,\psi}, \pi \otimes \pi')$ is non-zero (as $(\mathfrak{g}, K)$-modules or unitary representations), see [17] or also [1]. The main result is that the space of intertwining homomorphisms is always at most one-dimensional and hence one obtains a bijection between certain irreducible representations of $\mathrm{Mp}_2(\mathbb{R})$ and of $\mathrm{O}(V(\mathbb{R}))$. This bijection is called the Howe duality correspondence.

For V positive definite so that $\mathrm{O}(V(\mathbb{R}))$ is compact, an explicit description of the correspondence is given by the theory of spherical harmonics. In that case, let $\mathcal{H}_\ell(V)$ be the space of homogenous harmonic polynomials $p(x)$ on V of degree ℓ, that is, $\Delta p(x) = 0$. Then $\mathcal{H}_\ell(V)$ is an irreducible representation of $\mathrm{O}(V(\mathbb{R}))$ of highest weight $(l, 0, \ldots, 0)$. On the other hand, for $n \in \frac{1}{2}\mathbb{Z}$, let D_{n-1}^{+} be the holomorphic (limit of) discrete series representation for $\mathrm{Mp}_2(\mathbb{R})$ with holomorphic/lowest vector of weight n for K'. Then, as a unitary representation of $\mathrm{Mp}_2(\mathbb{R})\times\mathrm{O}(V(\mathbb{R}))$, $L^2(V(\mathbb{R}))$ is a Hilbert direct sum of the representations $\mathcal{H}_\ell(V)\otimes D^+_{\frac{p}{2}+\ell-1}$. The irreducible summands are generated by the vectors $p(x)\varphi_0(x)\in\mathcal{S}(V(\mathbb{R}))$ for $p(x)\in\mathcal{H}_\ell(V)$. Note that $p(x)\varphi_0(x)$ is holomorphic ($\omega(L)p(x)\varphi_0(x)=0$) and has weight $\frac{p}{2}+\ell$: $\omega(-H)p(x)\varphi_0(x)=\left(\frac{p}{2}+\ell\right)p(x)\varphi_0(x)$. This completely describes the theta correspondence for the dual pair $\mathrm{SL}_2(\mathbb{R})\times\mathrm{O}(V(\mathbb{R}))$ in the positive definite case.

In the indefinite case when $\mathrm{O}(V(\mathbb{R}))$ is non-compact, the situation is more complicated. The space $L^2(V(\mathbb{R}))$ then has both a discrete and a continuous spectrum and it is necessary to formulate the correspondence in terms of quotients, as above. Duality was proved in full generality in terms of Harish-Chandra modules, [17]. For a detailed description, see e.g. [16, 29, 28] (or also [1]).

Remark 2.2. The Weil representation exists in a *much* larger context, see Weil's original paper [33]. For example, the corresponding p-adic groups $\mathrm{O}(V(\mathbb{Q}_p))$ and $\mathrm{SL}_2(\mathbb{Q}_p)$ act on $S(V(\mathbb{Q}_p))$, the space of Bruhat-Schwartz functions on $V(\mathbb{Q}_p)$, which consists of the locally constant functions on $V(\mathbb{Q}_p))$. For us (see below), such a function arises by the choice of (a coset of) a lattice L on $V(\mathbb{Q})$, which is associated to the characteristic function of a translate of $L_p=L\otimes_{\mathbb{Z}}\mathbb{Z}_p$, indeed locally constant. The local theta or local Howe correspondence can be studied in this setting as well.

2.2. **Theta Series.** Let $\varphi\in S(V(\mathbb{R}))$ be an eigenfunction of weight $\frac{1}{2}r\in\frac{1}{2}\mathbb{Z}$ under the action of K'. We then set

$$\varphi(x,\tau,z):=j(g'_\tau,i)^{r/2}\omega(g'_\tau)\varphi(x)=v^{-r/4+m/4}\varphi^0(\sqrt{v}x)e^{\pi i(x,x)\tau}.$$

Here g'_τ is any element in $\mathrm{SL}_2(\mathbb{R})$ moving the basepoint i of the upper half plane $\mathbb{H}$ to $\tau=u+iv\in\mathbb{H}$; for example, $g'_\tau=\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)\begin{pmatrix}v^{1/2}&0\\0&v^{-1/2}\end{pmatrix}$. Given an even lattice[1] L and a coset $h\in L^\#/L$ we define the associated theta series by

$$\theta(\tau,\varphi)=\theta(\tau,\varphi,L,h)=\sum_{x\in L+h}\varphi(x,\tau,z)=v^{-r/4+m/4}\sum_{x\in L+h}\varphi^0(\sqrt{v}x)e^{\pi i(x,x)\tau}. \tag{2.3}$$

Then, by Poisson summation, one shows that $\theta(\tau,\varphi)$ transforms like a modular form of weight $r/2$ for the principal congruence subgroup $\Gamma(N)$ and for $\Gamma_0(N)$ if $h=0$. For an outline of the argument, see for example p99-100 in [15]. Here N is the level of the lattice L; that is, the smallest positive integer N such that $N(x,x)\in 2\mathbb{Z}$ for all $x\in L^\#$.

For example, if V is positive definite and $p(x)\in\mathcal{H}_\ell(V)$ harmonic of degree ℓ, then $\theta(\tau,p\varphi_0)=\sum_{x\in L+h}p(x)e^{\pi i(x,x)\tau}$ is a holomorphic modular form of weight $\frac{p}{2}+\ell$, and is cuspidal if $\ell>0$.

[1]Even means that $(x,x)\in 2\mathbb{Z}$ for all $x\in L$. In particular, $L\subset L^\#$, where $L^\#$ is the dual lattice

$$L^\#=\{x\in V\mid (x,L)\subset\mathbb{Z}\}$$

However, in general, $\theta(\tau,\varphi)$ is *not* holomorphic. We let $L_{r/2} = -2iv^2\frac{\partial}{\partial\bar\tau}$ be the Maass lowering operator which lowers the weight of forms by 2. Then a little calculation shows that the action of $L_{r/2}$ on theta series corresponds to the Weil representation action of $L \in \mathfrak{sl}_2(\mathbb{C})$ on $S(V(\mathbb{R}))$, that is,

$$L_{r/2}\theta(\tau,\varphi) = \theta(\tau,\omega(L)\varphi). \tag{2.4}$$

In particular, this 'explains' the holomorphicity of the theta series with harmonic coefficients from a representation-theoretic perspective.

Note that the Weil representation extends to the Hilbert space $L^2(V(\mathbb{R}))$ and hence L^2-eigenfunctions under the Fourier transform can be also used to construct modular objects. However, in order to obtain modularity one needs to ensure that Poisson summation holds (which is automatic for Schwartz functions but not for square integrable functions). We can summarize our discussion with the following theorem (see e.g. [32]):

Theorem 2.3. *Let $p(x)$ be a function on $V(\mathbb{R})$ such that $\varphi(x) := p(x)e^{-\pi(x,x)}$ and its first and second partial derivatives are in $L^2(V(\mathbb{R})) \cap L^1(V(\mathbb{R}))$. Assume* (2.2)*, that is, $(E - \frac{1}{4\pi}\Delta)p = \lambda p$ for some $\lambda \in \mathbb{Z}$. Then*

$$\theta(\tau,\varphi) = v^{-\lambda/2}\sum_{x\in L+h} p(\sqrt{v}x)e^{\pi i(x,x)\tau}$$

transforms like a modular form of weight $r/2+\lambda$ of level N as above.

We give now one example where Theorem 2.3 can be applied. This often amounts to the clever choice of a function $p(x)$ which (partially) restricts the summation of the theta series to the positive cone. For the general philosophy of this approach, see also section 2.3 in [15].

Example 2.4. [Zwegers' Mock theta function, [35]]

Let V be of signature $(p,1)$ and let $c_1, c_2 \in V(\mathbb{R})$ be two non-collinear vectors of negative length with $(c_1,c_2) < 0$. For simplicity, we assume $(c_1,c_1) = (c_2,c_2) = -1$. We let

$$E(t) = 2\int_0^t e^{-\pi u^2}du$$

be the (modified) error function. We then set

$$\begin{aligned} p(x) &= \frac{1}{2}\left(E(\sqrt{2}(x,c_1)) - E(\sqrt{2}(x,c_2))\right) \\ &= \frac{1}{2}\left(\operatorname{sgn}(x,c_1) - \operatorname{sgn}(x,c_2)\right) \\ &\quad + \frac{1}{2\sqrt{\pi}}\left[\operatorname{sgn}(x,c_2)\Gamma\left(\tfrac{1}{2}, 2\pi(x,c_2)^2\right) - \operatorname{sgn}(x,c_1)\Gamma\left(\tfrac{1}{2}, 2\pi(x,c_1)^2\right)\right]. \end{aligned}$$

Here $\Gamma(s,a) = \int_a^\infty e^{-t}t^{s-1}dt$ is the incomplete Γ-function, and we use the convention $\operatorname{sgn}(0) = 0$. Then $p(x)$ satisfies the hypothesis of Theorem 2.3 with $\lambda = 0$, and we conclude

$$\theta_Z(\tau, c_1, c_2) := \frac{1}{2} \sum_{x \in L+h} \Big(E(\sqrt{2v}(x, c_1)) - E(\sqrt{2v}(x, c_2)) \Big) e^{\pi i (x,x)\tau}$$

is a non-holomorphic modular form of level N of weight $m/2$. Now $(\operatorname{sgn}(x, c_1) - \operatorname{sgn}(x, c_2) = 0$ for all x of non-positive length. Hence we can view

$$\frac{1}{2\sqrt{\pi}} \sum_{x \in L+h} \left[\operatorname{sgn}(x, c_2) \Gamma\left(\tfrac{1}{2}, 2\pi v (x, c_2)^2\right) - \operatorname{sgn}(x, c_1) \Gamma\left(\tfrac{1}{2}, 2\pi v (x, c_1)^2\right) \right] e^{\pi i (x,x)\tau}$$

as the non-holomorphic modular completion of the Mock theta function

$$\frac{1}{2} \sum_{\substack{x \in L+h \\ (x,x)>0}} \left(\operatorname{sgn}(x, c_1) - \operatorname{sgn}(x, c_2) \right) e^{\pi i (x,x)\tau},$$

obtained by summation over the positive cone. Furthermore, for the shadow, we easily compute the action of the lowering operator $L_{m/2}$ and obtain

$$L_{m/2} \theta_Z(\tau, c_1, c_2) = \frac{v^{3/2}}{\sqrt{2}} \sum_{x \in L+h} \left((x, c_2) e^{-2\pi (x,c_2)^2 v} - (x, c_1) e^{-2\pi (x,c_1)^2 v} \right) e^{\pi i (x,x)\tau},$$

which defines a non-holomorphic form weight $m/2 - 2$.

2.2.1. *Siegel's theta series.* Indefinite theta series go back to Siegel [31] (and of course Hecke)) and originally arose in a more geometric setting which we now describe.

We let $G = \mathrm{SO}(V(\mathbb{R}))$ be the special orthogonal group, and let K be the compact subgroup of G stabilizing the oriented negative q-plane $z_0 := \operatorname{span}\{v_\mu;\ p+1 \leq \mu \leq p+q\}$. Then the symmetric space $D = D(V)$ associated to V is given by $D \simeq G/K$. Note that $D = D^+ \coprod D^-$ has two connected components and has dimension pq. We can realize D as the Grassmannian of oriented negative q-planes in $V(\mathbb{R})$:

$$D \simeq \{ z \subset V(\mathbb{R});\ z \text{ oriented};\ \dim z = q;\ (\ ,\ )|_z < 0 \}.$$

Indeed, by Witt's theorem G acts transitively on this Grassmannian, and the stabilizer of the base point z_0 is by definition K. We let $\Gamma \subset G$ be a congruence subgroup stabilizing $L + h$ (or even acting trivially on $L^\#/L$). Then $X = X_\Gamma = \Gamma \backslash D$ is a locally symmetric space of finite volume.

Example 2.5. (i) For signature $(p, 1)$, D is hyperbolic p-space. Then the connected component of the base point is given by

$$D^+ \simeq \{ z \in V(\mathbb{R}); (z, z) = -1; (z, v_m) < 0 \}.$$

We will make this identification henceforth.

(ii) For signature $(p, 2)$, D has a Hermitian structure and X is a quasi-projective variety. For $(1, 2)$, we have $D^+ \simeq \mathbb{H}$ and X is a modular or Shimura curve; for $(2, 2)$, we have $D^+ \simeq \mathbb{H} \times \mathbb{H}$ and X is a Hilbert modular curve; for $(3, 2)$, we have $D^+ \simeq \mathbb{H}_2$, the Siegel upper half space of genus 2.

For $z \in D$, we associate the standard majorant $(\ ,\ )_z$ given by

$$(x, x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z),$$

where $x = x_z + x_{z^\perp} \in V(\mathbb{R})$ is given by the orthogonal decomposition $V(\mathbb{R}) = z^\perp \oplus z$.

The standard Gaussian on $V(\mathbb{R})$ is given by

$$\varphi_0(x, z) = e^{-\pi(x,x)_z}.$$

In particular, at the base point $z_0 \in D$ we have $\varphi_0(x, z_0) = \varphi_0(x) = e^{-\pi \sum_{i=1}^{p+q} x_i^2}$, as already mentioned above. Note that $\varphi_0(x, z) = \varphi_0(g_z^{-1}x)$, where $g_z \in G$ is any element moving z_0 to z. In this way, we can view

$$\varphi_0 \in S(V(\mathbb{R}))^K \simeq [S(V(\mathbb{R})) \otimes C^\infty(D)]^G. \tag{2.5}$$

Here G acts diagonally and the inverse map is given by restriction at the base point z_0. Since the action of $SL_2(\mathbb{R})$ and $\mathrm{O}(V(\mathbb{R}))$ commute, we also see immediately that $\varphi_0(x, z)$ is an eigenfunction for K' of weight $(p - q)/2$ as well. Hence

$$\theta(\tau, z, \varphi_0) = v^{q/2} \sum_{x \in L+h} e^{-2\pi R(x,z)v} e^{\pi i(x,x)\tau} \in NonholM_{(p-q)/2}(\Gamma(N)) \otimes C^\infty(D)^\Gamma$$

defines a non-holomorphic modular form of weight $(p - q)/2$ and level N, taking values in the C^∞-functions on X. Here $R(x, z) := -(x_z, x_z) = \frac{1}{2}((x, x)_z - (x, x))$, so that $\varphi_0^0(x, z) = e^{-2\pi R(x,z)}$. Note that $R(x, z)$ is non-negative and zero if and only x is perpendicular to the negative q-plane z.

One can then use $\theta(\tau, z, \varphi_0)$ as an integral kernel to lift objects (automorphic forms) for SL_2 to the orthogonal group for V and vice versa. The celebrated Siegel-Weil formula (classically and its extensions) asserts that

$$E(\tau, \varphi_0) := \int_X \theta(\tau, z, \varphi_0) d\mu(z),$$

(in an adelic setting up) gives an Eisenstein series for SL_2. Here $d\mu(z)$ is the G-invariant measure on D. For an overview, see e.g. [18].

Conversely, the equally celebrated singular Borcherds lift arises by considering, for signature $(p, 2)$, the appropriately regularized theta integral

$$\Phi(z, f) := \int_{\Gamma(N)\backslash\mathbb{H}}^{reg} \theta(\tau, z, \varphi_0) f(\tau) d\mu(\tau),$$

where $f \in M^!_{1-p/2}(\Gamma(N))$ is a weakly holomorphic modular form (or more generally a weak Maass form) of weight $1 - p/2$, see [4, 14, 5]. The associated Borcherds product is roughly given by exponentiating $\Phi(z, f)$.

As another example, the Shimura-Shintani correspondence between holomorphic modular forms of weight $2k$ and $k + 1/2$ can be realized via integration against a certain theta series $\theta(\tau, z, \varphi_k)$ for signature $(2, 1)$, see [26, 30].

These theta liftings exist in *much* greater generality. For the general case, one considers the general theta series

$$\theta(g, g', \varphi, L, h) := \sum_{x \in L+h} \omega(g')\varphi(g^{-1}x) \qquad (g \in \mathrm{O}(V(\mathbb{R})), g' \in \mathrm{Mp}_2(\mathbb{R}))$$

as integral kernel. As one varies φ (and L and h) one obtains the theta correspondence between automorphic forms/representations of the two groups involved.

Remark 2.6. The global Weil representation acts on $S(V(\mathbb{A})) = \bigotimes'_{v\leq\infty} S(V(\mathbb{Q}_v))$. Then the global Howe correspondence is concerned with a correspondence between automorphic representations in a similar fashion as in the local case, and typically this correspondence can be realized by the above theta liftings/integrals. For example, the Shimura correspondence was studied from a representation-theoretic perspective by Waldspurger in the early 1980's.

3. Special Schwartz forms

3.1. **More on the orthogonal symmetric space.** Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ its Cartan decomposition. Then $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{k}$ is isomorphic to the tangent space $T_{z_0}(D)$ at the base point of D, and with respect to the above basis of $V(\mathbb{R})$ we have

$$\mathfrak{p} \simeq \operatorname{Hom}(z_0, z_0^{\perp}) \simeq \left\{ \begin{pmatrix} 0 & X \\ {}^tX & 0 \end{pmatrix};\ X \in M_{p,q}(\mathbb{R}) \right\}. \tag{3.1}$$

We let $X_{\alpha\mu}$ $(1 \leq \alpha \leq p,\ p+1 \leq \mu \leq p+q)$ denote the elements of the obvious basis of $\mathfrak{p}$ in (3.1), and let $\omega_{\alpha\mu}$ be the elements of the dual basis which pick out the $\alpha\mu$-th coordinate of $\mathfrak{p}$. For any point $z \in D$ we have $T_z(D) \simeq \operatorname{Hom}(z, z^{\perp})$. We orient $z^{\perp}$ such that its orientation followed by the given one of z gives the orientation of V. This then gives an orientation for $\operatorname{Hom}(z, z^{\perp})$ and thus for D as in [24], p. 130/131.

For $x \in V(\mathbb{R})$ with $(x,x) > 0$, we let

$$D_x = \{z \in D;\ z \perp x\}.$$

Note that D_x is a subsymmetric space of type $D_{p-1,q}$ attached to the orthogonal group G_x, the stabilizer of x in G. Note that $R(x,z) = 0$ exactly when $z \in D_x$. Again following [24], page 130/131, we orient the cycles D_x. Namely, we first orient the subspace $x^{\perp}$ in $V(\mathbb{R})$ such that x followed by an oriented basis of $x^{\perp}$ gives an oriented basis of $V(\mathbb{R})$ and then follow the procedure for D. Note that with these conventions we obtain $D_{-x} = (-1)^q D_x$. We let Γ_x be the stabilizer of x in Γ. Then we define the special cycle C_x as the image of $\Gamma_x\backslash D_x$ in $X = \Gamma\backslash D$.

Example 3.1. If D is hyperbolic and $(x,x) > 0$, the cycle $D_x^+ = \{z \in D^+; (z,x) = 0\}$ divides D^+ into two components defined by the sign of (z,x). On the other hand if $(x,x) < 0$, then (by definition) (x,z) doesn't change signs on each of the components D^+ and D^-. The same holds for x isotropic.

3.2. **Special Schwartz forms.** The second author and Millson (see [22, 23, 24]) constructed (in much more generality) Schwartz forms φ_{KM} on $V(\mathbb{R})$ taking values in $\mathcal{A}^q(D)$, the differential q-forms on D. More precisely,

$$\varphi_{KM} \in [\mathcal{S}(V(\mathbb{R})) \otimes {\textstyle\bigwedge}^q(\mathfrak{p}^*)]^K \simeq [\mathcal{S}(V(\mathbb{R})) \otimes \mathcal{A}^q(D)]^G. \tag{3.2}$$

The isomorphism in (3.2) generalizes the one indicated in (2.5) and can be described explicitly as follows. Let $g_z \in G$ be any element moving the base point z_0 to z. Then

g_z defines an isomorphism from $T_z(D) = \mathrm{Hom}(z, z^\perp)$ to $\mathfrak{p} = T_{z_0}(D) = \mathrm{Hom}(z_0, z_0^\perp)$ in the usual way, that is, for $T \in \mathrm{Hom}(z, z^\perp)$ we have $(g_z^{-1}T)(v) := g_z^{-1}T(g_z)v)$. We denote the dual map by $(g_z^{-1})^*$. We (initially) view φ_{KM} as a map from $V(\mathbb{R})$ to $\wedge^q \mathrm{Hom}(z_0, z_0^\perp)^* \simeq \wedge^q \mathfrak{p}^*$. Then

$$\varphi_{KM}(x, z) := (g_z^{-1})^* \varphi_{KM}(g_z^{-1}x).$$

By K-invariance this is independent of the choice of $g_z \in G$.

We define a differential operator acting on $\mathcal{S}(V(\mathbb{R}))$ by $\mathcal{D}_i = x_i - \frac{1}{2\pi}\frac{\partial}{\partial x_i}$. Then φ_{KM} is given by

$$\varphi_{KM} = \frac{1}{2^{q/2}} \sum_{\alpha_1, \dots, \alpha_q = 1}^{p} [\mathcal{D}_{\alpha_1} \cdots \mathcal{D}_{\alpha_q}]\varphi_0 \otimes \omega_{\alpha_1 p+1} \wedge \cdots \wedge \omega_{\alpha_q p+q}.$$

(This is $2^{q/2}$ times the corresponding quantity in [24].) A bit more explicitly, we have

$$\varphi^0_{KM}(x) = \sum_{\underline{\alpha}} P^{(q)}_{\underline{\alpha}}(x) e^{-2\pi R(x, z_0)} \otimes \omega_{\alpha_1 p+1} \wedge \cdots \wedge \omega_{\alpha_q p+q}, \tag{3.3}$$

where $P^{(q)}_{\underline{\alpha}}(x)$ is a polynomial of degree q and $\underline{\alpha} = (\alpha_1, \dots, \alpha_q) \in \{1, \dots, p\}^q$ denotes a multi index. In fact, for $\underline{\alpha} = (\alpha, \dots, \alpha)$, $P^{(q)}_{\underline{\alpha}}(x)$ is given by $P^{(q)}_{\underline{\alpha}}(x) = (4\pi)^{-q/2} H_q\left(\sqrt{2\pi} x_\alpha\right)$, where $H_q(t) = (-1)^q e^{t^2} \frac{d^q}{dt^q} e^{-t^2}$ is the q-th Hermite polynomial. For 'mixed' $\underline{\alpha}$, $P^{(q)}_{\underline{\alpha}}(x)$ is a product of Hermite polynomials in the x_α.

It is easy to see that φ_{KM} is K-invariant. Its key (non-trivial!) properties are

(1) $\varphi_{KM}(x, z)$ is a closed Γ_x-invariant differential form on D for all $x \in V(\mathbb{R})$.
(2) $\varphi_{KM}(x, z)$ is an eigenfunction of K' of weight $\frac{m}{2}$.
(3) $\varphi^0_{KM}(x, z)$ is a 'Thom form' for the cycle $C_x = \Gamma_x \backslash D_x$ on the 'tube' $\Gamma_x \backslash D$, that is,

$$\int_{\Gamma_x \backslash D} \eta \wedge \varphi^0_{KM}(x, z) = \int_{C_x} \eta$$

for any closed differential $(p-1)q$-form η on $\Gamma_x \backslash D$ with compact support.

We also define another form

$$\psi_{KM} = \psi \in [S(V(\mathbb{R})) \otimes \textstyle\bigwedge^{q-1}(\mathfrak{p}^*)]^K \simeq [S(V(\mathbb{R})) \otimes \mathcal{A}^{q-1}(D)]^G$$

by

(3.4)

$$\psi_{KM} = -\frac{1}{2^{(q+2)/2}} \sum_{j=1}^{q} \sum_{\alpha_1, \dots, \alpha_{q-1} = 1}^{p} (-1)^{j-1} [\mathcal{D}_{\alpha_1} \cdots \mathcal{D}_{\alpha_{q-1}} \mathcal{D}_{p+j}]\varphi_0 \otimes \omega_{\alpha_1 p+1} \wedge \cdots \wedge \widehat{\omega_{\alpha_j p+j}} \wedge \cdots \wedge \omega_{\alpha_{q-1} p+q}.$$

We can write
(3.5)

$$\psi^0(x) = -\frac{1}{\sqrt{2}}\sum_{j=1}^{q}\sum_{\underline{\alpha}}(-1)^{j-1}x_{p+j}P_{\underline{\alpha}}^{(q-1)}(x)e^{-2\pi R(x,z_0)} \otimes \omega_{\alpha_1 p+1}\wedge\cdots\wedge\widehat{\omega_{\alpha_j p+j}}\wedge\cdots\wedge\omega_{\alpha_{q-1}p+q},$$

where the sum extends over all multi-indices $\underline{\alpha} = (\alpha_1, \dots, \alpha_{q-1})$. The key relationship between φ_{KM} and ψ, see [24], is given by

(4) ψ is an eigenfunction of K' with weight $\frac{m}{2} - 2$.

(5) Let $L_\kappa = -2iv^2\frac{\partial}{\partial\bar{\tau}}$ be the Maass lowering operator and let d denotes the exterior differential in $\mathcal{A}^\bullet(D)$. Then

$$L_\kappa\varphi_{KM}(x,\tau,z) = d\psi(x,\tau,z).$$

Explicitly, this means

$$v\frac{\partial}{\partial v}\varphi_{KM}^0(\sqrt{v}x) = d\psi^0(\sqrt{v}x). \tag{3.6}$$

3.3. **A singular Schwartz form.** For $x \neq 0$, we set

$$\tilde{\psi}^0(x,z) := -\int_1^\infty \psi^0(x\sqrt{t},z)\frac{dt}{t}. \tag{3.7}$$

Note that $\tilde{\psi}^0$ has singularities when $R(x,z) = 0$, that is, exactly along the cycles D_x. In particular, for $(x,x) \leq 0$, $\tilde{\psi}^0$ it is smooth. More precisely, a little calculation gives (at the base point)

$$\tilde{\psi}^0(x,z_0) = \sum_{\underline{\alpha}} Q_{\underline{\alpha}}(x)\sum_{\ell=0}^{q-1}P_{\underline{\alpha},\ell}^{(q-1)}(x)(2\pi R(x,z_0))^{-(\ell+1)/2}\Gamma\left(\tfrac{\ell+1}{2}, 2\pi R(x,z_0)\right) \tag{3.8}$$

with

$$Q_{\underline{\alpha}}(x) = \frac{1}{2^{q/2}}\sum_{j=1}^{q}(-1)^{j-1}x_{p+j}\otimes\omega_{\alpha_1 p+1}\wedge\cdots\wedge\widehat{\omega_{\alpha_j p+j}}\wedge\cdots\wedge\omega_{\alpha_{q-1}p+q}.$$

Here $P_{\underline{\alpha},\ell}^{(q-1)}(x)$ denotes the homogeneous component of degree ℓ of $P_{\underline{\alpha}}^{(q-1)}(x)$. In particular, $R(x,z)^{q/2}\tilde{\psi}^0(x,z)$ extends to a smooth form on D.

In line with our conventions for Schwartz functions we also define

$$\tilde{\psi}(x,z) = \tilde{\psi}^0(x,z)e^{-\pi(x,x)}.$$

While $\tilde{\psi}$ is *not* a Schwartz function on $V(\mathbb{R})$, we can still define

$$\tilde{\psi}(x,\tau,z) = \tilde{\psi}^0(\sqrt{v}x,z)e^{\pi i(x,x)\tau}$$

for $\tau \in \mathbb{H}$ as if it was an eigenfunction under K' of weight $m/2$ (which it is not). Note

$$\tilde{\psi}(x,\tau,z) = -\left(\int_v^\infty \psi^0(\sqrt{t}x)\frac{dt}{t}\right)e^{\pi i(x,x)\tau}.$$

From the definition (3.7) of the singular form we immediately obtain

Proposition 3.2. *Outside the singularities we have*

(i)
$$d\tilde{\psi}(x,\tau,z) = \varphi_{KM}(x,\tau,z).$$

(ii)
$$L_\kappa \tilde{\psi}(x,\tau,z) = \psi(x,\tau,z).$$

Note that (ii) motivates to consider $\tilde{\psi}$ as if it had weight κ. The key geometric property of $\tilde{\psi}$ is given as follows.

Theorem 3.3. *The singular differential form $\tilde{\psi}(x)$ is a differential character for the cycle D_x. That is, $\tilde{\psi}(x)$ is locally integrable, and we have*

$$\int_D \eta \wedge \varphi^0_{KM}(x,z) = \int_{D_x} \eta + (-1)^{(p-1)q+1} \int_D d\eta \wedge \tilde{\psi}^0(x,z).$$

for any (not necessarily closed) $\eta \in \mathcal{A}_c^{(p-1)q}(D)$, the space of $(p-1)q$-forms D with compact support. Here we set $D_x = \emptyset$ if $(x,x) \leq 0$.

The theorem will be proved in Section 5 using the general Lie theoretic setting explained above.

Remark 3.4. Note that the theorem also implies a 'dual' statement for the integral $\int_C \varphi^0_{KM}(x,z)$ over a compact q-chain C in D with piecewise smooth boundary. For example, if C and D_x intersect transversely in the interior of C, we have

$$\int_C \varphi^0_{KM}(x,z) = [C, D_x] + (-1)^{(p-1)q+1} \int_{\partial C} \tilde{\psi}^0(x,z).$$

Here $[C, D_x]$ is the appropriately defined (local) intersection number of C and D_x. For $q = 2$, such a statement can be found in [21] for a particular geodesic square in D.

Example 3.5. For D hyperbolic p-space, all this can be seen explicitly, Namely, we naturally have $T_z(D) \simeq z^\perp$. Unraveling the definitions and basic formulas we see

$$\varphi_{KM}(x,z) = \sqrt{2}(x,\cdot)e^{-\pi(x,x)_z}$$

as functional on $z^\perp$ and

$$\psi^0(x,z) = \frac{1}{\sqrt{2}}(x,z)e^{-2\pi(x,z)^2} \qquad \text{and} \qquad \tilde{\psi}^0(x,z) = \frac{1}{2\sqrt{\pi}} \operatorname{sgn}(x,z)\Gamma\left(\tfrac{1}{2}, 2\pi(x,z)^2\right).$$

In particular, $\tilde{\psi}^0(x,z)$ has a singularity of type $\frac{1}{2}\operatorname{sgn}(x,z)$ along the cycle D_x, that is, $\tilde{\psi}^0(x,z) - \frac{1}{2}\operatorname{sgn}(x,z)$ extends to a smooth function on D. From that Theorem 3.3 is quite straightforward, see also Section 4.3 below.

4. Geometric Theta Integrals

In this section, we consider the theta series

$$\theta(\tau,z,\varphi_{KM}) = \sum_{x\in L+h} \varphi^0_{KM}(\sqrt{v}x)e^{\pi i(x,x)\tau} = \sum_{n\in\mathbb{Q}} \left(\sum_{\substack{x\in L+h\\(x,x)=2n}} \varphi^0_{KM}(\sqrt{v}x) \right) q^n$$

associated to φ_{KM} and use this theta series as an integral kernel. By the above discussion we immediately see

$$\theta(\tau,z,\varphi_{KM}) \in NonholM_{m/2}(\Gamma(N)) \otimes \mathcal{A}^q(D)^\Gamma,$$

that is, $\theta(\tau,z,\varphi_{KM})$ is a non-holomorphic modular forms of weight $(p-q)/2)$ and level N taking values in the closed differential q-forms on X.

4.1. **The cohomological lift.** For η a $(p-1)q$-form on X with compact support and C a compact q-chain on X, we set

$$I(\tau,\eta) = \int_X \eta\wedge\theta(\tau,z,\varphi_{KM}) \qquad \text{and} \qquad I(\tau,C) = \int_C \theta(\tau,z,\varphi_{KM}).$$

Then the n-th Fourier coefficient $I(\tau,\eta)$ for $n\neq 0$ given by

$$\int_{\Gamma\backslash D} \eta\wedge \sum_{\substack{x\in L+h\\(x,x)=2n}} \varphi^0_{KM}(\sqrt{v}x,z) = \sum_{\substack{x\in L+h\\(x,x)=2n\\ \bmod \Gamma}} \int_{\Gamma_x\backslash D} \eta\wedge\varphi^0_{KM}(\sqrt{v}x,z).$$

Here in the last step we unfolded the integral and used the fact that Γ acts for $n\neq 0$ on the set $\{x\in L+h; (x,x)=2n\}$ with finitely orbits. Then we set

$$C_n = \sum_{\substack{x\in L+h\\(x,x)=2n\\ \bmod \Gamma}} C_x \in H_{(p-1)q}(X,\partial X,\mathbb{Z}).$$

By the Thom property of φ^0_{KM} (Key property (3) above, or also by Theorem 3.3) one then obtains (in much greater generality)

Theorem 4.1 ([24])**.** *Let η as above be a <u>closed</u> $(p-1)q$-form and C be q-cycle on X. Then*

$$I(\tau,\eta) = \int_X \eta\wedge e_q + \sum_{n>0}\left(\int_{C_n}\eta\right)q^n \qquad \textit{and} \qquad I(\tau,C) = \int_C e_q + \sum_{n>0}[C,C_n]q^n.$$

Here $e_q = \varphi_{KM}(0)$ is the Euler form for D, a certain closed G-invariant q-form on D (which is zero for q odd) and $[C,C_n]$ is the cohomological intersection numbers of the cycles C and C_n.

In particular, these two geometric theta integrals define holomorphic modular forms of weight $m/2$. In particular, we obtain (co)homological maps

$$H_q(X,\mathbb{Z}) \to M_{m/2}(\Gamma(N)) \qquad \textit{and} \qquad H_c^{(p-1)q}(X,\mathbb{R}) \to M_{m/2}(\Gamma(N)),$$

where the Fourier coefficients are given by periods respectively intersection numbers.

Results of this kind were first famously found by Hirzebruch-Zagier [15] for Hilbert modular surfaces.

Remark 4.2. (i) Note that after integration all Fourier coefficients of negative index vanish. Hence after integration the resulting theta function is given by summation over the positive cone in the indefinite space V. However, in this construction one obtains these series very differently by employing some kind of 'universal" theta kernel arising from a Schwartz function, which simplifies some of the analytical considerations.

(ii) The holomorphicity of the (co)homological theta integral can be seen directly by the following amusing calculation: Using $L\varphi_{KM} = d\psi$ and Stokes' theorem we see

$$L_{m/2}I(\tau, C) = \int_C L_{m/2}\theta(\tau, z, \varphi_{KM}) = \int_C d\theta(\tau, z, \psi) = \int_{\partial C} \theta(\tau, z, \psi) = 0.$$

Here in the last step we used that C is a closed cycle. This highlights the role of the 'auxiliary" Schwartz form ψ for the theory.

Remark 4.3. In joint work with Millson, the first author has been studying generalizations of the theory. We mention two main aspects.

(i) It is a natural question to investigate the case when η is no longer of compact support respectively C is no longer compact. This natural question is particularly motivated by the original Hirzebruch-Zagier paper where this was also considered. This has been studied extensively in [9, 12, 11, 13].

(ii) In [10], the theory is extended to cycles and (co)homology with local coefficients in large generality. In the setting of this paper, the coefficient system involved is $\mathcal{H}^\ell(V)$, the harmonic polynomials of degree ℓ. This can be viewed as the analogue to the positive definite theta series with harmonic coefficients. The generating series of intersection numbers then gives rise to modular forms of higher weight $m/2 + \ell$. In particular, in [11] Section 6, the classical Eisenstein series

$$E_k(\tau) = \zeta(1-k) + \sum_{n,m>0} n^{k-1} e^{2\pi i n m \tau}$$

is explicitly realized as a theta series for signature $(1, 1)$.

Remark 4.4. We mention one significant application of the theory. In [3], the authors use the above construction to establish new cases of the Hodge conjecture when X arises from a space of signature $(p, 2)$.

4.2. **The general theta integral.** We now consider the theta integral for more general input. We work on the symmetric space D itself. We first note that for $n > 0$, the 'infinite cycle' $\coprod_{\substack{x \in L+h \\ (x,x)=2n}} D_x$ is locally finite in D, that is, the intersection with any compact set involves only finite many cycles D_x.

Theorem 4.5. *Let $\eta \in \mathcal{A}_c^{(p-1)q}(D)$ be a compactly supported differential form on D. Then*

$$\begin{aligned} I(\tau,\eta) &= \int_D \eta \wedge \theta(\tau, z, \varphi_{KM}) \\ &= \int_D \eta \wedge e_q + \sum_{n>0} \left(\sum_{\substack{x \in L+h \\ (x,x)=2n}} \int_{D_x} \eta \right) q^n \\ &\qquad\qquad + (-1)^{(p-1)q+1} \sum_{\substack{x \in L+h \\ x \neq 0}} \left(\int_D d\eta \wedge \tilde{\psi}^0(\sqrt{v}x) \right) e^{\pi i (x,x)\tau} \end{aligned}$$

is a non-holomorphic modular form of weight $m/2$. In particular, if η is a closed form, then $I(\tau,\eta)$ is a holomorphic modular form of weight $m/2$. Finally,

$$L_{m/2} I(\tau,\eta) = \int_D \eta \wedge \theta(\tau, z, \psi).$$

Proof. By the compact support of η we can compute the integral termwise

$$\int_D \eta \wedge \theta(\tau, z, \varphi_{KM}) = \sum_{n \in \mathbb{Q}} \sum_{\substack{x \in L+h \\ (x,x)=2n}} \left(\int_D \eta \wedge \varphi^0_{KM}(\sqrt{v}x, z) \right) q^n.$$

Now for $x = 0$, we have $\varphi^0_{KM} = e_q$. For all the other terms we use Theorem 3.3. $\square$

Remark 4.6. Let C be a compact q-chain in the symmetric space D with piecewise smooth boundary. Then

$$I(\tau, C) = \int_C \theta(\tau, z, \varphi_{KM}) = \sum_{n \in \mathbb{Q}} \sum_{\substack{x \in L+h \\ (x,x)=2n}} \left(\int_C \varphi^0_{KM}(\sqrt{v}x, z) \right) q^n.$$

defines a non-holomorphic form of weight $m/2$. If the image of C in X defines a closed cycle in C, then $I(\tau, C)$ is holomorphic. In general, we have

$$L_{m/2} I(\tau, C) = \int_{\partial C} \theta(\tau, z, \psi).$$

Of course, Theorem 3.4 and Remark 3.4 again provide a geometric formula.

In [8], we consider $I(\tau, C)$ for certain q-cubes in D in detail to recover the results of [27] in the same way as in [21]. We consider the case $q = 1$ in the next section.

4.3. **Hyperbolic Space.** We now explain how one can recover Zwegers' Mock theta function in this setting.

So let V be of signature $(p, 1)$ and let $c_1, c_2 \in V(\mathbb{R})$ be non-collinear as in Example 2.4, that is $(c_1, c_1) = (c_2, c_2) = -1$ and $(c_1, c_2) < 0$. Then c_1 and c_2 define two different points in the same component of D, say D^+, which by abuse of notation we also denote by c_1 and c_2. Let D_{c_1,c_2} be the geodesic arc segment in D^+ connecting c_1 with c_2. One can interpret D_{c_1,c_2} also as follows. The span of c_1 and c_2 defines a subspace U of signature $(1,1)$ of V. Let $U^\perp$ be its orthogonal complement in V which is positive definite of dimension $p-1$. Then $D^+_{U^\perp} := \{z \in D^+; z \in U\}$ defines the infinite geodesic in D passing through c_1 and c_2 and

$$D_{c_1,c_2} = \{\mathrm{span}(tc_1 + (1-t)c_2); t \in [0,1]\} \subset D^+_U.$$

Consider $x \neq 0$ in V. If $(x,x) \leq 0$ or if $(x,x) > 0$ with $D_x \cap D_{c_1,c_2} = \emptyset$, then $\varphi^0_{KM}(x,z)$ is an exact form on D_{c_1,c_2}, and we see by Stokes' theorem

$$\int_{D_{c_1,c_2}} \varphi^0_{KM}(x,z) = \tilde{\psi}^0(x,c_2) - \tilde{\psi}^0(x,c_1).$$

The hyperplane D_x intersects D_{c_1,c_2} transversely (in the interior of the arc) if and only if (x,c_1) and (x,c_2) have opposite signs. In that case the (local) intersection number of D_x and D_{c_1,c_2} is given by $\frac{1}{2}(\mathrm{sgn}(x,c_1) - \mathrm{sgn}(x,c_2))$. (This actually gives another characterization of D_{c_1,c_2} in terms the hyperplanes D_x). Assume $D_x \cap D_{c_1,c_2} = c$ and let c_{ϵ^-} and c_{ϵ^-+} be points on the arc with distance $\epsilon > 0$, say $\mathrm{sgn}(c,c_1) = \mathrm{sgn}(c,c_{\epsilon^-})$ and $\mathrm{sgn}(c,c_2) = \mathrm{sgn}(c,c_{\epsilon^+})$. We then compute

$$\begin{aligned} \int_{D_{c_1,c_2}} \varphi^0_{KM}(x,z) &= \tilde{\psi}^0(x,c_2) - \tilde{\psi}^0(x,c_1) + \lim_{\epsilon\to 0} \tilde{\psi}^0(x,c_{\epsilon^-}) - \tilde{\psi}^0(x,c_{\epsilon^+}) \\ &= \tilde{\psi}^0(x,c_2) - \tilde{\psi}^0(x,c_1) + \frac{1}{2}\left(\mathrm{sgn}(x,c_1) - \mathrm{sgn}(x,c_2)\right). \end{aligned} \tag{4.1}$$

Now this also holds if c is equal to c_1 or c_2 if we use the usual convention $\mathrm{sgn}(0) = 0$ and set $\tilde{\psi}^0(x,c) = 0$. Finally, if $D_{U^\perp} \subset D_x$, that is $x \in U^\perp$, then $x \in z^\perp$ for all $z \in D_{U^\perp}$. Hence the pullback of $\varphi_{KM}(x)$ to D_U is zero and thus $\int_{D_{c_1,c_2}} \varphi^0_{KM}(x,z) = 0$. In summary, (4.1) holds for all $x \neq 0$. We have shown

Theorem 4.7. *The theta integral $\int_{D_{c_1,c_2}} \theta(\tau,z,\varphi_{KM})$ is given by*

$$\begin{aligned} \int_{D_{c_1,c_2}} \theta(\tau,z,\varphi_{KM}) = \sum_{n>0} &\left[\sum_{\substack{x\in L+h \\ (x,x)=2n}} \frac{1}{2}\left(\mathrm{sgn}(x,c_1) - \mathrm{sgn}(x,c_2)\right) \right] q^n \\ &+ \sum_{\substack{x\in L+h \\ x\neq 0}} \left(\tilde{\psi}^0(\sqrt{v}x,c_2) - \tilde{\psi}^0(\sqrt{v}x,c_2) \right) e^{\pi i (x,x)\tau} \end{aligned}$$

and coincides with Zwegers' theta series $\theta_Z(\tau,c_1,c_2)$. Moreover, the "holomorphic coefficients" $\sum_{\substack{x\in L+h \\ (x,x)=2n}} \frac{1}{2}(\mathrm{sgn}(x,c_1) - \mathrm{sgn}(x,c_2))$ can be interpreted as the intersection number of D_{c_1,c_2} with the infinite cycle $\coprod_{\substack{x\in L+h \\ (x,x)=2n}} D_x$.

Furthermore, assume (for simplicity) $L+h = (L\cap U + h_U) \oplus (L \cap U^\perp + h_{U^\perp})$. Then we have the natural splitting

$$\int_{D_{c_1,c_2}} \theta(\tau,z,\varphi_{KM}) = \theta(\tau,U^\perp) \int_{D^U_{c_1,c_2}} \theta(\tau,z,\varphi^U_{KM}).$$

Here $\theta(\tau, U^\perp) = \sum_{y \in L\cap U^\perp + h_{U^\perp}} e^{\pi i (y,y)\tau}$ *is the usual theta series of weight* $(p-1)/2$ *for the positive definite space* $U^\perp$ *and* $\int_{D^U_{c_1,c_2}} \theta(\tau, z, \varphi^U_{KM})$ *is the indefinite theta series of signature* $(1,1)$ *for the space* $U = span(c_1, c_2)$ *and* $L \cap U + h_U$*.*

Remark 4.8. (i) The second part of the theorem follows from the pullback formula $\iota_U^* \varphi^V_{KM} = \varphi_0^{U^\perp} \varphi^U_{KM}$, where $\iota_U : D_U \hookrightarrow D_V$ is the natural embedding of the symmetric space for U into the one for V. The splitting of course also follows directly from considering the formula for Zwegers' theta series.

(ii) One doesn't have to pick D_{c_1,c_2} to be the connecting geodesic. Since D^+ is simply connected and $\theta(\tau, z, \varphi_{KM})$ is closed, any other piecewise smooth curve connecting c_1 and c_2 would do. Hence we have computed the integral $\int_C \theta(\tau, z, \varphi_{KM})$ over *any* compact piecewise smooth curve in D.

(iii) Assume c_1 and c_2 are rational vectors. Then the geodesic D_U connects two rational cusps of D, represented by two isotropic vectors u_1 and u_2 in V. In this case, the pointwise stabilizer of Γ_U of U in Γ is either trivial or infinitely cyclic. We let C_U be the image of $\Gamma_U \backslash D_U$ in X, which then either is an infinite geodesic in X connecting the two cusps or a closed geodesic defining a class in $H_1(X, \mathbb{Z})$.

In the latter case, c_1 and c_2 could be Γ-equivalent, in which case we have $\sum_{\substack{x \in L+h \\ x \neq 0}} \tilde{\psi}(x, \tau, c_1) = \sum_{\substack{x \in L+h \\ x \neq 0}} \tilde{\psi}(x, \tau, c_2)$. Furthermore, the image of D_{c_1,c_2} in X is an integral multiple of C_U, say kC_U. Hence

$$I(\tau, D_{c_1,c_2}) = I(\tau, kC_U) = \frac{1}{2} \sum_{\substack{x \in L+h \\ (x,x)>0}} (\operatorname{sgn}(x, c_1) - \operatorname{sgn}(x, c_2))\, e^{\pi i (x,x)\tau}.$$

In the first case, if C_U is infinite, then we can consider the limit case $c_1 = u_1$ and $c_2 = u_2$ isotropic. This was done in [9], see also Example 4.3 (i) above. One again obtains a holomorphic modular form and

$$I(\tau, C_U) = \lim_{c_i \to u_i} I(\tau, D_{c_1,c_2}) = \frac{1}{2} \sum_{\substack{x \in L+h \\ (x,x)>0}} (\operatorname{sgn}(x, u_1) - \operatorname{sgn}(x, u_2))\, e^{\pi i (x,x)\tau}.$$

(One now computes $\lim_{c_i \to u_i} \sum_{\substack{x \in L+h \\ x \neq 0}} \tilde{\psi}(x, \tau, c_i) = 0$).

(iv) Following [10, 11], one can also equip the geodesic arc segment D_{c_1,c_2} with coefficients, see Example 4.3 (ii) above. This then yields higher weight analogues of Zwegers' theta function. for $V = U$ of signature $(1,1)$, one then obtains in the limit as $c_i \to u_i$ isotropic, the classical holomorphic Eisenstein series.

Remark 4.9. The φ_{KM} also exist for higher Siegel genus n in which case the associated theta series define differential nq-forms on X. One can then consider integrals over compact nq-chains in D. For hyperbolic space this was done in the recent Toronto Ph.D. thesis of I. Livinskyi [25]. It can be then viewed as a higher genus analogue of Zwegers' construction.

5. The singular Schwartz form as a current

In this section, we prove Theorem 3.3. This result is already implicit in [5], Section 7, where the corresponding statement for the singular theta lift of Borcherds type for the Schwartz form ψ is given. We follow the line of reasoning given there. Since $q = 1$ was already discussed in the previous section, we assume $q \geq 2$.

Consider a top degree form $\phi \in A^{pq}(D)$. We then have

$$\int_D \phi(z) = \left(\int_G \phi(g)dg\right)(1_{\mathfrak{p}}),$$

where ϕ on the right hand side is considered as an element in $[C^\infty(G) \otimes \bigwedge^{pq} \mathfrak{p}^*]^K$. (We will frequently use this identification without further comment.) Here $1_{\mathfrak{p}}$ is a properly oriented basis vector for $\bigwedge^{pq} \mathfrak{p} \simeq \mathbb{R}$ of length one with respect to the Killing form. Moreover, $dg = dz\, dk$, where dz is the measure on D coming from the Killing form and $\operatorname{vol}(K, dk) = 1$.

We now pick appropriate coordinates for D. We set $H = G_{v_1}$, the stabilizer in G of the first basis vector v_1 of V. Let $a_t = \exp(tX_{1p+q})$ for $t \in \mathbb{R}$, and let $A = \{a_t;\ t \in \mathbb{R}\}$ be the associated one-parameter subgroup of G We write $A_\varepsilon = \{a_t;\ t \geq \varepsilon\}$. We have a decomposition $G = HAK$ and, with a positive constant C depending on the normalizations of the invariant measures, the integral formula (see [7], section 2)

$$\int_G \phi(g)dg = C \int_{A_0} \int_H \phi(ha_t)|\sinh(t)|^{q-1} \cosh(t)^{p-1} dh\, dt.$$

We first show that $\tilde{\psi}^0(x)$ is locally integrable, i.e., $\int_D \eta \wedge \tilde{\psi}^0(x) < \infty$ for any compactly supported differential form η on D of degree $(p-1)q + 1$. We can assume $x = \kappa\sqrt{m}v_1$ for some $m > 0$ and $\kappa = \pm 1$. We have

$$\begin{aligned}(5.1)\quad &\int_G \eta(g) \wedge \tilde{\psi}^0(g^{-1}\sqrt{m}v_1)\, dg \\ &\qquad = C \int_H \int_0^\infty \eta(ha_t) \wedge \tilde{\psi}^0(\kappa a_t^{-1}h^{-1}\sqrt{m}v_1) \sinh(t)^{q-1} \cosh(t)^{p-1}\, dt\, dh.\end{aligned}$$

We have

$$a_t^{-1}h^{-1}\sqrt{m}v_1 = \cosh(t)\sqrt{m}v_1 - \sinh(t)\sqrt{m}v_{p+q}$$

so that, see (3.8),

$$\begin{aligned}(5.2)\quad &\tilde{\psi}^0(\kappa a_t^{-1}h^{-1}\sqrt{m}v_1) = \\ &\frac{\kappa(-1)^q}{2^{q/2}}\sqrt{m}\sinh(t) \sum_{\ell=0}^{q-1} \left(2\pi m \sinh^2(t)\right)^{-(\ell+1)/2} \Gamma\left(\tfrac{\ell+1}{2}, 2\pi m \sinh^2(t)\right) \\ &\qquad \times \sum_{\underline{\alpha}} P^{(q-1)}_{\underline{\alpha},\ell}\left(\kappa\sqrt{m}\cosh(t)v_1\right) \otimes \omega_{\alpha p+1} \wedge \cdots \wedge \omega_{\alpha_{q-1}p+q-1}.\end{aligned}$$

Therefore the integrand in (5.1) is bounded as $t \to 0$. On the other hand, $\tilde{\psi}^0(\kappa a_t^{-1}h^{-1}v_1)$ is exponentially decreasing in e^t (uniformly in h). Since η has compact support, we conclude that (5.1) converges.

Now let $\eta \in \mathcal{A}_c^{(p-1)q}(D)$. Using $d\psi^0 = \varphi^0_{KM}$ we have

$$\eta \wedge \varphi^0_{KM}(x) = (-1)^{(p-1)q} d\left(\eta \wedge \tilde{\psi}^0(x)\right) + (-1)^{(p-1)q+1} d\eta \wedge \tilde{\psi}^0(x).$$

Hence we only need to show

$$(-1)^{(p-1)q} \int_D d\left(\eta \wedge \tilde{\psi}^0(x)\right) = \int_{D_x} \eta.$$

Again, we assume $x = \kappa\sqrt{m}v_1$. For $\varepsilon > 0$, we let U_ε be the open neighborhood of the cycle D_{v_1} defined in terms of the HAK-coordinates by $\{a_t; t < \varepsilon\}$. Then by Stokes' theorem we obtain

$$\int_D d\left(\eta \wedge \tilde{\psi}^0(\kappa\sqrt{m}v_1)\right) = \lim_{\varepsilon\to 0} \int_{\partial(D-U_\varepsilon)} \eta \wedge \tilde{\psi}^0(\kappa\sqrt{m}v_1). \tag{5.3}$$

By the analogue for (5.1) (which follows from the considerations in [7], section 2), we see that (5.3) is equal to

$$C \lim_{\varepsilon\to 0} \int_H \eta(ha_\varepsilon) \wedge \tilde{\psi}^0(\kappa a_{-\varepsilon} h^{-1}\sqrt{m}v_1) \sinh(\varepsilon)^{q-1} \cosh(\varepsilon)^{p-1}\, dh \left(1_{\mathfrak{p}/\mathbb{R}X_{1p+q}}\right). \tag{5.4}$$

for some universal constant $C \neq 0$. We consider (5.2) (with $t = \varepsilon$). For (5.4) only the terms with $\ell = q-1$ can contribute. But

$$P^{(q-1)}_{\underline{\alpha},q-1}(\kappa\sqrt{m}\cosh(\varepsilon)v_1) = \begin{cases} \left(\kappa\sqrt{2m}\cosh(\varepsilon)\right)^{q-1}, & \underline{\alpha} = (1,1,\dots,1), \\ 0, & \text{otherwise}. \end{cases}$$

We obtain

$$C \lim_{\varepsilon\to 0} \int_H \eta(ha_\varepsilon) \wedge \tilde{\psi}^0(\kappa a_{-\varepsilon} h^{-1}\sqrt{m}v_1) \sinh(\varepsilon)^{q-1} \cosh(\varepsilon)^{p-1}\, dh = C'\kappa^q \int_H \eta(h)\, dh$$

with a constant C'. Therefore, the theorem holds with a certain constant C'' independent of η. (The factor κ^q arises from the orientation of D_x). We conclude that the constant is equal to 1 by noting that this is the case for η closed, since then$\int_D \eta \wedge \varphi_{KM} = \int_{D_x} \eta$ by KM-theory.

References

[1] *J. Adams*, The Theta Correspondence over $\mathbb{R}$, Harmonic analysis, group representations, automorphic forms and invariant theory, 1-39, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 12, World Sci. Publ., Hackensack, NJ, 2007

[2] *S Alexandrov, S. Banerjee, J. Manschot, B. Pioline*, Indefinite theta series and generalised error functions, preprint (2016), arXiv:1606.05495v1.

[3] *N. Bergeron, J. Millson, C. Moeglin*, Hodge type theorems for arithmetic manifolds associated to orthogonal groups, preprint (2011), arXiv:1110.3049.

[4] *R. E. Borcherds*, Automorphic forms with singularities on Grassmannians, Invent. Math. **132** (1998), 491-562.

[5] *J. Bruinier and J. Funke*, On two geometric theta lifts, Duke Math. J. **125** (2004), 45-90.

[6] *J. Duncan, M. Griffin, K. Ono*, Proof of the umbral moonshine conjecture, Research in the Mathematical Sciences **2**, doi:10.1186/s40687-015-0044-7.

[7] *M. Flensted-Jensen*, Discrete series for semi-simple symmetric spaces, Ann. Math. **111** (1980), 253-311.

[8] *J. Funke and S. Kudla*, Theta integrals and generalized error functions II, in preparation.
[9] *J. Funke and J. Millson*, Cycles in hyperbolic manifolds of non-compact type and Fourier coefficients of Siegel modular forms, Manuscripta Math. **107** (2002), 409-449.
[10] *J. Funke and J. Millson*, Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms, American J. Math. **128** (2006), 899-948.
[11] *J. Funke and J. Millson*, Spectacle cycles with coefficients and modular forms of half-integral weight, in: Arithmetic Geometry and Automorphic forms, Volume in honor of the 60th birthday of Stephen S. Kudla, Advanced Lectures in Mathematics series. International Press and the Higher Education Press of China, (2011), 91-154.
[12] *J. Funke and J. Millson*, Boundary behavior of special cohomology classes arising from the Weil representation, Jussieu Math. J. **12** (2013), 571-634.
[13] *J. Funke and J. Millson*, The geometric theta correspondence for Hilbert modular surfaces, Duke Math. J. **163** (2014), 65-116.
[14] *J. Harvey and G. Moore*, Algebras, BPS states, and strings, Nuclear Phys. B **463** (1996), no. 2-3, 315-368.
[15] *F. Hirzebruch and D. Zagier*, Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus, Inv. Math. **36** (1976), 57-113.
[16] *R. Howe*, On some results of Strichartz and Rallis and Schiffman. J. Funct. Anal., **32** (1979), 297-30.
[17] *R. Howe*, Transcending classical invariant theory. J. Amer. Math. Soc. **2** (1989), 535-552.
[18] *S. Kudla*, Some extensions of the Siegel-Weil formula, in Eisenstein Series and Applications, Wee Teck Gan, S. Kudla and Y. Tschinkel, eds., Progress in Math. **258**, Birkhäuser, Boston, 2008, 205-238.
[19] *S. Kudla*, A note about special cycles on moduli spaces of K3 surfaces, in Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, Fields Inst. Commun. **67**, Springer, New York, 2013, 411-427.
[20] *S. Kudla*, A note on Zwegers' theta function, preprint (2013).
[21] *S. Kudla*, Theta integrals and generalized error functions, preprint (2016), arXiv:1608.03534v1.
[22] *S. Kudla and J. Millson*, The theta correspondence and harmonic forms I, Math. Ann. **274**, (1986), 353–378.
[23] *S. Kudla and J. Millson*, The theta correspondence and harmonic forms II, Math. Ann. **277**, (1987), 267–314.
[24] *S. Kudla and J. Millson*, Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables, IHES Publi. Math. **71** (1990), 121-172.
[25] *I. Livinskyi*, On the integrals of the Kudla-Millson theta series, Thesis, Univ. of Toronto, (2016).
[26] *S. Niwa*, Modular forms of half integral weight and the integral of certain theta functions, Nagoya Math. J. **56** (1974), 147-161.
[27] *C. Nazaroglu*, r-tuple error functions and indefinite theta series of higher depth, preprint (2016), arXiv:1609.01224v1.
[28] *A. Paul*, On the Howe correspondence for symplectic-orthogonal dual pairs, J. Funct. Anal., **228** (2005), 270-310.
[29] *S. Rallis and G. Schiffmann*, On a relation between SL_2 cusp forms and cusp forms on tube domains associated to orthogonal groups, Trans. AMS **263** (1981), 1-58.
[30] *T. Shintani*, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J., **58** (1975), 83-126.
[31] *C. L. Siegel*, On the theory of indefinite quadratic forms. Ann. of Math. **45** (1944), 577-622.
[32] *M.-F. Vignéras*, Séries theta des formes quadratiques indéfinies in: Modular functions of one variable VI, Springer Lecture Notes **627** (1977), 227-239.

[33] *A. Weil*, Sur certains groupes d'opérateurs unitaires, Acta. Math. **111** (1964), 143-211.
[34] *M. Westerholt-Raum*, Indefinite Theta Series on Tetrahedral Cones, preprint (2016), arXiv:1608.08874.
[35] *S. Zwegers*, Mock theta functions, Ph.D. thesis, Utrecht University (2002).

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