COMPLEX HYPERBOLIC $(3,3,n)$ TRIANGLE GROUPS

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Abstract. Complex hyperbolic triangle groups are representations of a hyperbolic (p,q,r) reflection triangle group to the group of holomorphic isometries of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$, where the generators fix complex lines. In this paper, we obtain all the discrete and faithful complex hyperbolic $(3,3,n)$ triangle groups. Our result solves a conjecture of Schwartz [16] in the case when $p=q=3$.

1. Introduction

An abstract (p,q,r) reflection triangle group for positive integers p, q, r is the group

$$\Delta_{p,q,r} = \Big\langle \sigma_1,\, \sigma_2,\, \sigma_3 \;\Big|\; \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_2\sigma_3)^p = (\sigma_3\sigma_1)^q = (\sigma_1\sigma_2)^r = id \Big\rangle.$$

We sometimes take (at least) one of p, q, r to be ∞, in which case the corresponding relation does not appear.

It is interesting to seek geometrical representations of $\Delta_{p,q,r}$. An extremely well known fact is that $\Delta_{p,q,r}$ may be realised geometrically as the reflections in the side of a geodesic triangle with internal angles π/p, π/q, π/r. Furthermore, if $1/p + 1/q + 1/r > 1$, $= 1$ or < 1 then this triangle is spherical, Euclidean or hyperbolic respectively. Moreover, up to isometries (or similarities in the Euclidean case) there is a unique such triangle and the representation is rigid. In the case where (at least) one of p, q, r is ∞ then we omit the relevant term from $1/p + 1/q + 1/r$ and we insist that the sides of the triangle are asymptotic. Thus the (∞,∞,∞) triangle is a triangle in the hyperbolic plane with all three vertices on the boundary.

In contrast, if we choose a geometrical representation of $\Delta_{p,q,r}$ in a space of non-constant curvature then more interesting things can happen; see for example Brehm [1]. In this paper, we consider representations of $\Delta_{p,q,r}$ to $\mathrm{SU}(2,1)$ which is (a triple cover of) the group of holomorphic isometries of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$. A convenient model of $\mathbf{H}^2_{\mathbb{C}}$ is the unit ball in $\mathbb{C}^2$ with the Bergman metric, having constant holomorphic sectional curvature and 1/4-pinched real sectional curvatures.

A *complex hyperbolic triangle group* will be a representation of $\Delta_{p,q,r}$ to $\mathrm{SU}(2,1)$ where the generators fix complex lines. Note we could have made other choices. For example, we could choose the generators to be anti-holomorphic isometries, or we could choose reflections in three complex lines but with higher order. These choices lead to interesting results, but we will not consider them here. A crucial observation is that when $\min\{p,q,r\} \geq 3$ there is a one (real) dimensional representation space

2010 Mathematics Subject Classification: Primary 20H10; Secondary 22E40, 51M10.

Key words and Phrases: complex hyperbolic geometry, complex hyperbolic triangle groups.

Wang and Xie were supported by NSFC (grant numbers 11071059 and 11371126). Xie was also supported by NSFC (grant number 11201134) and the young teachers support program of Hunan University.

of complex hyperbolic triangle groups with $1/p+1/q+1/r<1$ (either make a simple dimension count or see Brehm [1] for example). This means that the representation is determined up to conjugacy by p, q, r and one extra variable. This variable is determined by certain traces; see for example Pratoussevitch [13].

In order to state our main results we need a little terminology. Elements of $\mathrm{SU}(2,1)$ act on complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ and its boundary (see below). An element $A\in\mathrm{SU}(2,1)$ is called *loxodromic* if it fixes two points, both of which lie on $\partial\mathbf{H}^2_{\mathbb{C}}$; *parabolic* if it fixes exactly one point, and this point lies on $\partial\mathbf{H}^2_{\mathbb{C}}$; *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$. Discrete groups cannot contain elliptic elements of infinite order. Therefore in a representation of an abstract group to $\mathrm{SU}(2,1)$, if an element of infinite order in the abstract group is represented by an elliptic map then the representation is not discrete or not faithful (or both), compare [7].

Complex hyperbolic triangle groups have a rich history; see Schwartz's ICM survey [16] for an overview. In particular, he presented the following conjectural picture:

Conjecture 1.1 (Schwartz [16]). *Let $\Delta_{p,q,r}$ be a triangle group with $p\le q\le r$. Then any complex hyperbolic representation Γ of $\Delta_{p,q,r}$ is discrete and faithful if and only if $W_A=I_1I_3I_2I_3$ and $W_B=I_1I_2I_3$ are not elliptic. Furthermore:*

- (i) *If $p<10$ then Γ is discrete and faithful if and only if $W_A=I_1I_3I_2I_3$ is non-elliptic.*
- (ii) *If $p>13$ then Γ is discrete and faithful if and only if $W_B=I_1I_2I_3$ is non-elliptic.*

The initial step towards solving this conjecture is the following result of Grossi.

Proposition 1.2 (Grossi [8]). *Let $\Delta_{p,q,r}$ be a triangle group with $p\le q\le r$. Define $W_A=I_1I_3I_2I_3$ and $W_B=I_1I_2I_3$. Then for complex hyperbolic representations of $\Delta_{p,q,r}$:*

- (i) *If $p<10$ and $W_A=I_1I_3I_2I_3$ is non-elliptic then W_B is non-elliptic.*
- (ii) *If $p>13$ and $W_B=I_1I_2I_3$ is non-elliptic then W_A is non-elliptic.*

A motivating example, initially considered by Goldman and Parker [7] and completed by Schwartz [14, 17], concerns complex hyperbolic ideal triangle groups, that is representations of $\Delta_{\infty,\infty,\infty}$. This result may be summarised as follows:

Theorem 1.3 (Goldman, Parker [7], Schwartz [14, 17]). *Let $\Gamma=\langle I_1,I_2,I_3\rangle$ be a complex hyperbolic (∞,∞,∞) triangle group. Then Γ is a discrete and faithful representation of $\Delta_{\infty,\infty,\infty}$ if and only if $I_1I_2I_3$ is non-elliptic.*

Note that this gives a complete solution to Schwartz's conjecture in the case $p=q=r=\infty$. Furthermore, Schwartz [15] gives an elegant description of the group where $I_1I_2I_3$ is parabolic.

Theorem 1.4 (Schwartz [15]). *Let $\Gamma=\langle I_1,I_2,I_3\rangle$ be the (∞,∞,∞) complex hyperbolic triangle group for which $I_1I_2I_3$ is parabolic. Let $\Gamma_2=\langle I_1I_2,I_1I_3\rangle$ be the index 2 subgroup of Γ with no complex reflections. Then $\mathbf{H}^2_{\mathbb{C}}/\Gamma_2$ is a complex hyperbolic orbifold whose boundary is a triple cover of the Whitehead link complement.*

In his book [18], Schwartz proves his conjecture for $\min\{p,q,r\}$ sufficiently large (but unfortunately with no effective bound on this minimum).

Theorem 1.5 (Schwartz [18]). *Let $\Gamma = \langle I_1, I_2, I_3\rangle$ be a complex hyperbolic (p,q,r) triangle group with $p \le q \le r$. If p is sufficiently large, then Γ is a discrete and faithful representation of $\Delta_{p,q,r}$ if and only if $I_1I_2I_3$ is non-elliptic.*

Our main result solves Schwartz's conjecture in the case when $p = q = 3$.

Theorem 1.6. *Let n be an integer at least 4. Let $\Gamma = \langle I_1, I_2, I_3\rangle$ be a complex hyperbolic $(3,3,n)$ triangle group. Then Γ is a discrete and faithful representation of $\Delta_{3,3,n}$ if and only if $I_1I_3I_2I_3$ is non-elliptic.*

Note that the 'only if' part is a consequence of our observation about elliptic elements above. The 'if' part will follow from Corollary 4.4 below.

For the representation where $I_1I_3I_2I_3$ is parabolic, when $n = 4$ and 5 we have the following description of the quotient orbifold from the census of Falbel, Koseleff and Rouillier [5]. The case $n = 4$ combines work of Deraux, Falbel and Wang [3, 6]. The cleanest statement may be found in Theorem 4.2 of Deraux [2], which also treats the case $n = 5$.

Theorem 1.7 (Theorem 4.2 of Deraux [2]).

(i) *Let $\Gamma = \langle I_1, I_2, I_3\rangle$ be the complex hyperbolic $(3,3,4)$ triangle group for which $I_1I_3I_2I_3$ is parabolic. Let $\Gamma_2 = \langle I_1I_2, I_1I_3\rangle$ be the index 2 subgroup of Γ with no complex reflections. Then Γ_2 is conjugate to both $\rho_{1-1}(\pi_1(M_4))$ and $\rho_{4-1}(\pi_1(M_4))$ from [5]. In particular, $\mathbf{H}^2_{\mathbb{C}}/\Gamma_2$ is a complex hyperbolic orbifold whose boundary is the figure eight knot complement.*

(ii) *Let $\Gamma = \langle I_1, I_2, I_3\rangle$ be the complex hyperbolic $(3,3,5)$ triangle group for which $I_1I_3I_2I_3$ is parabolic. Let $\Gamma_2 = \langle I_1I_2, I_1I_3\rangle$ be the index 2 subgroup of Γ with no complex reflections. Then Γ_2 is conjugate to both $\rho_{4-3}(\pi_1(M_9))$ and $\rho_{3-3}(\pi_1(M_{15}))$ from [5].*

It should be possible to give a similar description of the other complex hyperbolic $(3,3,n)$ triangle groups for which $I_1I_3I_2I_3$ is parabolic.

Note that Theorem 1.6 holds in the case $n = \infty$. This follows from recent work of Parker and Will [11] (see also [10]). Furthermore, if as above $\Gamma_2 = \langle I_1I_2, I_1I_3\rangle$ is the index two subgroup of representation of the $(3,3,\infty)$ triangle group for which $I_1I_3I_2I_3$ is parabolic, then $\mathbf{H}^2_{\mathbb{C}}/\Gamma_2$ is a complex hyperbolic orbifold whose boundary is the Whitehead link complement. This is one of the representations in the census of Falbel, Koseleff and Rouillier [5].

Finally, we note some further interesting groups in this family.

Theorem 1.8 (Thompson [19]). *The complex hyperbolic $(3,3,4)$ triangle group with $I_1I_3I_2I_3$ of order 7 and the complex hyperbolic $(3,3,5)$ triangle group with $I_1I_3I_2I_3$ of order 5 are both lattices.*

Our method of proof will be to construct a Dirichlet domain based at the fixed point of the order n elliptic map I_1I_2. Since this point has non-trivial stabiliser, this domain is not a fundamental domain for Γ, but it is a fundamental domain for the coset space of the stabiliser of this point in Γ. Of course, in order to prove directly that this is a Dirichlet domain we would have to check infinitely many inequalities. Instead, we construct a candidate Dirichlet domain and then use the Poincaré polyhedron theorem for coset decompositions (see Theorem 6.3.2 of Mostow [9] or Theorem 3.2 of Deraux, Parker, Paupert [4] for example).

In the case of a Fuchsian $(3,3,n)$ triangle group acting on the hyperbolic plane, a fundamental domain is a hyperbolic triangle with internal angles $\pi/3$, $\pi/3$ and π/n. The Dirichlet domain with centre the fixed point of an order n elliptic map is a regular hyperbolic $2n$-gon with internal angles $2\pi/3$. This $2n$-gon is made up of $2n$ copies of the triangular fundamental domain for the $(3,3,n)$ group. The stabiliser of the order n fixed point, which is a dihedral group of order $2n$, fixes the $2n$-gon and permutes the triangles.

For the complex hyperbolic $(3,3,n)$ triangle groups we will see that the combinatorial structure of the Dirichlet domain D is the same as that in the Fuchsian case. Namely, D has $2n$ sides, each of which is contained in a bisector. Each side meets exactly two other sides (in the case where $I_1I_3I_2I_3$ is parabolic, there are some additional tangencies between sides on the ideal boundary). The sides are permuted by the dihedral group $\langle I_1, I_2\rangle$.

In Section 2 we give the necessary background on complex hyperbolic geometry and the Poincaré polyhedron theorem. In Section 3 we normalise the generators of Γ and discus the parameters this involves. Finally, in Section 4 we consider the bisectors and their intersection properties. This is the heart of the paper.

2. Background

2.1. **Complex hyperbolic space.** Let $\mathbb{C}^{2,1}$ be the three dimensional complex vector space equipped with a Hermitian form H of signature $(2,1)$. In this paper we consider the diagonal Hermitian form $H = \mathrm{diag}(1,1,-1)$. Thus if $\mathbf{u} = (u_1, u_2, u_3)^t$ and $\mathbf{v} = (v_1, v_2, v_3)^t$ then the Hermitian form is given by

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{v}^* H \mathbf{u} = u_1\bar{v}_1 + u_2\bar{v}_2 - u_3\bar{v}_3.$$

Define

$$V_- = \Big\{\mathbf{v} \in \mathbb{C}^{2,1} \ : \ \langle \mathbf{v}, \mathbf{v}\rangle < 0\Big\}, \quad V_0 = \Big\{\mathbf{v} \in \mathbb{C}^{2,1} - \{0\} \ : \ \langle \mathbf{v}, \mathbf{v}\rangle = 0\Big\}.$$

There is a natural projection map $\mathbb{P}$ from $\mathbb{C}^{2,1} - \{0\}$ to $\mathbb{CP}^2$ that identifies all non-zero (complex) scalar multiples of a vector in $\mathbb{C}^{2,1}$. *Complex hyperbolic space* is defined to be $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_-$ and its boundary is $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_0$. Clearly, if $\mathbf{v}$ lies in V_- or V_0 then $v_3 \neq 0$ and so $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ is contained in the affine chart of $\mathbb{CP}^2$ with $v_3 \neq 0$. We canonically identify this chart with $\mathbb{C}^2$ by setting $z = v_1/v_3$ and $w = v_2/v_3$. Thus a vector $(z, w) \in \mathbb{C}^2$ corresponds to $[z : w : 1]^t$ in $\mathbb{CP}^2$. Evaluating the Hermitian form at this point gives $|z|^2 + |w|^2 - 1 = (|v_1|^2 + |v_2|^2 - |v_3|^2)/|v_3|^2$. Therefore

$$\mathbf{H}^2_{\mathbb{C}} = \Big\{(z,w) \in \mathbb{C}^2 \ : \ |z|^2 + |w|^2 < 1\Big\}, \quad \partial\mathbf{H}^2_{\mathbb{C}} = \Big\{(z,w) \in \mathbb{C}^2 \ : \ |z|^2 + |w|^2 = 1\Big\}.$$

In other words, $\mathbf{H}^2_{\mathbb{C}}$ is the unit ball in $\mathbb{C}^2$ and its boundary is the unit sphere S^3.

The Bergman metric on $\mathbf{H}^2_{\mathbb{C}}$ is given in terms of the Hermitian form. Let u and v be points in $\mathbf{H}^2_{\mathbb{C}}$ and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V_- so that $\mathbb{P}\mathbf{u} = u$ and $\mathbb{P}\mathbf{v} = v$. The Bergman metric is given as a Riemannian metric ds^2 or a distance function $\rho(u,v)$ by the formulae:

$$ds^2 = \frac{-4}{\langle \mathbf{u}, \mathbf{v}\rangle^2} \det\begin{pmatrix} \langle \mathbf{u}, \mathbf{v}\rangle & \langle d\mathbf{u}, \mathbf{v}\rangle \\ \langle \mathbf{u}, d\mathbf{v}\rangle & \langle d\mathbf{u}, d\mathbf{v}\rangle \end{pmatrix}, \quad \cosh^2\left(\frac{\rho(u,v)}{2}\right) = \frac{\langle \mathbf{u}, \mathbf{v}\rangle\langle \mathbf{v}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle}.$$

The formulae for the Bergman metric are homogeneous and so the ambiguity in the choice of $\mathbf{u}$ and $\mathbf{v}$ does not matter.

Let $\mathrm{SU}(2,1)$ be the group of unimodular matrices preserving the Hermitian form H. An element A of $\mathrm{SU}(2,1)$ acts on $\mathbf{H}^2_{\mathbb{C}}$ as $A(u) = \mathbb{P}(A\mathbf{u})$ where $\mathbf{u}$ is any vector in V_- with $\mathbb{P}\mathbf{u} = u$. It is clear that scalar multiples of the identity act trivially. Since the determinant of A is 1, such a scalar multiple must be a cube root of unity. Therefore, we define $\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{\omega I : \omega^3 = 1\}$. Since the Bergman metric is given in terms of the Hermitian form, it is clear that elements of $\mathrm{SU}(2,1)$ or $\mathrm{PU}(2,1)$, act as isometries of $\mathbf{H}^2_{\mathbb{C}}$. Indeed, $\mathrm{PU}(2,1)$ is the full group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$. In what follows we choose to work with matrices in $\mathrm{SU}(2,1)$.

2.2. **Bisectors and Dirichlet domains.** We will consider subgroups of $\mathrm{SU}(2,1)$ acting on $\mathbf{H}^2_{\mathbb{C}}$ and we want to show they are discrete. We will do this by constructing a fundamental polyhedron and using the Poincaré polyhedron theorem. There are no totally geodesic real hypersurfaces in $\mathbf{H}^2_{\mathbb{C}}$ and so we must choose hypersurfaces for the sides of our polyhedra. We choose to work with bisectors. A *bisector* in $\mathbf{H}^2_{\mathbb{C}}$ is the locus of points equidistant (with respect to the Bergman metric) from a given pair of points in $\mathbf{H}^2_{\mathbb{C}}$. Suppose that these points are u and v. Choose lifts $\mathbf{u} = (u_1, u_2, u_3)^t$ and $\mathbf{v} = (v_1, v_2, v_3)^t$ to V_- so that $\langle \mathbf{u}, \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle$. Then the bisector equidistant from u and v is

$$\begin{aligned} \mathcal{B} = \mathcal{B}(u,v) &= \Big\{(z,w) \in \mathbf{H}^2_{\mathbb{C}} \ :\ \rho\big((z,w),u\big) = \rho\big((z,w),v\big)\Big\} \\ &= \Big\{(z,w) \in \mathbf{H}^2_{\mathbb{C}} \ :\ |z\bar{u}_1 + w\bar{u}_2 - \bar{u}_3| = |z\bar{v}_1 + w\bar{v}_2 - \bar{v}_3|\Big\}. \end{aligned}$$

Suppose that we are given three points u, v_1 and v_2 in $\mathbf{H}^2_{\mathbb{C}}$. If the three corresponding vectors $\mathbf{u}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ in V_- form a basis for $\mathbb{C}^{2,1}$ then the intersection $\mathcal{B}(u,v_1)\cap\mathcal{B}(u,v_2)$ is called a Giraud disc. This is a particularly nice type of bisector intersection (see Section 2.5 of [4]).

Suppose that Γ is a discrete subgroup of $\mathrm{PU}(2,1)$. Let u be a point of $\mathbf{H}^2_{\mathbb{C}}$ and write Γ_u for the stabiliser of u in Γ (that is the subgroup of Γ comprising all elements fixing u). Then the *Dirichlet domain* $D_u(\Gamma)$ for Γ with centre u is defined to be

$$D_u(\Gamma) = \Big\{v \in \mathbf{H}^2_{\mathbb{C}} \ :\ \rho(v,u) < \rho(v, A(u)) \text{ for all } A \in \Gamma - \Gamma_u\Big\}.$$

Dirichlet domains for certain cyclic groups are particularly simple.

Proposition 2.1. *Let A be a regular elliptic element of* $\mathrm{PU}(2,1)$ *of order* 3. *Then for any point u not fixed by A the Dirichlet domain $D_u(\langle A\rangle)$ for the cyclic group $\langle A\rangle$ with centre u has exactly two sides.*

Proof. Since there are only two non-trivial elements in $\langle A\rangle$, neither of which fix u, the Dirichlet domain $D_u(\langle A\rangle)$ is

$$D_u(\langle A\rangle) = \Big\{v \in \mathbf{H}^2_{\mathbb{C}} \ :\ \rho(v,u) < \rho(v,A(u)),\ \rho(v,u) < \rho(v,A^{-1}(u))\Big\}.$$

Its images under A and A^{-1} are:

$$\begin{aligned} A\big(D_u(\langle A\rangle)\big) &= \Big\{v \ :\ \rho(v,A(u)) < \rho(v,u),\ \rho(v,A(u)) < \rho(v,A^{-1}(u))\Big\}, \\ A^{-1}\big(D_u(\langle A\rangle)\big) &= \Big\{v \ :\ \rho(v,A^{-1}(u)) < \rho(v,u),\ \rho(v,A^{-1}(u)) < \rho(v,A(u))\Big\}. \end{aligned}$$

By considering the minimum of $\rho(v,u)$, $\rho(v,A(u))$, $\rho(v,A^{-1}(u))$ as v varies over $\mathbf{H}^2_{\mathbb{C}}$, it is clear these three domains are disjoint and their closures cover $\mathbf{H}^2_{\mathbb{C}}$. $\square$

Proposition 2.2 (Phillips [12]). *Let $A \in \mathrm{SU}(2,1)$ have real trace which is at least 3. Then for any $u \in \mathbf{H}^2_{\mathbb{C}}$ the bisectors $\mathcal{B}(u, A(u))$ and $\mathcal{B}(u, A^{-1}(u))$ are disjoint. Thus, the Dirichlet domain $D_u(\langle A \rangle)$ has exactly two sides.*

2.3. **The Poincaré polyhedron theorem.** Our goal is to construct the Dirichlet domain for a complex hyperbolic representation Γ of the $(3,3,n)$ triangle group with centre the fixed point of an order n elliptic map. If we use the definition of Dirichlet domain, then we need to check infinitely many inequalities. Thus, we need to use another method. This method is to construct a candidate Dirichlet domain and then use the Poincaré polyhedron theorem.

The main tool we use to show discreteness is the Poincaré polyhedron theorem. The version of this theorem that we use is for polyhedra D with a finite stabiliser; see Theorem 6.3.2 of Mostow [9] or Theorem 3.2 of Deraux, Parker, Paupert [4]. Rather than give a general statement of this theorem we will state it in the particular case we are interested in, namely Dirichlet polyhedra for reflection groups.

Let u be a point in $\mathbf{H}^2_{\mathbb{C}}$ and let Υ be a finite subgroup of $\mathrm{PU}(2,1)$ fixing u. Let $A_1, \ldots, A_n$ be a finite collection of involutions in $\mathrm{PU}(2,1)$ (so A_i^2 is the identity for each i). Suppose that no A_i fixes u. Suppose that the group Υ preserves this collection of involutions under conjugation. That is, for each A_i with $1 \leq i \leq n$ and each $P \in \Upsilon$ we suppose that $PA_iP^{-1} = A_j$ for some $1 \leq j \leq n$. Let $\mathcal{B}_i = \mathcal{B}(u, A_i(u))$ be the bisector equidistant from u and $A_i(u)$. If $P \in \Upsilon$ satisfies $PA_iP^{-1} = A_j$ then $PA_i(u) = A_j(u)$ (since $P(u) = u$) and so P maps $\mathcal{B}_i$ to $\mathcal{B}_j$. We define D to be the component of $\mathbf{H}^2_{\mathbb{C}} - \bigcup_{i=1}^n \mathcal{B}_i$ containing u and we suppose that there are points from each of the $\mathcal{B}_i$ on the boundary of D (that is, the $\mathcal{B}_i$ are not nested). This construction makes D open. Note that, by construction, Υ maps D to itself.

For each $1 \leq i \leq n$ let $s_i = \mathcal{B}_i \cap \overline{D}$. We call s_i a *side* of D. Such a side can be given a cell structure based on how it intersects other sides. We suppose that the involutions A_i for $1 \leq i \leq n$ satisfy the following conditions, and so form a *side pairing* of D:

(1) For each $1 \leq i \leq n$ the involution A_i sends s_i to itself, preserving the cell structure. The relation $A_i^2 = id$ is a called *reflection relation*.
(2) For each $1 \leq i \leq n$ we have $\overline{D} \cap A_i(\overline{D}) = s_i$ and $D \cap A(D) = \emptyset$.
(3) If v is a point in s_i and in no other side (that is v lies in the relative interior of s_i) then there is an open neighbourhood U_v of v lying in $\overline{D} \cup A_i(\overline{D})$.

Note that, unlike the case of reflection groups in constant curvature, A_i does not fix s_i pointwise. Therefore, we could have subdivided s_i into two sets (each of dimension 3) that are interchanged by A_i. In practice this would cause unnecessary complication.

Suppose that s_i and s_j are two sides with non-empty intersection. Their intersection $r = s_i \cap s_j$ is called a *ridge* of D. Since A_i preserves the cell structure of s_i, we see that $A_i(r) = s_i \cap s_k$ is another ridge of D. Applying A_k gives another ridge in s_k. Continuing in this way gives a *ridge cycle*:

$$r_1 = s_{i_0} \cap s_{i_1} \xrightarrow{A_{i_1}} r_2 = s_{i_1} \cap s_{i_2} \xrightarrow{A_{i_2}} r_3 = s_{i_2} \cap s_{i_3} \cdots .$$

Since there are finitely many Υ orbits of r_1, eventually we find

$$r_{m+1} = s_{i_m} \cap r_{i_{m+1}} = P^{-1} r_1, \quad s_{i_m} = P^{-1} s_{i_0}, \quad s_{i_{m+1}} = P^{-1} s_{i_1}.$$

for some $P \in \Upsilon$. We call $T_1 = PA_{i_m} \cdots A_{i_1}$ the *cycle transformation* associated to r_1. Clearly T_1 maps r_1 to itself. Of course, T_1 may not act as the identity on r_1 and even if it does, then it may not act as the identity on $\mathbf{H}^2_{\mathbb{C}}$. Nevertheless, we suppose T_1 has finite order ℓ. The relation $T_i^\ell = id$ is called at *cycle relation*.

In the example we are interested in, the ridge cycle is

$$r_1 = s_{i_0} \cap s_{i_1} \xrightarrow{A_{i_1}} r_2 = s_{i_1} \cap s_{i_2} \xrightarrow{P} r_1$$

and, in fact, $s_{i_2} = s_{i_0}$ and so $r_2 = r_1$. Moreover, P is an involution. Hence the cycle transformation is PA_{i_1} and the cycle relation is $(PA_{i_1})^3 = id$.

We suppose that D satisfies the *cycle condition* which means that copies of D tessellate a neighbourhood if each ridge r. Furthermore, the relevant copies of D are its preimages under suffix subwords of T^ℓ. The full statement is explained in Deraux, Parker, Paupert [4]. For brevity, we state this condition only in the special case we are interested in. Let r be a ridge and let $T = PA_i$ be its cycle transformation with cycle relation $(PA_i)^3 = id$. Let $\mathcal{C} = \{id,\, PA_i,\, (PA_i)^2\}$. Then the cycle condition states that:

(1)
$$r = \bigcap_{C \in \mathcal{C}} C^{-1}(\overline{D}).$$
(2) If $C_1,\, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ then $C_1^{-1}(D) \cap C_2^{-1}(D) = \emptyset$.
(3) If v is a point in r and in no other ridge (that is v lies in the relative interior of r) then there is an open neighbourhood U_v of v with
$$U_v \subset \bigcup_{C \in \mathcal{C}} C^{-1}(\overline{D}).$$

Finally, if two sides of D are asymptotic at a point v of $\partial \mathbf{H}^2_{\mathbb{C}}$ then there is a horoball H_v so that H_v intersects $\overline{D}$ only in facets of D containing v and H_v is preserved by the stabiliser of v in Γ. We say that H_v is a *consistent horoball* at v. In particular, if v is a fixed point of a parabolic element of Γ then there exists a consistent horoball at v.

The the Poincaré polyhedron theorem states that:

Theorem 2.3 (Theorem 6.3.2 of [9], Theorem 3.2 of [4]). *Suppose that D is a polyhedron on $\mathbf{H}^2_{\mathbb{C}}$ with sides contained in bisectors together with a side pairing. Let $\Upsilon < \mathrm{PU}(2,1)$ be a discrete group of automorphisms of D. Let Γ be the group generated by Υ and the side pairing maps. Suppose that the cycle condition holds at all ridges of D and that there is a consistent horoball at all points (if any) where sides of D are asymptotic. Then:*

(1) *Γ is discrete.*
(2) *The images of D under the cosets of Υ in Γ tessellate $\mathbf{H}^2_{\mathbb{C}}$.*
(3) *A fundamental domain for Γ may be obtained by intersecting D with a fundamental domain for Υ.*
(4) *A presentation for Γ is given as follows. The generators are a generating set for Υ together with all side pairing maps. The relations are generated by all relations in Υ, all reflection relations and all cycle relations.*

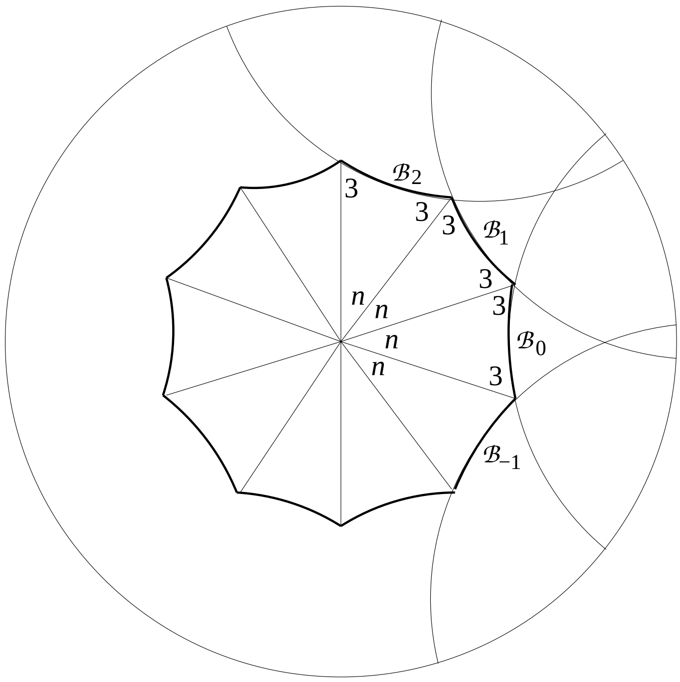

FIGURE 1. *The $2n$-gon in the hyperbolic plane made up of $2n$ copies of a $(3,3,n)$ triangle.*

3. THE GENERATORS

Consider complex reflections I_1 and I_2 in SU(2, 1) so that I_1I_2 has order n and fixes the origin o. Writing $c = \cos(\pi/n)$ and $s = \sin(\pi/n)$, we may choose I_1 and I_2 to be:

$$I_1 = \begin{bmatrix} -c & s & 0 \\ s & c & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} -c & -s & 0 \\ -s & c & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.1}$$

Note that polar vectors of I_1 and I_2 are

$$\mathbf{n}_1 = \begin{bmatrix} s \\ 1+c \\ 0 \end{bmatrix}, \quad \mathbf{n}_3 = \begin{bmatrix} -s \\ 1+c \\ 0 \end{bmatrix}.$$

We want to find I_3 so that I_1I_3 and I_2I_3 both have order 3. Let

$$\mathbf{n}_3 = \begin{bmatrix} \alpha \\ \beta \\ d-1 \end{bmatrix}$$

where $|\alpha|^2 + |\beta|^2 - (d-1)^2 = 2(d-1)$, that is $|\alpha|^2 + |\beta|^2 - d^2 = -1$. Then

$$I_3 = \begin{bmatrix} -1 + |\alpha|^2/(d-1) & \alpha\bar{\beta}/(d-1) & -\alpha \\ \beta\bar{\alpha}/(d-1) & -1 + |\beta|^2/(d-1) & -\beta \\ \bar{\alpha} & \bar{\beta} & -d \end{bmatrix}. \tag{3.2}$$

(Our notation α, β, d is supposed to reflect the fact that α and β are complex numbers and d is (positive) real number.) It is easy to check that I_3 lies in $\mathrm{SU}(2,1)$, has order 2 and polar vector $\mathbf{n}_3$.

Lemma 3.1. *Let I_1, I_2 and I_3 be given by* (3.1) *and* (3.2)*. If I_1I_3 and I_2I_3 have order 3 then*

$$c(|\alpha|^2-|\beta|^2)=d(d-1),\quad \alpha\bar\beta+\beta\bar\alpha=0. \tag{3.3}$$

Proof. The condition that I_1I_3 and I_2I_3 have order 3 is equivalent to $\mathrm{tr}(I_1I_3)=\mathrm{tr}(I_2I_3)=0$. That is

$$\frac{-c(|\alpha|^2-|\beta|^2)+s(\alpha\bar\beta+\beta\bar\alpha)}{d-1}+d=\frac{-c(|\alpha|^2-|\beta|^2)-s(\alpha\bar\beta+\beta\bar\alpha)}{d-1}+d=0.$$

The result follows directly. □

Corollary 3.2. *Using the notation of Lemma 3.1 we have*

$$\begin{aligned}
&(3.4) & |\alpha|^2 &= (d-1)(1+d+d/c)/2, \quad |\beta|^2 = (d-1)(1+d-d/c)/2,\\
&(3.5) & -(2\alpha\bar\beta)^2 &= 4|\alpha|^2|\beta|^2 = (d-1)^2\big((d+1)^2-d^2/c^2\big),\\
&(3.6) & \frac{\alpha}{\alpha^2+\beta^2} &= \frac{\bar\alpha}{|\alpha|^2-|\beta|^2} = \frac{c\bar\alpha}{d(d-1)},\\
&(3.7) & \frac{\beta}{\alpha^2+\beta^2} &= \frac{-\bar\beta}{|\alpha|^2-|\beta|^2} = \frac{-c\bar\beta}{d(d-1)}.
\end{aligned}$$

Proof. The first part follows from $|\alpha|^2+|\beta|^2=d^2-1$ and (3.3). For the second part, since $\alpha\bar\beta+\beta\bar\alpha=0$ we have $\alpha^2\bar\beta^2=-|\alpha|^2|\beta|^2$. For the last two parts multiply top and bottom by $\bar\alpha^2$ and $\bar\beta^2$ respectively, and use the previous parts to simplify. □

Corollary 3.3. *Let*

$$\iota:\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}\longmapsto\begin{bmatrix}\bar z_1\alpha^2/|\alpha|^2\\ \bar z_2\beta^2/|\beta|^2\\ \bar z_3\end{bmatrix}.$$

Then ι has order 2 and

$$\iota I_1\iota=I_2,\quad \iota I_2\iota=I_1,\quad \iota I_3\iota=I_3.$$

Proof. It is easy to see that ι^2 is the identity. A simple calculation shows $\iota(\mathbf{n}_3)=\mathbf{n}_3$. Finally, we claim $\iota(\beta\mathbf{n}_1)=\beta\mathbf{n}_2$. To see this, we use $\alpha\bar\beta=-\beta\bar\alpha$ as follows:

$$\iota:\begin{bmatrix}s\beta\\(1+c)\beta\\0\end{bmatrix}\longmapsto\begin{bmatrix}s\bar\beta\alpha^2/|\alpha|^2\\(1+c)\bar\beta\beta^2/|\beta|^2\\0\end{bmatrix}=\begin{bmatrix}-s\beta\bar\alpha\alpha/|\alpha|^2\\(1+c)\beta\bar\beta\beta/|\beta|^2\\0\end{bmatrix}=\begin{bmatrix}-s\beta\\(1+c)\beta\\0\end{bmatrix}.$$

The result follows. □

Lemma 3.4. *The group $\langle I_1,I_2,I_3\rangle$ is determined up to conjugacy by the variable d, which lies in the interval $1<d\le c/(1-c)$. Moreover, $\langle I_1,I_2,I_3\rangle$ lies in* $\mathrm{SO}(2,1)$ *when $d=c/(1-c)$.*

Proof. We have conjugated so that I_1 and I_2 have the form (3.1). We still have the freedom to conjugate by a complex reflection in the origin, that is a matrix $\mathrm{diag}(e^{i\psi}, e^{i\psi}, e^{-2i\psi})$ or by complex conjugation (the antiholomorphic isometry fixing real points of $\mathbf{H}^2_{\mathbb{C}}$). Such a conjugation would send the pair (α, β) to $(e^{3i\psi}\alpha, e^{3i\psi}\beta)$ or $(\bar{\alpha}, \bar{\beta})$ respectively. Therefore, the only conjugation invariant quantities are $|\alpha|$, $|\beta|$ and $(\alpha\bar{\beta})^2$. Using Corollary 3.2 these are completely determined by d.

Moreover, using (3.4) we see that $|\alpha|^2$ and $|\beta|^2$ are positive provided $d \geq 1$ and $d \leq c/(1-c)$. We cannot have $d = 1$ or else $\mathbf{n}_3$ is the zero vector. Thus $1 < d \leq c/(1-c)$. It is not hard to check that for all these values of the parameter d, we may define I_3 by (3.2), where α and β satisfy the conditions of Lemma 3.3.

Finally, when $d = c/(1-c)$ we have $\beta = 0$ and the entries of I_3 are all real. □

Lemma 3.5. *Let I_1, I_2 and I_3 be given by* (3.1) *and* (3.2). *Suppose I_1I_3 and I_2I_3 have order* 3. *Then $I_1I_3I_2I_3$ is elliptic if and only if $d < 3/(4s^2)$.*

Proof. Calculating directly, we see that

$$\begin{aligned}\mathrm{tr}(I_1I_3I_2I_3) &= \frac{c^2(|\alpha|^2-|\beta|^2)^2}{(d-1)^2} - \frac{s^2(\alpha\bar{\beta}+\beta\bar{\alpha})^2}{(d-1)^2} \\ &\quad + \frac{2(c^2-s^2)(d-1-|\alpha|^2-|\beta|^2)}{d-1} - 2c(|\alpha|^2-|\beta|^2) + d^2 \\ &= 4s^2 d.\end{aligned}$$

(We could have derived this using the formulae in Pratoussevitch [13].) The condition that $I_1I_3I_2I_3$ is elliptic is that $3 > \mathrm{tr}(I_1I_3I_2I_3) = 4s^2d$. □

Therefore, our parameter space for $\langle I_1, I_2, I_3\rangle$ with $I_1I_3I_2I_3$ non-elliptic is given by

$$\frac{3}{4s^2} \leq d \leq \frac{c}{1-c}. \tag{3.8}$$

Note that the condition $n > 3$ implies both $3/(4s^2) > 1$ and $c/(1-c) > 1$. For example, when $n = 4$ we have $c = s = 1/\sqrt{2}$ and our range becomes

$$3/2 \leq d \leq \sqrt{2} + 1.$$

4. The bisectors

We define a polyhedron D bounded by sides contained in $2n$ bisectors.

Definition 4.1. For $k \in \mathbb{Z}$, define the involution $A_k \in \langle I_1, I_2, I_3\rangle$ as follows:

(1) If $k = 2m$ is an even integer then $A_k = (I_2I_1)^{k/2}I_3(I_1I_2)^{k/2}$.
(2) If $k = 2m+1$ is an odd integer then $A_k = (I_2I_1)^{(k-1)/2}I_2I_3I_2(I_1I_2)^{(k-1)/2}$.

Let o be the fixed point of I_1I_2 in $\mathbf{H}^2_{\mathbb{C}}$. For all integers k, the bisector $\mathcal{B}_k$ is defined to be the bisector equidistant from o and $A_k(o)$. Note that in both cases $A_{k+2n} = A_k$ and so $\mathcal{B}_{k+2n} = \mathcal{B}_k$. This gives $2n$ bisectors $\mathcal{B}_{-n+1}$ to $\mathcal{B}_n$ and we may take the index k mod $2n$.

Note $\iota A_k \iota = A_{-k}$. The following lemma follows immediately from the definition.

Lemma 4.2. *Let $\mathcal{B}_{-n+1}$ to $\mathcal{B}_n$ be as defined in Definition 4.1. Then for each k mod $2n$ and each m mod n:*

(1) *The map* $(I_2I_1)^m$ *sends* $\mathcal{B}_k$ *to* $\mathcal{B}_{2m+k}$.
(2) *The map* $(I_2I_1)^m I_2$ *sends* $\mathcal{B}_k$ *to* $\mathcal{B}_{2m+1-k}$. *In particular,* $(I_2I_1)^k I_2$ *sends* $\mathcal{B}_k$ *to* $\mathcal{B}_{k+1}$.
(3) *The map* ι *sends* $\mathcal{B}_k$ *to* $\mathcal{B}_{-k}$. *In particular* $(I_2I_1)^m I_2\iota$ *sends* $\mathcal{B}_k$ *to* $\mathcal{B}_{2m+1+k}$.

The main result of this section is the following.

Theorem 4.3. *Let* $\mathcal{B}_{-n+1}$ *to* $\mathcal{B}_n$ *be as defined in Definition 4.1. Suppose that* $3/(4s^2) \le d \le c/(1-c)$. *Then, taking the indices mod* $2n$, *for each* k:

(1) *The bisector* $\mathcal{B}_k$ *intersects* $\mathcal{B}_{k\pm1}$ *in a Giraud disc. This Giraud disc is preserved by* $A_kA_{k\pm1}$, *which has order 3.*
(2) *The intersection of* $\mathcal{B}_k$ *with* $\mathcal{B}_{k\pm2}$ *is contained in the halfspace bounded by* $\mathcal{B}_{k\pm1}$ *not containing* o.
(3) *The bisector* $\mathcal{B}_k$ *does not intersect* $\mathcal{B}_{k\pm\ell}$ *for* $3 \le \ell \le n$. *Moreover, the boundaries of these bisectors are disjoint except for when* $\ell = 3$ *and* $d = 3/(4s^2)$ *in which case the boundaries intersect in a single point, which is a parabolic fixed point.*

As a corollary to this theorem, we can use the Poincaré polyhedron theorem to prove the 'if' part of Theorem 1.6.

Corollary 4.4. *Let* A_{-n+1} *to* A_n *and* $\mathcal{B}_{-n+1}$ *to* $\mathcal{B}_n$ *be as in Theorem 4.3. Suppose that* $3/(4s^2) \le d \le c/(1-c)$. *Let* D *be the polyhedron in* $\mathbf{H}^2_{\mathbb{C}}$ *containing* o *and bounded by* $\mathcal{B}_{-n+1}$ *to* $\mathcal{B}_n$. *Then the maps* A_{-n+1} *to* A_n *form a side paring for* D *that satisfies the conditions of the Poincaré polyhedron theorem, Theorem 2.3. In particular,* $\langle I_1, I_2, I_3\rangle$ *is a discrete and faithful representation of* $\Delta_{3,3,n}$.

Proof. Since A_k is an involution, it is clear that the $\{A_k\}$ form a side pairing for D. Now consider the ridge $r_k = \mathcal{B}_k \cap \mathcal{B}_{k+1}$. Applying either of the side pairing maps A_k or A_{k+1} sends this ridge to itself. We then apply $P_k = (I_2I_1)^k I_2$ to obtain the cycle transformation P_kA_k. When k is even:

$$P_kA_k = (I_2I_1)^k I_2 (I_2I_1)^{k/2} I_3 (I_1I_2)^{k/2} = (I_2I_1)^{k/2} I_2 I_3 (I_1I_2)^{k/2};$$

and when k is odd:

$$\begin{aligned} P_kA_k &= (I_2I_1)^k I_2 (I_2I_1)^{(k-1)/2} I_2 I_3 I_2 (I_1I_2)^{(k-1)/2} \\ &= (I_2I_1)^{(k+1)/2} I_3 I_1 (I_1I_2)^{(k+1)/2}. \end{aligned}$$

In both cases, $P_kA_k = A_kA_{k+1}$ which has order 3. Therefore, we have local tessellation around all the ridges of D using the argument of Proposition 2.1.

All the other sides of D are disjoint, apart from when $d = 3/(4s^2)$, in which case $\mathcal{B}_k$ and $\mathcal{B}_{k\pm3}$ are asymptotic at a point of $\partial\mathbf{H}^2_{\mathbb{C}}$. This point is a parabolic fixed point, as required.

Finally, each side yields the reflection relation A_k^2, which is conjugate to I_3^2. The cycle relations give $(P_kA_k)^3$, which are conjugate to $(I_2I_3)^3$ when k is even and $(I_3I_1)^3$ when k is odd. In addition we have the relations from $\Upsilon = \langle I_1, I_2\rangle$, which are I_1^2, I_2^2 and $(I_1I_2)^n$. From the Poincaré theorem, all other relations may be deduced from these. Thus $\langle I_1, I_2, I_3\rangle$ is a faithful representation of $\Delta_{3,3,n}$. □

Write $c_k = \cos(k\pi/n)$ and $s_k = \sin(k\pi/n)$. Then

$$(I_2I_1)^m = \begin{bmatrix} c_{2m} & -s_{2m} & 0 \\ s_{2m} & c_{2m} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (I_2I_1)^m I_2 \begin{bmatrix} -c_{2m+1} & -s_{2m+1} & 0 \\ -s_{2m+1} & c_{2m+1} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We have

$$(I_2I_1)^mI_3(o)=\begin{bmatrix}-c_{2m}\alpha+s_{2m}\beta\\ -s_{2m}\alpha-c_{2m}\beta\\ -d\end{bmatrix},\quad (I_1I_2)^mI_3(o)=\begin{bmatrix}-c_{2m}\alpha-s_{2m}\beta\\ s_{2m}\alpha-c_{2m}\beta\\ -d\end{bmatrix}.$$

Also

$$(I_2I_1)^mI_2I_3(o)=\begin{bmatrix}c_{2m+1}\alpha+s_{2m+1}\beta\\ s_{2m+1}\alpha-c_{2m+1}\beta\\ d\end{bmatrix},\ (I_1I_2)^mI_1I_3(o)=\begin{bmatrix}c_{2m+1}\alpha-s_{2m+1}\beta\\ -s_{2m+1}\alpha-c_{2m+1}\beta\\ d\end{bmatrix}.$$

We begin by proving Theorem 4.3 (1).

Proposition 4.5. *Suppose that $d\geq 3/(4s^2)$. For each $-n+1\leq k\leq n$ the bisectors $\mathcal{B}_k$ and $\mathcal{B}_{k+1}$ (with indices taken mod $2n$) intersect in $\mathbf{H}^2_{\mathbb{C}}$ in a Giraud disc. This Giraud disc is preserved by $(I_2I_1)^{k/2}(I_2I_3)(I_1I_2)^{k/2}$ when k is even and $(I_2I_1)^{(k+1)/2}(I_3I_1)(I_1I_2)^{(k+1)/2}$ when k is odd.*

Proof. Using Lemma 4.2 we need only consider $k=0$ and $k=1$. The bisectors $\mathcal{B}_0$ and $\mathcal{B}_1$ are equidistant from o and from $I_3(o)=I_3I_2(o)$ and from $I_2I_3(o)$ respectively. Since the map I_2I_3 has order 3, the Dirichlet domain with centre o for the cyclic group $\langle I_2I_3\rangle$ only contains faces contained in these two bisectors. The intersection is a Giraud disc invariant under powers of I_2I_3 by construction. □

Next we prove Theorem 4.3 (3) in the case where $\ell=2m+1$ is odd.

Proposition 4.6. *Suppose that $3/(4s^2)\leq d\leq c/(1-c)$. For each $-n+1\leq k\leq n$ and $1\leq m\leq (n-1)/2$ the bisectors $\mathcal{B}_k$ and $\mathcal{B}_{k\pm(2m+1)}$ (with indices taken mod $2n$) do not intersect in $\mathbf{H}^2_{\mathbb{C}}$. Moreover, their closures intersect on $\partial\mathbf{H}^2_{\mathbb{C}}$ if and only if $d=3/(4s^2)$ and $m=1$. In the latter case, the closures intersect in a unique point, which is a parabolic fixed point.*

Proof. Using Lemma 4.2 we need only consider $\mathcal{B}_0$ and $\mathcal{B}_{2m+1}$. These bisectors are equidistant from o and $I_3(o)=I_3I_2(I_1I_2)^m(o)$ and from $(I_2I_1)^mI_2I_3(o)$ respectively. Consider the Dirichlet domain with centre o for the cyclic group $\langle (I_2I_1)^mI_2I_3\rangle$. We claim that this Dirichlet domain has exactly two sides and these sides are disjoint. To do so, we use Phillips's theorem, Proposition 2.2.

A brief calculation shows that

$$\begin{aligned}\mathrm{tr}\big((I_2I_1)^mI_2I_3\big) &= -c_{2m+1}\frac{|\alpha|^2-|\beta|^2}{d-1}-s_{2m+1}\frac{\alpha\bar\beta+\beta\bar\alpha}{d-1}+d\\ &= \frac{d(c-c_{2m+1})}{c}=\frac{2ds_{m+1}s_m}{c}.\end{aligned}$$

When $1\leq m\leq n-2$ we have

$$s_ms_{m+1}\geq ss_2=2s^2c$$

with equality if and only if $m=1$. Therefore

$$\mathrm{tr}\big((I_2I_1)^mI_2I_3\big)=2ds_{m+1}s_m/c\geq 4ds^2$$

with equality if and only if $m=1$. Hence when $4ds^2\geq 3$ we have $(I_2I_1)^mI_2I_3$ is non-elliptic with real trace, and is loxodromic unless $m=1$ and $d=3/(4s^2)$. By Phillips's theorem be see that any Dirichlet domain for $\langle (I_2I_1)^mI_2I_3\rangle$ has two faces and these faces do not intersect in $\mathbf{H}^2_{\mathbb{C}}$.

In fact, when $d = 3/(4s^2)$ and $m = 1$ the bisectors $\mathcal{B}_0$ and $\mathcal{B}_3$ are asymptotic on the boundary of $\mathbf{H}^2_{\mathbb{C}}$ at the (parabolic) fixed point of $I_2 I_1 I_2 I_3$. □

Proposition 4.7. (i) *Suppose $p = [z, w, 1]^t$ lies on $\mathcal{B}_{2\ell} \cap \mathcal{B}_{-2\ell}$. Then for some angles θ, ϕ we have*

$$\begin{aligned} z &= \frac{s_{2\ell}\alpha(\cos(\theta)e^{i\phi} + d) + c_{2\ell}\beta i \sin(\theta)e^{i\phi}}{c_{2\ell}s_{2\ell}(|\alpha|^2 - |\beta|^2)}, \\ w &= \frac{-s_{2\ell}\beta(\cos(\theta)e^{i\phi} + d) + c_{2\ell}\alpha i \sin(\theta)e^{i\phi}}{c_{2\ell}s_{2\ell}(|\alpha|^2 - |\beta|^2)} \end{aligned}$$

(ii) *Suppose $p = [z, w, 1]^t$ lies on $\mathcal{B}_{2\ell+1} \cap \mathcal{B}_{-2\ell-1}$. Then for some angles θ, ϕ we have*

$$\begin{aligned} z &= \frac{s_{2\ell+1}\alpha(\cos(\theta)e^{i\phi} + d) + c_{2\ell+1}\beta i \sin(\theta)e^{i\phi}}{c_{2\ell+1}s_{2\ell+1}(|\alpha|^2 - |\beta|^2)}, \\ w &= \frac{s_{2\ell+1}\beta(\cos(\theta)e^{i\phi} + d) - c_{2\ell+1}\alpha i \sin(\theta)e^{i\phi}}{c_{2\ell+1}s_{2\ell+1}(|\alpha|^2 - |\beta|^2)}. \end{aligned}$$

Proof. First consider the bisector intersection from (i). Then z and w satisfy

$$\begin{aligned} 1 &= \left| z(-c_{2\ell}\bar{\alpha} - s_{2\ell}\bar{\beta}) + w(s_{2\ell}\bar{\alpha} - c_{2\ell}\bar{\beta}) + d \right|, \\ 1 &= \left| z(-c_{2\ell}\bar{\alpha} + s_{2\ell}\bar{\beta}) + w(-s_{2\ell}\bar{\alpha} - c_{2\ell}\bar{\beta}) + d \right|. \end{aligned}$$

Expanding out, adding and subtracting yields

$$\begin{aligned} 1 &= \left| zc_{2\ell}\bar{\alpha} + wc_{2\ell}\bar{\beta} - d \right|^2 + \left| -zs_{2\ell}\bar{\beta} + ws_{2\ell}\bar{\alpha} \right|^2, \\ 0 &= 2\mathrm{Re}\Big(\big(zc_{2\ell}\bar{\alpha} + wc_{2\ell}\bar{\beta} - d\big)\big(-\bar{z}s_{2\ell}\beta + \bar{w}s_{2\ell}\alpha\big) \Big). \end{aligned}$$

Thus we can write

$$\begin{aligned} zc_{2\ell}\bar{\alpha} + wc_{2\ell}\bar{\beta} - d &= \cos(\theta)e^{i\phi}, \\ -zs_{2\ell}\bar{\beta} + ws_{2\ell}\bar{\alpha} &= i\sin(\theta)e^{i\phi}. \end{aligned}$$

Inverting these equations yields

$$\begin{aligned} z &= \frac{s_{2\ell}\bar{\alpha}(\cos(\theta)e^{i\phi} + d) - c_{2\ell}\bar{\beta} i \sin(\theta)e^{i\phi}}{c_{2\ell}s_{2\ell}(\bar{\alpha}^2 + \bar{\beta}^2)}, \\ w &= \frac{s_{2\ell}\bar{\beta}(\cos(\theta)e^{i\phi} + d) + c_{2\ell}\bar{\alpha} i \sin(\theta)e^{i\phi}}{c_{2\ell}s_{2\ell}(\bar{\alpha}^2 + \bar{\beta}^2)}. \end{aligned}$$

Finally use (3.6) and (3.7) to obtain the required equations.

For the second bisector intersection, we have

$$\begin{aligned} 1 &= \left| z(c_{2\ell+1}\bar{\alpha} - s_{2\ell+1}\bar{\beta}) + w(-s_{2\ell+1}\bar{\alpha} - c_{2\ell+1}\bar{\beta}) - d \right|^2, \\ 1 &= \left| z(c_{2\ell+1}\bar{\alpha} + s_{2\ell+1}\bar{\beta}) + w(s_{2\ell+1}\bar{\alpha} - c_{2\ell+1}\bar{\beta}) - d \right|^2. \end{aligned}$$

Expanding out, adding and subtracting yields

$$\begin{aligned} 1 &= \left| zc_{2\ell+1}\bar{\alpha} - wc_{2\ell+1}\bar{\beta} - d \right|^2 + \left| zs_{2\ell+1}\bar{\beta} + ws_{2\ell+1}\bar{\alpha} \right|^2, \\ 0 &= 2\mathrm{Re}\Big(\big(zc_{2\ell+1}\bar{\alpha} - wc_{2\ell+1}\bar{\beta} - d\big)\big(\bar{z}s_{2\ell+1}\beta + \bar{w}s_{2\ell+1}\alpha\big) \Big). \end{aligned}$$

So once again we have

$$\begin{aligned} zc_{2\ell+1}\bar{\alpha} - wc_{2\ell+1}\bar{\beta} - d &= \cos(\theta)e^{i\phi}, \\ zs_{2\ell+1}\bar{\beta} + ws_{2\ell+1}\bar{\alpha} &= -i\sin(\theta)e^{i\phi}. \end{aligned}$$

Thus

$$\begin{aligned} z &= \frac{s_{2\ell+1}\bar{\alpha}(\cos(\theta)e^{i\phi}+d) - c_{2\ell+1}\bar{\beta}i\sin(\theta)e^{i\phi}}{c_{2\ell+1}s_{2\ell+1}(\bar{\alpha}^2+\bar{\beta}^2)}, \\ w &= \frac{-s_{2\ell+1}\bar{\beta}(\cos(\theta)e^{i\phi}+d) - c_{2\ell+1}\bar{\alpha}i\sin(\theta)e^{i\phi}}{c_{2\ell+1}s_{2\ell+1}(\bar{\alpha}^2+\bar{\beta}^2)}. \end{aligned}$$

Again using (3.6) and (3.7) we get the result. □

We can now prove Theorem 4.3 (3) in the case where $\ell = 2m$ is even.

Proposition 4.8. *Suppose that* $3/(4s^2) \le d \le c/(1-c)$. *For each* $-n+1 \le k \le n$ *and* $2 \le m \le n/2$ *the bisectors* $\mathcal{B}_k$ *and* $\mathcal{B}_{k\pm 2m}$ *(with indices taken mod* $2n$*) do not intersect in complex hyperbolic space.*

Proof. Using Lemma 4.2 we need only consider $\mathcal{B}_m$ and $\mathcal{B}_{-m}$ where $2 \le m \le n/2$.

Using Proposition 4.7 we see that an intersection point $p = [z, w, 1]^t$ of $\mathcal{B}_m$ and $\mathcal{B}_{-m}$ must satisfy:

$$\begin{aligned} z &= \frac{s_m\alpha(\cos(\theta)e^{i\phi}+d) + c_m\beta i\sin(\theta)e^{i\phi}}{c_m s_m(|\alpha|^2-|\beta|^2)}, \\ w &= \pm\frac{-s_m\beta(\cos(\theta)e^{i\phi}+d) + c_m\alpha i\sin(\theta)e^{i\phi}}{c_m s_m(|\alpha|^2-|\beta|^2)}. \end{aligned}$$

We claim that $|z|^2+|w|^2 \ge 1$ and so such a point does not lie in $\mathbf{H}^2_{\mathbb{C}}$. We have

$$\begin{aligned} & c_m^2 s_m^2(|\alpha|^2-|\beta|^2)^2(|z|^2+|w|^2-1) \\ &= \Big|s_m\alpha(\cos(\theta)e^{i\phi}+d) + c_m\beta i\sin(\theta)e^{i\phi}\Big|^2 \\ &\quad + \Big|-s_m\beta(\cos(\theta)e^{i\phi}+d) + c_m\alpha i\sin(\theta)e^{i\phi}\Big|^2 - c_m^2 s_m^2(|\alpha|^2-|\beta|^2)^2 \\ &= s_m^2(|\alpha|^2+|\beta|^2)\big(\cos^2(\theta)+2d\cos(\theta)\cos(\phi)+d^2\big) \\ &\quad + c_m s_m(\beta\bar{\alpha}-\alpha\bar{\beta})\big(2i\cos(\theta)\sin(\theta)+2di\sin(\theta)\cos(\phi)\big) \\ &\quad + c_m^2(|\alpha|^2+|\beta|^2)\sin^2(\theta) - c_m^2 s_m^2\big(|\alpha|^2+|\beta|^2\big)^2 + c_m^2 s_m^2|\alpha\bar{\beta}-\beta\bar{\alpha}|^2 \\ &= s_m^2(d^2-1)\big(\cos^2(\theta)+2d\cos(\theta)\cos(\phi)+d^2\big) \\ &\quad \pm c_m s_m|\beta\bar{\alpha}-\alpha\bar{\beta}|\big(2\cos(\theta)\sin(\theta)+2d\sin(\theta)\cos(\phi)\big) \\ &\quad + c_m^2(d^2-1)\sin^2(\theta) - c_m^2 s_m^2\big(d^2-1\big)^2 + c_m^2 s_m^2|\alpha\bar{\beta}-\beta\bar{\alpha}|^2 \\ &= \Big(\cos(\theta)\sin(\theta)+d\sin(\theta)\cos(\phi) \pm c_m s_m|\alpha\bar{\beta}-\beta\bar{\alpha}|\Big)^2 + d^2\sin^2(\theta)\sin^2(\phi) \\ &\quad + \Big(s_m^2(d^2-1)-\sin^2(\theta)\Big)\Big(\cos^2(\theta)+2d\cos(\theta)\cos(\phi)+d^2-c_m^2(d^2-1)\Big) \\ &\ge \Big(s_m^2(d^2-1)-\sin^2(\theta)\Big)\Big(\cos^2(\theta)+2d\cos(\theta)\cos(\phi)+d^2-c_m^2(d^2-1)\Big). \end{aligned}$$

Therefore it is sufficient to prove

$$0 < s_m^2(d^2-1) - \sin^2(\theta), \tag{4.1}$$

$$0 < \cos^2(\theta)+2d\cos(\theta)\cos(\phi)+d^2-c_m^2(d^2-1).. \tag{4.2}$$

In order to prove these inequalities we need to use the lower bound on d. Using $m \geq 2$ and $d \geq 3/(4s^2)$ we have

$$(1-c_m)d \geq (1-c_2)d = 2s^2 d \geq 3/2. \tag{4.3}$$

We also use $s_m^2 = 1 - c_m^2 = (1-c_m)(1+c_m)$ and $c_m \geq 0$ (the latter uses $m \leq n/2$).

First, we consider (4.1):

$$\begin{aligned} s_m^2(d^2-1) - \sin^2(\theta) &= \frac{1+c_m}{1-c_m}\big((1-c_m)d\big)^2 - 2 + c_m^2 + \cos^2(\theta) \\ &\geq \big((1-c_m)d)\big)^2 - 2 \\ &\geq 1/4, \end{aligned}$$

where the last inequality follows from (4.3). This proves (4.1).

Now consider (4.2):

$$\begin{aligned} &\cos^2(\theta) + 2d\cos(\theta)\cos(\phi) + d^2 - c_m^2(d^2-1) \\ = \ & \frac{\big(d(1-c_m) + \cos(\theta)\cos(\phi)\big)^2 + \cos^2(\theta)\sin^2(\phi)}{1-c_m} \\ & + \frac{c_m}{1-c_m}\Big(\big(d(1-c_m)\big)^2 - \cos^2(\theta)\Big) + c_m^2 \\ \geq \ & \frac{c_m}{1-c_m}\Big(9/4 - \cos^2(\theta)\Big) \\ > \ & 0. \end{aligned}$$

Again we used (4.3). This proves (4.2) and so establishes the result. □

In Proposition 4.8 we excluded the case $m = \pm 1$. It is not hard to show that the fixed point of $I_3I_1I_2I_3$ (that is $I_3(o)$) lies on $\mathcal{B}_1 \cap \mathcal{B}_{-1}$. To see this, observe that

$$\rho(I_3(o), I_jI_3(o)) = \rho(I_3(o), I_3I_jI_3(o)) = \rho(o, I_jI_3(o)).$$

The first equality follows using $I_jI_3(o) = I_jI_3I_j(o) = I_3I_jI_3(o)$ and the second since I_3 is an isometry.

Therefore, we need to investigate $\mathcal{B}_1 \cap \mathcal{B}_{-1}$, more carefully. By symmetry, this comment also applies to the intersection of $\mathcal{B}_k$ and $\mathcal{B}_{k\pm 2}$. Finally, we prove Theorem 4.3 (2).

Proposition 4.9. *Suppose that $3/(4s^2) \leq d \leq c/(1-c)$. For each $-n+1 \leq k \leq n$ all points of $\mathcal{B}_k \cap \mathcal{B}_{k\pm 2}$ lie in the halfspace bounded by $\mathcal{B}_{k\pm 1}$ not containing o.*

Proof. Using Lemma 4.2 as before, it suffices to consider $\mathcal{B}_1$ and $\mathcal{B}_{-1}$. We need to show that all points of $\mathcal{B}_1 \cap \mathcal{B}_{-1}$ lie in the halfspace closer to $I_3(o)$ than to o.

Suppose that $p = [z, w, 1]^t$ lies on $\mathcal{B}_1 \cap \mathcal{B}_{-1}$. Using Proposition 4.7 (ii) with $m = 0$, and using (3.3) to write $c(|\alpha|^2 - |\beta|^2) = d(d-1)$, we find

$$z = \frac{s\alpha(\cos(\theta)e^{i\phi} + d) + c\beta i\sin(\theta)e^{i\phi}}{sd(d-1)}, \tag{4.4}$$

$$w = \frac{s\beta(\cos(\theta)e^{i\phi} + d) - c\alpha i\sin(\theta)e^{i\phi}}{sd(d-1)}. \tag{4.5}$$

Note that we used equation (3.3) to simplify the denominator.

The point $p = [z, w, 1]^t$ lies in the halfspace closer to $I_3(o)$ than to o if and only if $1 > |z\bar{\alpha} + w\bar{\beta} - d|$. We want to give this inequality in terms of θ, ϕ and d. Suppose z and w satisfy (4.4) and (4.5) and consider $z\bar{\alpha} + w\bar{\beta} - d$:

$$\begin{aligned} z\bar{\alpha} + w\bar{\beta} - d &= \frac{s|\alpha|^2(\cos(\theta)e^{i\phi} + d) + c\beta\bar{\alpha}i\sin(\theta)e^{i\phi}}{sd(d-1)} \\ &\quad + \frac{s|\beta|^2(\cos(\theta)e^{i\phi} + d) - c\alpha\bar{\beta}i\sin(\theta)e^{i\phi}}{sd(d-1)} - d \\ &= \frac{s(|\alpha|^2 + |\beta|^2)\cos(\theta)e^{i\phi}}{sd(d-1)} \pm \frac{2c|\alpha\bar{\beta}|\sin(\theta)e^{i\phi}}{sd(d-1)} + \frac{s(|\alpha|^2 + |\beta|^2)d}{sd(d-1)} - d \\ &= \frac{s(d^2-1)\cos(\theta)e^{i\phi}}{sd(d-1)} \pm \frac{2c|\alpha\bar{\beta}|\sin(\theta)e^{i\phi}}{sd(d-1)} + \frac{s(d^2-1)d}{sd(d-1)} - d \\ &= \frac{(d+1)\cos(\theta)e^{i\phi}}{d} \pm \frac{\sqrt{c^2(d+1)^2 - d^2}\,\sin(\theta)e^{i\phi}}{sd} + 1. \end{aligned}$$

Therefore

$$\begin{aligned} &|z\bar{\alpha} + w\bar{\beta} - d|^2 - 1 \\ &= \frac{(d+1)^2\cos^2(\theta)}{d^2} + \frac{c^2(d+1)^2\sin^2(\theta)}{s^2d^2} - \frac{\sin^2(\theta)}{s^2} \\ &\quad \pm \frac{2(d+1)\sqrt{c^2(d+1)^2 - d^2}\,\cos(\theta)\sin(\theta)}{sd^2} \\ &\quad + \frac{2(d+1)\cos(\theta)\cos(\phi)}{d} \pm \frac{2\sqrt{c^2(d+1)^2 - d^2}\,\sin(\theta)\cos(\phi)}{sd}. \end{aligned}$$

Arguing as in the proof of Proposition 4.8 we have

$$\begin{aligned} &|z|^2 + |w|^2 - 1 \\ &= \left|\frac{s\alpha(\cos(\theta)e^{i\phi} + d) + c\beta i\sin(\theta)e^{i\phi}}{sd(d-1)}\right|^2 + \left|\frac{s\beta(\cos(\theta)e^{i\phi} + d) - c\alpha i\sin(\theta)e^{i\phi}}{sd(d-1)}\right|^2 - 1 \\ &= \frac{s^2(|\alpha|^2 + |\beta|^2)|\cos(\theta)e^{i\phi} + d|^2}{s^2d^2(d-1)^2} + \frac{c^2(|\alpha|^2 + |\beta|^2)\sin^2(\theta)}{s^2d^2(d-1)^2} - 1 \\ &\quad + \frac{isc(\beta\bar{\alpha} - \alpha\bar{\beta})\big(2\cos(\theta)\sin(\theta) + 2d\sin(\theta)\cos(\phi)\big)}{s^2d^2(d-1)^2} \\ &= \frac{(d+1)\cos^2(\theta)}{d^2(d-1)} + \frac{2(d+1)\cos(\theta)\cos(\phi)}{d(d-1)} + \frac{d+1}{d-1} + \frac{c^2(d+1)\sin^2(\theta)}{s^2d^2(d-1)} - 1 \\ &\quad \pm \frac{2\sqrt{c^2(d+1)^2 - d^2}\,\cos(\theta)\sin(\theta)}{sd^2(d-1)} \pm \frac{2\sqrt{c^2(d+1)^2 - d^2}\,\sin(\theta)\cos(\phi)}{sd(d-1)} \\ &= \frac{2}{d-1} + \frac{(d+1)\cos^2(\theta)}{d^2(d-1)} + \frac{c^2(d+1)\sin^2(\theta)}{s^2d^2(d-1)} \pm \frac{2\sqrt{c^2(d+1)^2 - d^2}\,\cos(\theta)\sin(\theta)}{sd^2(d-1)} \\ &\quad + \frac{2(d+1)\cos(\theta)\cos(\phi)}{d(d-1)} \pm \frac{2\sqrt{c^2(d+1)^2 - d^2}\,\sin(\theta)\cos(\phi)}{sd(d-1)}. \end{aligned}$$

Now we eliminate $\cos(\phi)$ using the equation for $|z\bar{\alpha}+w\bar{\beta}-d|^2$ derived above:

$$
\begin{aligned}
&|z|^2+|w|^2-1\\
&= \frac{1}{d-1}\Big(|z\bar{\alpha}+w\bar{\beta}-d|^2-1\Big)+\frac{2\cos^2(\theta)}{d-1}+\frac{2\sin^2(\theta)}{d-1}\\
&\quad +\frac{(d+1)\cos^2(\theta)}{d^2(d-1)}+\frac{c^2(d+1)\sin^2(\theta)}{s^2d^2(d-1)}\pm\frac{2\sqrt{c^2(d+1)^2-d^2}\,\cos(\theta)\sin(\theta)}{sd^2(d-1)}\\
&\quad -\frac{(d+1)^2\cos^2(\theta)}{d^2(d-1)}-\frac{c^2(d+1)^2\sin^2(\theta)}{s^2d^2(d-1)}+\frac{\sin^2(\theta)}{s^2(d-1)}\\
&\quad \mp\frac{2(d+1)\sqrt{c^2(d+1)^2-d^2}\,\cos(\theta)\sin(\theta)}{sd^2(d-1)}\\
&= \frac{1}{d-1}\Big(|z\bar{\alpha}+w\bar{\beta}-d|^2-1\Big)\\
&\quad +\frac{1}{d}\left(\cos(\theta)\mp\frac{\sqrt{c^2(d+1)^2-d^2}\,\sin(\theta)}{s(d-1)}\right)^2+\frac{(4s^2d-3)\sin^2(\theta)}{s^2(d-1)^2}.
\end{aligned}
$$

Since the last two terms are non-negative, all points $p=[z,w,1]^t$ with z and w given by (4.4) and (4.5) and that satisfy $|z|^2+|w|^2<1$ must also satisfy $|z\bar{\alpha}+w\bar{\beta}-d|<1$. Geometrically, this means that all points in $\mathbf{H}^2_{\mathbb{C}}$ that are on $\mathcal{B}_1\cap\mathcal{B}_{-1}$ are in the halfspace closer to $I_3(o)$ than to o. This proves the result. □

This completes the proof of Theorem 4.3.

Acknowledgments. This research was carried out while John Parker and Jieyan Wang were visiting the Department of Mathematics at Hunan University. We would like to thank Yueping Jiang for making this visit possible.

References

[1] Ulrich Brehm, *The shape invariant of triangles and trigonometry in two-point homogeneous spaces*, Geom. Dedicata **33** (1990) 59–76.

[2] Martin Deraux, *On spherical CR uniformization of 3-manifolds*, `arXiv:1410.0659`.

[3] Martin Deraux & Elisha Falbel, *Complex hyperbolic geometry of the figure eight knot.* `arXiv:1303.7096`.

[4] Martin Deraux, John R. Parker & Julien Paupert, *New non-arithmetic complex hyperbolic lattices.* `arXiv:1401.0308`.

[5] Elisha Falbel, Pierre-Vincent Koseleff & Fabrice Rouillier, *Representations of fundamental groups of 3-manifolds into* $\mathrm{PGL}(3,\mathbb{C})$*: Exact computations in low complexity*, `arXiv:1307.6697`.

[6] Elisha Falbel & Jieyan Wang, *Branched spherical CR structures on the complement of the figure eight knot*, Michigan Math. J. **63** 2014 635–667.

[7] William M. Goldman & John R. Parker, *Complex hyperbolic ideal triangle groups*, J. reine angewandte Math. **425** (1992) 71-86.

[8] Carlos H. Grossi, *On the type of triangle groups*, Geom. Dedicata **130** (2007) 137–148.

[9] G. D. Mostow, *On a remarkable class of polyhedra in complex hyperbolic space*, Pacific J. Math. **86** (1980) 171–276.

[10] John R. Parker & Pierre Will, *Complex hyperbolic free groups with many parabolic elements*, To appear in 'Groups, Geometry and Dynamics', Contemporary Mathematics, AMS.

[11] John R. Parker & Pierre Will, *A complex hyperbolic Riley slice*, in preparation.

[12] Mark B. Phillips, *Dirichlet polyhedra for cyclic groups in complex hyperbolic space*, Proc. Amer. Math. Soc. **115** (1992) 221–228.

[13] Anna Pratoussevitch, *Traces in complex hyperbolic triangle groups*, Geom. Dedicata **111** (2005) 159–185.

[14] Richard Evan Schwartz, *Ideal triangle groups, dented tori, and numerical analysis*, Annals of Mathematics **153** (2001) 533–598.

[15] Richard Evan Schwartz, *Degenerating the complex hyperbolic ideal triangle groups.* Acta Math. **186** (2001) 105–154.

[16] Richard Evan Schwartz, *Complex hyperbolic triangle groups*, Proceedings of the International Congress of Mathematicians (2002) Volume **1**: Invited Lectures, 339–350.

[17] Richard Evan Schwartz, *A better proof of the Goldman-Parker conjecture*, Geometry and Topology **9** (2005) 1539–1602.

[18] Richard Evan Schwartz, Spherical CR Geometry and Dehn Surgery, Annals of Maths. Studies **165**, 2007.

[19] James M. Thompson, *Complex Hyperbolic Triangle Groups.* PhD Thesis, Durham University, 2010.

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