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NOTES ON COMPLEX HYPERBOLIC TRIANGLE GROUPS

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ABSTRACT. We first demonstrate a family of isomorphisms between complex hyperbolic triangle groups and outline a systematic approach classifying the groups. Then we describe conditions that determine the discreteness of certain groups, in particular we prove a slightly weaker version of a conjecture given by Schwartz in [11]. Finally we collect together a list of known discrete triangle groups and propose some good candidates for discrete groups.

1. INTRODUCTION

1.1. **Deformed $\mathbb{R}$-Fuchsian groups.** The setting of this paper is 2 dimensional, complex hyperbolic space, that is the complex projectivisation of the negative vectors in $\mathbb{C}^{2,1}$ with respect to the Hermitian inner product of signature $(2, 1)$. For further background see [5].

In the real hyperbolic case one of the simplest types of discrete group acting on $\mathbf{H}^2_{\mathbb{R}}$ are the Fuchsian triangle groups. To define a Fuchsian triangle group we chose a geodesic triangle in the hyperbolic plane and then look at the group generated by the anti-holomorphic involutions in the edges of the triangle. If we chose the triangle to have angles π/p, π/q and π/r (with $p, q, r \in \mathbb{N} \cup \{\infty\}$ and $1/p + 1/q + 1/r < 1$), then the group will be discrete and have presentation

$$(p, q, r) = \langle I_1, I_2, I_3 | I_i^2, (I_2I_3)^p, (I_3I_1)^q, (I_1I_2)^r\rangle.$$

In the real hyperbolic case angle two triangles with the same angles are isometric, so the representation of (p, r, q) is independent of the triangle we chose, since there is always conjugate one representation to the another.

We can carry out a similar construction in $\mathbf{H}^2_{\mathbb{C}}$, i.e. pick a triangle with angles π/p, π/q and π/r (again with $1/p + 1/q + 1/r < 1$) and then look at the group generated by the order two complex reflections in the complex geodesics between pairs of vertices of the triangle. This will give a representation of (p, q, r) in $\mathrm{SU}(2, 1)$. For a triple (p, q, r), in $\mathbf{H}^2_{\mathbb{C}}$, the space of non-isometric triangles with angles π/p, π/q and π/r, is of one real dimension. This leads to a one real dimensional family of non-conjugate triangle representations for the (p, q, r) triangle group in $\mathrm{SU}(2, 1)$. This is the deformation space of the (p, q, r) triangle group. The short words I_i and I_iI_k will still satisfy the group relations of the Fuchsian (p, q, r) triangle groups but extra relations for longer words may occur as the group is deformed. In particular we can choose a generating triangle so that the word $I_1I_3I_2I_3$ is regular elliptic of order n (for sufficiently large n). Fixing the order of $I_1I_3I_2I_3$ specifies a single point

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in the deformation space and uniquely determines our triangle group, we call this the $(p, q, r; n)$ group.

1.2. **Notation.** Triangle groups are generated by three complex involutions (order 2 complex reflections) I_1, I_2, I_3. These involutions fix two real dimensional, totally geodesics subspaces called $\mathbb{C}$-lines. Compositions of reflections, for example $I_1I_2I_1$ will be written as I_{121} or simply 121, we call this *word notation* and such a string of numbers is called a *word*.

When a word is palindromic, the corresponding isometry is a complex reflection, since it can be thought of as conjugate to one of the generating reflections. For example, the complex reflection $I_{121} = I_1I_2I_1^{-1}$. The composition of two word isometries is simply the composition of the words, e.g.

$$I_{121}I_{131} = I_{121131}.$$

Recall that the reflections are of order two so $I_jI_j = Id$, in word notation this condition implies that any double letters in a word are deleted, so the above example would become,

$$I_{121}I_{131} = I_{1231}.$$

The other group relations, $(ij)^p = Id$ and $(ijkj)^n = Id$, may be also used to simplify words (we use Id to denote the identity element in order to avoid confusion with the word 1).

1.3. $(p, q, r; n)$ **groups.** Let $p, q, r, n \in \mathbb{N} \cup \{\infty\}$, then we define the triangle group $(p, q, r; n)$ to be the unique tirangle group with they following partial presentation

$$\langle I_1, I_2, I_3 \ : \ I_i^2, (I_2I_3)^p, (I_3I_1)^q, (I_1I_2)^r, (I_1I_3I_2I_3)^n \rangle,$$

when p, q, r or $n = \infty$, the corresponding group element is parabolic. There may be more relations, but these seven are sufficient to uniquely determine the group. This differs from slightly from Schwartz's convention given in [11], since we do not require that $p \le q \le r$. There are six possible orderings of p, q, r (or equivalently six orderings on the indices 1,2,3). Each ordering on the generating set will lead to the same underlying group, but it will change the presentation to another triangle group. In particular permuting p, q, r will have the effect of changing n. This section describes a method for determining if two triangle groups with different presentations are isomorphic. We have chosen to define n in terms of the order of I_{1323} so that when $p \le q \le r$, our notation agrees with Schwartz's.

Lemma 1.1. *If I_{23} and I_{31} have order three and I_{1323} has order n (i.e. a $(3, 3, *; n)$ group), then for any distinct $i, j, k \in \{1, 2, 3\}$, I_{ijkj} has order n.*

Proof. By hypothesis, I_{1323} has order n. The word I_{3132} is equal to $I_3I_{1323}I_3^{-1}$, so $\mathrm{ord}(I_{2313}) = \mathrm{ord}(I_{1323}) = n$. Also since $I_{23}^3 = Id$, we have $I_{323} = I_{232}$, so $I_{1323} = I_{1232}$ and I_{1232} has order n. The order of the other I_{ijkj} words follow by similar arguments. □

1.4. **Isomorphisms between** $(p, q, r; n)$ **groups.** The group $\Gamma = (p, q, r; n)$ is generated by three order two complex reflections I_1, I_2 and I_3. We define ι_1 be the involution of group that acts on the generating set (I_1, I_2, I_3) as follows,

$$\iota_1(I_1) = I_1, \quad \iota_1(I_2) = I_{121}, \quad \iota_1(I_3) = I_3.$$

i.e. we conjugate the second generator by the first. The map ι_1 extends to the rest of the group the obvious way. Since I_1 has order two it is clear ι_1^2 acts trivially. The other analogous two group involutions ι_2 and ι_3 as follows

$$\iota_2(I_1, I_2, I_3) = (I_1, I_2, I_{232}) \text{ and } \iota_3(I_1, I_2, I_3) = (I_{313}, I_2, I_3).$$

Let $A_1 = \iota_i(I_1)$, $A_2 = \iota_i(I_2)$ and $A_3 = \iota_i(I_3)$ for some ι_i. Then ι_i produces another triangle group presentation for Γ in terms of A_j, namely

$$\langle A_1, A_2, A_3 \mid A_i^2, A_{23}^{p'}, A_{31}^{q'}, A_{12}^{r'}, A_{1323}^{n'} \rangle.$$

This is the same group, $(p, q, r; n)$, that we started with, all we have done is changed the generating set and found the required relations in terms of these new generators. This produces an isometry $(p, q, r; n) \cong (p', q', r'; n')$. In general there is no reason to expect A_{23}, A_{31}, A_{12} or A_{1323} to be finite order elliptic or parabolic. When one or more of the A_{ij} is loxodromic we have a **generalised triangle group** presentation, that is a group generated by reflections in three geodesics that do not intersect to form a triangle. We don't consider these groups here.

Lemma 1.2. *The involution ι_3 sends $(p, q, r; n)$ to $(p, q, n; r)$.*

Proof. Let (I_1, I_2, I_3) be a triple of reflections that generate $(p, q, r; n)$ with the necessary relations, I_{23}^n, I_{31}^p, I_{12}^n and I_{1323}^n. Then the involution ι_3 sends this set to $(A_1 = I_{313}, A_2 = I_2, A_3 = I_3)$. Then the group generated by the A_i has the following partial presentation,

$$\langle A_1, A_2, A_3 \mid A_i^2, A_{23}^p, A_{31}^q, A_{12}^n, A_{1323}^r \rangle.$$

By partial presentation, we mean there might be more relations, but these seven relations are sufficient to uniquely determine a complex hyperbolic triangle group. □

The involution ι_3 has this special property due the appearance of I_{1323} in the relations that we chose to classify triangle groups. For ι_1 and ι_2 the situation is more complicated (see tables 1 and 2 for details).

Corollary 1.3. The triangle groups $(p, q, r; n)$ and $(p, q, n; r)$ are isomorphic for all p, q, r, n.

We can think of these maps as order 2 isomorphisms between triangle groups. Applying ι_i repeatedly will produce new generating sets and new presentations for the group. There is no reason to expect two different sequences of involutions to produce the same generating set so, in general, the graph of all possible generating sets is the valency three tree, part of which is shown in figure 1.

Each vertex of the tree corresponds to a triple of the words in the generating set which give a presentation of the triangle group, so this could instead be thought of as a tree of isomorphisms between triangle groups. Closer analysis shows that the relations of the original group will cause many vertices to collapse to the same point so this tree becomes a graph.

Example 1:$(4, 4, 4; 5)$

For the sake of clarity we work though a concrete example,
$(4, 4, 4; 5) = \langle I_1, I_2, I_3 | I_i^2, I_{ij}^4, I_{1323}^5 \rangle$. Under ι_1 this group is sent to

$$\langle \iota_1(I_1), \iota_1(I_2), \iota_1(I_3) | (\iota_1(I_i))^2, (\iota_1(I_{23}))^5, (\iota_1(I_{31}))^4, (\iota_1(I_{12}))^4, (\iota_1(I_{1323}))^5 \rangle.$$

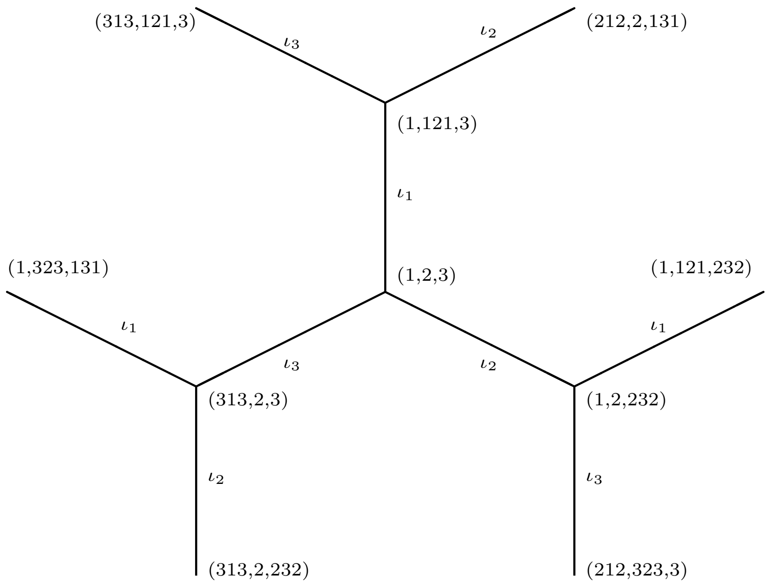

FIGURE 1. Graph of generating sets

This is a partial representation of $(5,4,4;5)$, however since $(5,4,4;5)$ is a complex hyperbolic triangle groups, knowing the order of $\iota_1(I_{1323}) = I_{131213}$ is sufficient to uniquely determine the group. So we conclude that $\iota_1 : (4,4,4;5) \leftrightarrow (5,4,4;5)$ is an isomorphism between two triangle groups. Since ι_1 is an involution, ι_1^2 acts trivially and will preserve the original presentation. The other involutions give the following isometries

$$\iota_2 : (4,4,4;5) \leftrightarrow (4,5,4;5), \iota_3 : (4,4,4;5) \leftrightarrow (4,4,5;4)$$

Although these groups have different $(p,q,r;n)$ presentations, when we permute the generating reflections to put them into the standard form (i.e. $p \le q \le r$) we see that they are in fact the same group, namely $(4,4,5;4)$.

The group relations for $(4,4,4;5)$ cause the corresponding tree to reduce to a finite graph. To see this, notice that $\iota_3\iota_1(I_1, I_2, I_3) = (I_{313}, I_{121}, I_3)$ and $\iota_1\iota_3(I_1, I_2, I_3) = (I_1, I_{323}, I_{131})$, conjugating the second triple by I_{13} produces $(I_{13131}, I_{121}, I_{1313131})$. In $(4,4,4;5)$, $I_{31313} = I_{131}$, so the vertices corresponding to ι_{13} and ι_{31} collapse to a single vertex. There are similar relations related to I_{12}^4 and I_{23}^4. This has only used the fact that we're in a $(4,4,4;n)$ group, so the same collapsing will occur for all n. Now we list the groups produced by ι_{12}, ι_{23} and ι_{31}.

$$\iota_{12} : (4,4,4;5) \leftrightarrow (5,5,4;6),$$
$$\iota_{23} : (4,4,4;5) \leftrightarrow (4,5,5;4),$$
$$\iota_{31} : (4,4,4;5) \leftrightarrow (5,4,5;4).$$

The next level of the graph will only consist of 3 points corresponding to $\iota_{312} = \iota_{321}$, $\iota_{123} = \iota_{132}$ and $\iota_{231} = \iota_{213}$. These involutions send $(I_1, I_{2,3})$ to the following generating sets

$$\iota_{312}(I_1, I_2, I_3) = (I_{212}, I_{31213}, I_3),$$
$$\iota_{123}(I_1, I_2, I_3) = (I_1, I_{323}, I_{12321}),$$
$$\iota_{231}(I_1, I_2, I_3) = (I_{23132}, I_2, I_{131}).$$

and from this we can work out the new group presentations from these generators, namely

$$\iota_{312} : (4,4,4;5) \leftrightarrow (5,5,6;4),$$
$$\iota_{123} : (4,4,4;5) \leftrightarrow (6,5,5;5),$$
$$\iota_{231} : (4,4,4;5) \leftrightarrow (5,6,5;5).$$

After permuting indices to put the group in the standard form, we see that all these groups are isomorphic to $(5,5,6;4)$.
At this point, the graph terminates, in the sense that any longer word in ι_i take us to a generating set which can be reduced (using the group relations) to another generating set arising from a shorter ι word. In particular this uses the $I_{ijik}^5 = Id$ relation from the original group, for general $(4,4,4;n)$ groups the graph will be larger. The graph of presentations for $(4,4,4;5)$ after collapsing is shown in figure 2. After putting the groups in the standard form, they are all isomorphic to one of

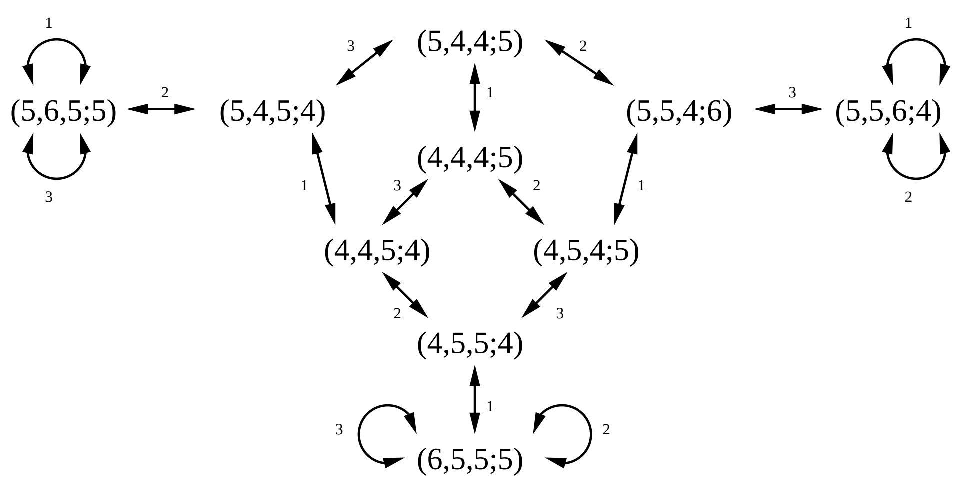

FIGURE 2. Graph of isometries of $(4,4,4;5)$

the following presentations.

$$(4,4,4;5),\ (4,4,5;4),\ (4,5,5;4),\ (5,5,6;4).$$

This is a rather special case, in that no sequence of ι maps send the group to a generalised triangle group. The only triangle groups with this property appear to be lattices (and finite groups).

Corollary 1.4. These are the only triangle groups isomorphic, under some ι_w, to $(4,4,4;5)$.

Example 2:$(3,3,4;n)$ (with $n>7$)
We now work through a non-lattice example to show how generalised triangle groups occur. Following the process as before, we produce the graph of isometries shown in figure 3. The generalised triangle groups are the groups contained in boxes, we terminate the graph at these groups since there is no obvious way of extending the notation. If the group were a lattice, at these points the edges of the graphs would form loops, i.e. two of the ι isomorphisms would send the group to itself (as in the $(4,4,4;5)$ graph, figure 2) This suggests another approach to find lattices amongst complex hyperbolic triangle groups, namely find groups where $\mathrm{ord}(23)=\mathrm{ord}(2131)$ and $\mathrm{ord}(31)=\mathrm{ord}(3212)$.

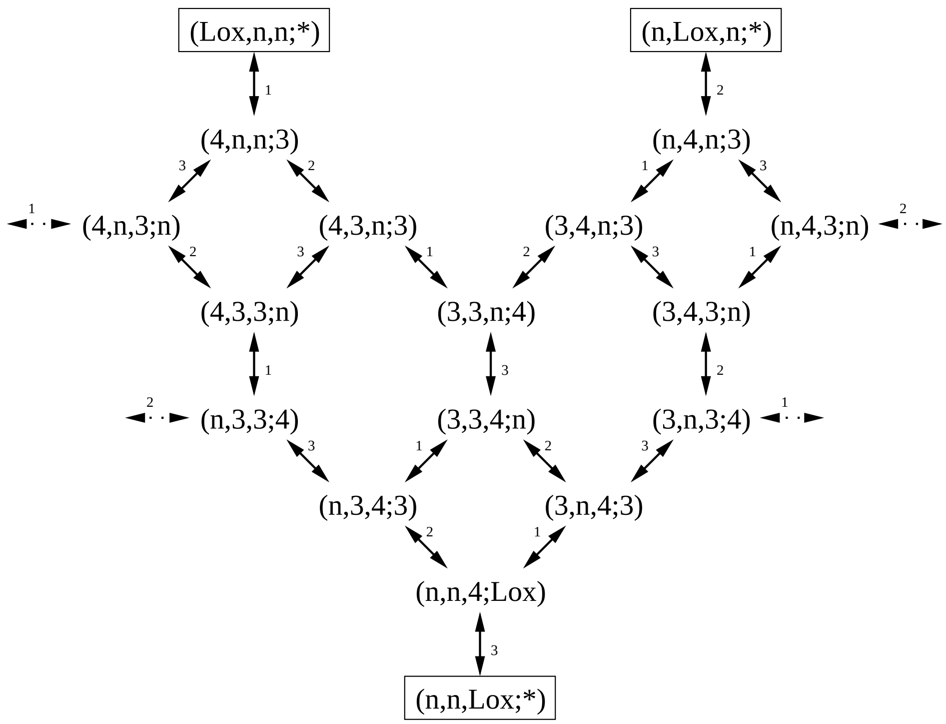

FIGURE 3. Graph of isometries of $(3,3,4;n)$

1.5. **The parameter space of** $(p,q,r;n)$**.** We explicitly define a presentation for $(p,q,r;n)$ triangle groups as follows. We can choose a basis such that the polar vectors, $\mathbf{n}_i$ to the fixed complex lines of I_i are

$$\mathbf{n}_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{n}_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \mathbf{n}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

satisfying

$$\langle \mathbf{n}_1, \mathbf{n}_1 \rangle = \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = \langle \mathbf{n}_3, \mathbf{n}_3 \rangle = 2, \quad \langle \mathbf{n}_2, \mathbf{n}_1 \rangle = \rho, \quad \langle \mathbf{n}_3, \mathbf{n}_2 \rangle = \sigma, \quad \langle \mathbf{n}_1, \mathbf{n}_3 \rangle = \tau.$$

Then the Hermitian form is given by

$$H = \begin{bmatrix} 2 & \rho & \overline{\tau} \\ \overline{\rho} & 2 & \sigma \\ \tau & \overline{\sigma} & 2 \end{bmatrix}.$$

In order to have signature $(2,1)$ we must have $\det(H) < 0$. That is

$$\rho\sigma\tau + \overline{\rho}\,\overline{\sigma}\,\overline{\tau} - 2|\rho|^2 - 2|\sigma|^2 - 2|\tau|^2 + 8 < 0. \tag{1.1}$$

Let I_j be the complex reflection of order 2 in the complex line orthogonal to $\mathbf{n}_j$. Then using the formula

$$I_j(\mathbf{z}) = -\mathbf{z} + 2\frac{\langle \mathbf{z}, \mathbf{n}_j \rangle}{\langle \mathbf{n}_j, \mathbf{n}_j \rangle}\mathbf{n}_j$$

we have

$$I_1 = \begin{bmatrix} 1 & \rho & \overline{\tau} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} -1 & 0 & 0 \\ \overline{\rho} & 1 & \sigma \\ 0 & 0 & -1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \tau & \overline{\sigma} & 1 \end{bmatrix}.$$

Note that

$$\mathrm{tr}(I_1 I_2 I_3) = \rho\sigma\tau - |\rho|^2 - |\sigma|^2 - |\tau|^2 + 3.$$

Combining this with (1.1) we see that

$$\mathrm{Re}\big(\mathrm{tr}(I_1 I_2 I_3)\big) < -1.$$

Note that a 1-eigenvector for $I_1 I_2$ is $\mathbf{v}_{12}$ where

$$\mathbf{v}_{12} = \begin{bmatrix} \rho\sigma - 2\overline{\tau} \\ \overline{\rho}\,\overline{\tau} - 2\sigma \\ 4 - |\rho|^2 \end{bmatrix}.$$

Then

$$\mathrm{tr}(I_1 I_2) = |\rho|^2 - 1, \quad \mathrm{tr}(I_2 I_3) = |\sigma|^2 - 1, \quad \mathrm{tr}(I_3 I_1) = |\tau|^2 - 1.$$

We suppose that

$$|\sigma| = 2\cos(\pi/p), \quad |\tau| = 2\cos(\pi/q), \quad |\rho| = 2\cos(\pi/r),$$

this ensures that $I_1 I_2$, $I_2 I_3$ and $I_3 I_1$ have the required order.
Having made this restriction then, up to conjugation, there is a one parameter family of these groups. The parameter being the argument of $\rho\sigma\tau$. We have

$$\mathrm{tr}(I_1 I_2 I_1 I_3) = |\rho\tau - \overline{\sigma}|^2 - 1, \ \mathrm{tr}(I_2 I_3 I_2 I_1) = |\rho\sigma - \overline{\tau}|^2 - 1, \ \mathrm{tr}(I_3 I_1 I_3 I_2) = |\sigma\tau - \overline{\rho}|^2 - 1.$$

Suppose that

$$|\rho\tau - \overline{\sigma}| = 2\cos(\pi/s), \quad |\rho\sigma - \overline{\tau}| = 2\cos(\pi/t), \quad |\sigma\tau - \overline{\rho}| = 2\cos(\pi/u).$$

Then

$$\begin{aligned} \rho\sigma\tau + \overline{\rho}\,\overline{\sigma}\,\overline{\tau} &= 16\cos^2(\pi/p)\cos^2(\pi/r) + 4\cos^2(\pi/q) - 4\cos^2(\pi/t) \\ &= 16\cos^2(\pi/p)\cos^2(\pi/q) + 4\cos^2(\pi/r) - 4\cos^2(\pi/u) \\ &= 16\cos^2(\pi/q)\cos^2(\pi/r) + 4\cos^2(\pi/p) - 4\cos^2(\pi/s). \end{aligned} \tag{1.2}$$

Lemma 1.5. *Let $(p, q, r; n)$ be a complex hyperbolic triangle group and define ρ, σ, τ as above. Then,*

$$\mathrm{Re}\,(\rho\sigma\tau) = \frac{-2\cos(2\pi/n) - 2 + |\rho|^2 + |\sigma\tau|^2}{2}$$

Definition 1.6.

$$\mathbf{K} := \mathrm{Re}\,(\rho\sigma\tau) - |\sigma|^2 - |\tau|^2 - |\rho|^2,$$
$$\mathbf{L} := |\sigma\tau\rho|^2 - (\mathrm{Re}\,(\rho\sigma\tau))^2.$$

Remark 1.7. The determinant of the Hermitian form H is $\mathbf{K}+4$, so H has signature $(2, 1)$, and $(p, r, q; n)$ corresponds to a complex hyperbolic triangle group, if and only if $\mathbf{K} < -4$.

Lemma 1.8. *The quantities $\boldsymbol{K}$ and $\boldsymbol{L}$ are fixed under the isomorphisms ι_i*

Proof. We only show this for ι_1, the other cases are essentially the same. Assume we have a $(p, q, r; n)$ group with associated ρ, σ, τ. Let $(p', q', r'; n')$ be image of $(p, q, r; n)$ under ι_1 with the associated parameters ρ', σ', τ'. Under the image of ι_1 there is the following identification of words $(I_{23}, I_{31}, I_{12}, I_{1213}) \leftrightarrow (I_{1213}, I_{31}, I_{21}, I_{23})$. Using the trace formulae we see that $|\sigma'| = |\rho\tau - \overline{\sigma}|$, $|\tau'| = |\tau'|$, $|\rho'| = |\rho'|$ and $|\rho'\tau' - \overline{\sigma'}| = |\sigma|$. Then

$$\begin{aligned}
2\mathbf{K} - 2\mathbf{K}' &= 2\mathrm{Re}\,(\rho\sigma\tau) - 2|\sigma|^2 - 2|\tau|^2 - 2|\rho|^2 - 2\mathrm{Re}\,(\rho'\sigma'\tau') + 2|\sigma'|^2 + 2|\tau'|^2 + 2|\rho'|^2 \\
&= |\rho\tau|^2 + |\sigma|^2 - |\sigma'|^2 - 2|\sigma|^2 - 2|\tau|^2 - 2|\rho|^2 - |\rho'\tau'|^2 \\
&\qquad - |\sigma'|^2 + |\sigma|^2 + 2|\sigma'|^2 + 2|\tau'|^2 + 2|\rho'|^2 \\
&= |\rho\tau|^2 - 2|\tau|^2 - 2|\rho|^2 - |\rho'\tau'|^2 + 2|\tau'|^2 + 2|\rho'|^2 \\
&= 0
\end{aligned}$$

Hence $\mathbf{K} = \mathbf{K}'$. Using this equality we get $\mathrm{Re}\,(\rho'\sigma'\tau') = -\mathrm{Re}\,(\rho\sigma\tau) + |\rho\tau|^2$. Then

$$\begin{aligned}
\mathbf{L} - \mathbf{L}' &= |\sigma\tau\rho|^2 - (\mathrm{Re}\,(\rho\sigma\tau))^2 - |\sigma'\tau'\rho'|^2 - (\mathrm{Re}\,(\rho'\sigma'\tau'))^2 \\
&= |\tau\rho|^2(|\sigma|^2 - |\sigma'|^2) - (\mathrm{Re}\,(\rho\sigma\tau))^2 + (\mathrm{Re}\,(\rho\sigma\tau))^2 - 2|\tau\rho|^2\mathrm{Re}\,(\rho\sigma\tau) + |\tau\rho|^4 \\
&= |\tau\rho|^2(|\sigma|^2 - |\sigma'|^2 - 2\mathrm{Re}\,(\rho\sigma\tau) + |\tau\rho|^2) \\
&= |\tau\rho|^2(|\sigma|^2 - |\sigma'|^2 - |\sigma|^2 - |\tau\rho|^2 + |\sigma'|^2 + |\tau\rho|^2) \\
&= 0.
\end{aligned}$$

Hence $\mathbf{L} = \mathbf{L}'$. □

Remark 1.9. The terms $\mathbf{K}$ and $\mathbf{L}$ individually, are not enough to distinguish triangle groups. For example $(4, 4, 4; 6)$ and $(4, 4, \infty; 3)$ both have $\mathbf{K} = -4.5$, but since they have different values for $\mathbf{L}$ they a not isomorphic.

Conjecture 1.10. Two triangle groups are isomorphic if and only if they have the same $\mathbf{K}$ and $\mathbf{L}$.

For the rest of the article, we assume $p \leq q \leq r$.

2. Discreteness results

In this section we describe some technical results which allow us to quickly determine the discreteness of some groups, then we use these results to prove our main result, theorem 2.6, which says that for $p > 31$, there are no discrete $(p, q, r; n)$ groups.

Theorem 2.1. *Let* $\Gamma = (p, q, r; n)$ *with* $p, q, r, n \in \{2, 3, 4, 6, \infty\}$. *Then* Γ *is discrete.*

Proof. Using Pratoussevitch's formula [10] the trace of each element of the group is an integer polynomial in the following variables

$$|\sigma|^2,\ |\tau|^2,\ |\rho|^2,\ \rho\sigma\tau,\ \overline{\rho\sigma\tau}. \tag{2.1}$$

Using the presentation described above and lemma 1.5, it is clear, for $\{p, q, r, n\} \subset \{2, 3, 4, 6, \infty\}$, that $|\rho|^2$, $|\sigma|^2$, $|\tau|^2$ and $2\mathrm{Re}\,(\rho\sigma\tau)$ are integers. There are two possibilities, either $\mathrm{Re}\,(\rho\sigma\tau) = m \in \mathbb{Z}$ or $\mathrm{Re}\,(\rho\sigma\tau) = m/2$ for some odd $m \in \mathbb{Z}$.

If $\mathrm{Re}\,(\rho\sigma\tau) = m \in \mathbb{Z}$, then $\mathbf{L} = \Im^2(\rho\sigma\tau) = |\rho\sigma\tau|^2 - \mathrm{Re}^{\,2}(\rho\sigma\tau)$ is also an integer. Then $\rho\sigma\tau = m + i\sqrt{\mathbf{L}}$ and clearly $\overline{\rho\sigma\tau} = m - i\sqrt{\mathbf{L}}$. So by Pratoussevitch's result, the trace of any word in the group can be written as an integer polynomial in $m + i\sqrt{\mathbf{L}}$, so the trace of every word in $(p, q, r; n)$ lies in the ring $\mathbb{Z}[i\sqrt{\mathbf{L}}]$. This is a discrete ring, so the group $(p, q, r; n)$ is discrete.

If $\mathrm{Re}\,(\rho\sigma\tau) = m/2$ for m odd, then $\mathbf{L} = \Im^2(\rho\sigma\tau) = l/4$. We can express l in terms of $|\rho|^2$, $|\sigma|^2$, $|\tau|^2$ and m as follows, $l = 4|\rho\sigma\tau|^2 - m^2$, in particular, l is congruent to 3 modulo 4. Then $\rho\sigma\tau = \frac{m}{2} + \frac{i\sqrt{l}}{2}$ and the trace of any word can therefore be written as some element of $\mathbb{Z}[(1 + i\sqrt{l})/2]$, so the group is discrete. $\square$

This result is essentially the same a corollary 18 of Pratoussevitch's paper [10] rewritten in our terminology.

Example 1: $(4, 4, 6; 6)$

$\mathbf{K} = -5$, $\mathrm{ord}(23) = 4$, $\mathrm{ord}(31) = 4$, $\mathrm{ord}(12) = 6$.
We have

$$|\rho|^2 = 3, \quad |\sigma|^2 = 2, \quad |\tau|^2 = 2, \quad \mathrm{Re}\,(\rho\sigma\tau) = 2.$$

From this we deduce that

$$\Im^2(\rho\sigma\tau) = |\rho|^2|\sigma|^2|\tau|^2 - \mathrm{Re}^{\,2}(\rho\sigma\tau) = 8.$$

Therefore $\rho\sigma\tau = 2 \pm 2i\sqrt{2}$. A solution is

$$\rho = 1 + i\sqrt{2}, \quad \sigma = i\sqrt{2}, \quad \tau = -i\sqrt{2}.$$

We can choose the following generators whose entries lie in $\mathbb{Z}[i\sqrt{2}]$

$$I_1 = \begin{bmatrix} 1 & 1 + i\sqrt{2} & i\sqrt{2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 - i\sqrt{2} & 1 & i\sqrt{2} \\ 0 & 0 & -1 \end{bmatrix},\ I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -i\sqrt{2} & -i\sqrt{2} & 1 \end{bmatrix}.$$

Example 2: $(4, 6, 6; 6)$
$\mathbf{K} = -5$, $\mathrm{ord}(23) = 4$, $\mathrm{ord}(31) = 6$, $\mathrm{ord}(12) = 6$.
We have

$$|\rho|^2 = 3, \quad |\sigma|^2 = 2, \quad |\tau|^2 = 3, \quad \mathrm{Re}\,(\rho\sigma\tau) = 3.$$

From this we deduce that

$$\Im^2(\rho\sigma\tau) = |\rho|^2|\sigma|^2|\tau|^2 - \mathrm{Re}^2(\rho\sigma\tau) = 9.$$

Therefore $\rho\sigma\tau = 3 \pm 3i$. A solution is

$$\rho = i\sqrt{3}, \quad \sigma = 1+i, \quad \tau = -i\sqrt{3}.$$

Putting these into the standard generators gives

$$I_1 = \begin{bmatrix} 1 & i\sqrt{3} & i\sqrt{3} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} -1 & 0 & 0 \\ -i\sqrt{3} & 1 & 1+i \\ 0 & 0 & -1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -i\sqrt{3} & 1-i & 1 \end{bmatrix}.$$

Conjugating by $C = \mathrm{diag}(-i\sqrt{3}, 1, 1)$ gives the following matrices whose entries lie in $\mathbb{Z}[i]$.

$$CI_1C^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad CI_2C^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1+i \\ 0 & 0 & -1 \end{bmatrix},$$

$$CI_3C^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1-i & 1 \end{bmatrix}.$$

These preserve the Hermitian form

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1+i \\ 1 & 1-i & 2 \end{bmatrix}.$$

Example 3: $(3,3,4;7)$
$\mathbf{K} = 3/2 - 2\cos^2(\pi/7)$, $\mathrm{ord}(23) = 3$, $\mathrm{ord}(31) = 3$, $\mathrm{ord}(12) = 4$.
We have

$$|\rho|^2 = 2, \quad |\sigma|^2 = 1, \quad |\tau|^2 = 1, \quad \mathrm{Re}\,(\rho\sigma\tau) = 3/2 - 2\cos^2(\pi/7) = 1/2 - \cos(2\pi/7).$$

This leads to

$$\rho = e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7} = \frac{-1+i\sqrt{7}}{2}, \quad \sigma = e^{-4\pi i/7}, \quad \tau = e^{-4\pi i/7}.$$

Therefore the matrix entries are all in $\mathbb{Z}[e^{2\pi i/7}]$. This is an imaginary quadratic extension of $\mathbb{Z}[2\cos(2\pi/7)]$.

The determinant of H is $\mathbf{K} + 4 = 1/2 - \cos(2\pi/7)$. For the two non-trivial Galois conjugations in $\mathbb{Z}[2\cos(2\pi/7)]$, namely $2\cos(2\pi/7) \longmapsto 2\cos(4\pi/7)$ and $2\cos(2\pi/7) \longmapsto 2\cos(6\pi/7)$ the Hermitian form is positive definite. Therefore this group is a subgroup of an arithmetic lattice.

Example 4: $(3,3,5;5)$
$\mathbf{K} = -7/2 - 2\cos(2\pi/5)$, $\mathrm{ord}(23) = 3$, $\mathrm{ord}(31) = 3$, $\mathrm{ord}(12) = 5$.
We have

$$|\rho|^2 = (3+\sqrt{5})/2, \quad |\sigma|^2 = 1, \quad |\tau|^2 = 1, \quad \mathrm{Re}\,(\rho\sigma\tau) = 1/2.$$

This leads to

$$\rho = -1 - e^{8\pi i/5}, \quad \sigma = e^{4\pi i/5}, \quad \tau = e^{4\pi i/5}.$$

Therefore the matrix entries are all in $\mathbb{Z}[e^{2\pi i/5}]$. This is an imaginary quadratic extension of $\mathbb{Z}[2\cos(2\pi/5)]$.

The determinant of H is $\mathbf{K}+4 = 1/2 - 2\cos(2\pi/5)$. For the non-trivial Galois conjugations in $\mathbb{Z}[2\cos(2\pi/5)]$, $2\cos(2\pi/5) \longmapsto 2\cos(4\pi/5)$ the Hermitian form is positive definite. Therefore this group is a subgroup of an arithmetic lattice.

Lemma 2.2 (Jørgensen's inequality)**.** *Let $A \in \mathrm{SU}(2,1)$ be a regular elliptic map of order $n \geq 7$ that preserves a Lagrangian plane (i.e. $\mathrm{tr}(A)$ is real). Suppose that A fixes a point $z \in \boldsymbol{H}^2_{\mathbb{C}}$. Let B be any element of $\mathrm{PU}(2,1)$ with $B(z) \neq z$. If*

$$\cosh\left(\frac{\rho(B(p),p)}{2}\right)\sin\left(\frac{\pi}{n}\right) < \frac{1}{2}, \tag{2.2}$$

then $\langle A, B\rangle$ is not discrete and consequently any group containing A and B is not discrete.

Corollary 2.3. Let $\Gamma = (p,q,r;n)$ with $p \leq q \leq r$. Let ρ, σ, τ be defined as in section 1.5. If $\mathrm{ord}(I_{12}) = r \geq 7$ and

$$\left(\rho\sigma\tau + \overline{\rho\sigma\tau} - 2|\sigma|^2 - 2|\tau|^2 - |\rho|^2 + 4\right)^2 < 4 - |\rho|^2 \tag{2.3}$$

then Γ is not discrete.
If $\mathrm{ord}(I_{23}) = p \geq 7$ and

$$\left(\rho\sigma\tau + \overline{\rho\sigma\tau} - |\sigma|^2 - 2|\tau|^2 - 2|\rho|^2 + 4\right)^2 < 4 - |\sigma|^2 \tag{2.4}$$

then Γ is not discrete.
If $\mathrm{ord}(I_{31}) = q \geq 7$ and

$$\left(\rho\sigma\tau + \overline{\rho\sigma\tau} - 2|\sigma|^2 - |\tau|^2 - 2|\rho|^2 + 4\right)^2 < 4 - |\tau|^2 \tag{2.5}$$

then Γ is not discrete.

Proof. Let $I_{12} = A$, $I_3 = B$ and $z = \mathbf{v}_{12}$ (fixed point of I_{12}) in lemma 2.2. Then (2.2) becomes

$$\left|\frac{\langle I_3(\mathbf{v}_{12}), \mathbf{v}_{12}\rangle}{\langle \mathbf{v}_{12}, \mathbf{v}_{12}\rangle}\right| \sin\left(\frac{\pi}{r}\right) < \frac{1}{2}.$$

Squaring both sides we obtain

$$\left|\frac{\langle I_3(\mathbf{v}_{12}), \mathbf{v}_{12}\rangle}{\langle \mathbf{v}_{12}, \mathbf{v}_{12}\rangle}\right|^2 (3 - \mathrm{tr}(I_{12})) < 1$$

which is equivalent to

$$\left(\rho\sigma\tau + \overline{\rho\sigma\tau} - 2|\sigma|^2 - 2|\tau|^2 - |\rho|^2 + 4\right)^2 < 4 - |\rho|^2$$

as required. The other inequalities arise from identical arguments. □

Corollary 2.4. We can rewrite these inequalities respectively as

$$(2\mathrm{Re}\,(\mathrm{tr}\,(I_{123})) + \mathrm{tr}\,(I_{12}) - 1)^2 < 3 - \mathrm{tr}\,(I_{12}),$$
$$(2\mathrm{Re}\,(\mathrm{tr}\,(I_{123})) + \mathrm{tr}\,(I_{23}) - 1)^2 < 3 - \mathrm{tr}\,(I_{23}),$$
$$(2\mathrm{Re}\,(\mathrm{tr}\,(I_{123})) + \mathrm{tr}\,(I_{31}) - 1)^2 < 3 - \mathrm{tr}\,(I_{31}).$$

Remark 2.5. These inequalities are the best possible, in the sense that there are discrete groups where we get equality, in particular $(18,18,18;18)$ and $(7,7,14;4)$.

Theorem 2.6. *Assume $p \leq q \leq r < \infty$, if $p > 31$ then the group $(p,q,r;n)$ is not discrete.*

The outline for proof is as follows, we first prove a technical inequality in lemma 2.8, which we then use to prove lemma 2.10. This lemma tells us that if $(p, q, r; N)$ fails one of the Jørgensen discreteness tests ((4), (5) or (6)), then so will $(p, q, r; n)$ for any $n \leq N$. Finally, we show that if $p > 31$, the group $(p, q, r; \infty)$ fails the discreteness test for any q and r. Then by lemma 2.10 $(p, q, r; n)$ is non-discrete for all q, r and n. For the rest of this proof we assume $7 \leq p$.

Lemma 2.7. *If $7 \leq p \leq q \leq r < \infty$, then $2\cos(\pi/7) \leq |\sigma| \leq |\tau| \leq |\rho| < 2$.*

Lemma 2.8. *Let ρ, σ, τ be defined in terms of a 4-tuple $(p, q, r; n)$ as in section 2. Then*

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \leq 0$$

Proof. In section 1.5 it was shown that for a group where I_{1323} is regular elliptic of order n, we have the following equality.

$$\mathrm{Re}\,(\rho\sigma\tau) = -\frac{2\cos(2\pi/n) + 2 - |\rho|^2 - |\sigma\tau|^2}{2}$$

Then we can rewrite $-1 \leq \dfrac{\mathrm{Re}\,(\rho\sigma\tau)}{|\rho\sigma\tau|} \leq 1$ as

$$-1 \leq \frac{2\cos(2\pi/n) + 2 - |\rho|^2 - |\sigma\tau|^2}{2|\rho\sigma\tau|} \leq 1$$

Rearranging this gives us

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \leq 2|\rho\sigma\tau| - |\rho|^2 - 2|\tau|^2 - 2|\sigma|^2 + 4 \tag{2.6}$$

We can rearrange the right hand side to get

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \leq -2(|\tau| - |\sigma|)^2 - 2|\sigma||\tau|(2 - |\rho|) - |\rho|^2 + 4 \tag{2.7}$$

Since $(|\tau| - |\sigma|)^2$ is always non-negative, we have

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \leq -2|\sigma||\tau|(2 - |\rho|) - |\rho|^2 + 4 \tag{2.8}$$

The right hand side of this inequality is a quadratic in $|\rho|$

$$-|\rho|^2 + 2|\sigma\tau||\rho| + 4(1 - |\sigma\tau|).$$

This quadratic has roots 2 and $2|\sigma\tau| - 2$. Since $p \geq 7$, then $|\sigma\tau| \geq 4\cos^2(\pi/7)$ and $(2|\sigma\tau| - 2) \geq 2$. So on the interval $0 \leq |\rho| \leq 2$ the quadratic is always negative. Since these are the only values that $|\rho|$ can take, the right hand side of (2.8) is always negative, which proves the lemma. $\square$

Corollary 2.9. Let ρ, σ, τ be defined in terms of a 4-tuple $(p, q, r; n)$ as in section 2. Then we have the following inequalities

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - |\sigma|^2 - |\rho|^2) \leq 0,$$
$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - |\tau|^2 - 2|\sigma|^2 - |\rho|^2) \leq 0.$$

Proof. For the first inequality notice that

$$\begin{aligned}(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - |\sigma|^2 - |\rho|^2) - (-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)\\ = |\sigma|^2 - |\rho|^2.\end{aligned}$$

By lemma 2.7 $|\sigma|^2 - |\rho|^2 \leq 0$, so using lemma 2.8,

$$\begin{aligned} 0 &\geq (-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \\ &\geq (-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - |\sigma|^2 - |\rho|^2) \end{aligned}$$

as required. The second inequality follow by essentially the same argument but using $|\tau|^2 - |\rho|^2 \leq 0$. □

Lemma 2.10. *If the group $(p, q, r; N)$ satisfies any of the Jørgensen nondiscreteness conditions above (inequalities (2.3), (2.4) and (2.5)) for some $N \in \mathbb{N} \cup \{\infty\}$, then $(p, q, r; n)$ will also satisfy them for $n < N$.*

Proof. First recall that $\rho\sigma\tau + \overline{\rho\sigma\tau} = -2\cos(2\pi/n) - 2 + |\rho|^2 + |\sigma\tau|^2$. Substituting this into (2.3) gives

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)^2 < 4 - |\rho|^2. \tag{2.9}$$

For $n < N$, $2\cos(2\pi/n) < 2\cos(2\pi/N)$, so we have following inequality

$$\begin{aligned} 0 &\geq (-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \\ &> (-2\cos(2\pi/N) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \end{aligned} \tag{2.10}$$

The less than zero inequality comes from lemma 2.8. Squaring both sides and combining with (2.9) gives

$$\begin{aligned} 4 - |\rho|^2 &> (-2\cos(2\pi/N) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)^2 \\ &> (-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)^2 \end{aligned} \tag{2.11}$$

Therefore if $(p, q, r; N)$ satisfies (2.3), then so does $(p, q, r; n)$ for all $n < N$.
For (2.4) and (2.5) we can use the inequalities from corollary 2.9. □

Proof of proposition 2.6. Using lemma 2.10, we only need to find conditions on $(p, q, r; \infty)$ groups. We know that if a $(p, q, r; \infty)$ group satisfies inequalities (2.3), (2.4) and (2.5), then the group is non-discrete and then the lemma tells us that $(p, q, r; n)$ also satisfy the inequalities and are also non-discrete. So let $n = \infty$. Then $\rho\sigma\tau + \overline{\rho\sigma\tau} = -4 + |\rho|^2 + |\sigma\tau|^2$. Substituting this into inequality (2.4) gives

$$(|\sigma\tau|^2 - |\rho|^2 - |\sigma|^2 - 2|\tau|^2)^2 < 4 - |\sigma|^2. \tag{2.12}$$

By lemma 2.8 the term inside the brackets on the left hand side is always negative and bounded from below by $|\sigma|^4 - 3|\sigma|^2 - 4$, to see this we use lemma 2.7 to obtain the following inequality (in particular we use the facts $(|\sigma|^2 - 2) > 0$ and $|\rho|^2 \leq 4$).

$$\begin{aligned} |\sigma\tau|^2 - |\rho|^2 - |\sigma|^2 - 2|\tau|^2 &= (|\sigma|^2 - 2)|\tau|^2 - |\sigma|^2 - |\rho|^2 \\ &\geq (|\sigma|^2 - 2)|\sigma|^2 - |\sigma|^2 - 4 \\ &= |\sigma|^4 - 3|\sigma|^2 - 4. \end{aligned}$$

By lemma 2.8, $|\sigma\tau|^2 - |\rho|^2 - |\sigma|^2 - 2|\tau|^2$ is negative so squaring both sides will give,

$$(|\sigma|^4 - 3|\sigma|^2 - 4)^2 \geq (|\sigma\tau|^2 - |\rho|^2 - |\sigma|^2 - 2|\tau|^2)^2. \tag{2.13}$$

Combining (2.12) with (2.13), it is clear that if $|\sigma|$ satisfies

$$(|\sigma|^4 - 3|\sigma|^2 - 4)^2 < 4 - |\sigma|^2. \tag{2.14}$$

then then $|\sigma|$, $|\tau|$ and $|\rho|$ will satisfy (2.12) for any permitted $|\tau|$ and $|\rho|$, so the corresponding group $(p, q, r; \infty)$ group is non-discrete. Then using lemma 2.10, all $(p, q, r; n)$ will also be non-discrete. Expanding out the brackets and collecting terms in (2.14) gives

$$(|\sigma|^2-4)(|\sigma|^6-2|\sigma|^4-7|\sigma|^2-3) > 0. \tag{2.15}$$

By hypothesis, $\operatorname{ord}(I_{23}) \geq 7$, so $4\cos^2(\pi/7) \leq |\sigma|^2 \leq 4$. As a polynomial in $|\sigma|^2$, the left hand side of (2.15) has two roots in the interval $\left[4\cos^2(2\pi/7), 4\right]$, namely $3.9593\ldots$ and 4. When $|\sigma|^2$ lies between these roots the polynomial is negative, so (2.15) is not satisfied and the group is not discrete. So any $(p, q, r; \infty)$ group with $3.9593\ldots \leq |\sigma|^2 \leq 4$, fails a Jørgensen discreteness test. Since $|\sigma| = 2\cos(\pi/p)$, and $4\cos^2(\pi/31) < 3.9593\ldots < 4\cos^2(\pi/32)$, then for all $p > 31$, the group $(p, q, r; \infty)$ satisfies the inequality (2.4) and is not discrete. Then applying lemma 2.10, it follows for $p > 31$, $(p, q, r; n)$ will satisfy (2.4) for all q, r, n. Therefore $(p, q, r; n)$ is not discrete if $p > 31$. □

Conjecture 2.11. Computer calculations strongly suggest that for $p > 22$ the group $(p, q, r; n)$ will always fail at least one of the Jørgensen discreteness tests described in corollary 2.3. Consequently, we conjecture that there are no discrete groups with $p > 22$.

Remark 2.12. This result should be compared with conjectures 5.1, 5.2 and 5.3 of [11] which taken together, in part conjecture that there are no discrete groups $(p, q, r; n)$ with $p > 13$. In [8] Parker discovered two counter examples to this conjecture, namely $(18, 18, 18; 18)$ and a $(14, 14, 14)$ group. In a forthcoming paper ([6]) it is shown, using Shimizu's lemma, that for $p > 28$, the group $(p, p, \infty; n)$ is non-discrete.

It is unlikely that are similar bounds for q or r. Conjecturally $(4, 4, 4; n)$ is discrete for all $n > 4$ and this group is isomorphic to $(4, 4, n; 4)$ and $(4, n, n; 4)$. If the conjecture is correct, we can always find a discrete group $(p, q, r; n)$ with arbitrarily large values for q or r.

Proposition 2.13. For all $n \geq 4$, the triangle group $(3, 3, 4; n)$ contains a deformed (and not necessarily discrete) (n, n, n) group.

Proof. Consider the subgroup generated by the reflections $A_1 = I_1$, $A_2 = I_{32123}$, $A_3 = I_{23132}$, by checking the necessary words we see that A_i has order 2 and A_{ij} has order n for all i, j. So the group $\langle A_1, A_2, A_3\rangle$ is an (n, n, n) triangle group. The forth number in the 4-tuple defining a triangle group is the order of the word I_{1323}, in this case we look at the order of

$$A_{1323} = I_1 I_{23132} I_{32123} I_{23132} = I_{123123132132}.$$

A priori there is no reason to expect this word to have finite order or to even be elliptic. □

The $(3, 3, 4; n)$ groups appear to be good candidates for discreteness. For $n > 7$, the word A_{1323} described above is loxodromic, so the (n, n, n) triangle subgroup is from inside the critical interval, in addition they have a large number of 'extra' symmetries. In particular $(3, 3, 4; n)$ is isomorphic to $(3, 3, n; 4)$, $(3, 4, n; 3)$ and $(4, n, n; 3)$. For $n < 7$ the groups are finite. The group $(3, 3, 4; 7)$ is a lattice and it is described in [13].

3. Overview of results

In tables 1 and 2 we collect some known results about triangle groups, from this paper and elsewhere. In table 1 the first column records the value of $\mathbf{K}$ described above, the next seven columns record the order of their respective elements, these are arranged so that the first four columns are exactly the numbers $(p, q, r; n)$, the next three are not immediately obvious from the presentation and were obtained from trace calculations. A finite number indicates the element is regular elliptic of finite order, ∞ indicates a parabolic and lox indicates loxodromic. For table 2 we omit the $\mathbf{K}$, the eighth column records whether the group is discrete or a lattice. The ninth column records which ring of integers the traces of group element lie in, calculated using theorem 2.1 above and the tenth give references for articles where these groups are studied in greater detail.

Horizontal lines separate isomorphism classes of groups. Within isomorphism classes we list all the groups in the 'tree' of generators which correspond to triangle groups i.e. p,q,r,$n \in \mathbb{N} \cup \{\infty\}$. Generally there will be other generalised triangle group presentations, but we don't included these groups.

For example table 2 tells us $(3, 3, 5; 5)$ and $(3, 5, 5; 3)$ and $(5, 5, 10; 3)$ are isomorphic groups with I_{123} an order 15 regular elliptic element, the group is a lattice, the traces of all group elements lie in the ring $\mathbb{Z}[\zeta_5]$. If (2.1) appears in the *cite* column, that indicates we have used proposition 2.1 of this paper to show discreteness.

$\mathbf{K}$	23	31	12	1323	3212	2131	123
-4.309016995	3	4	4	5	5	10	lox
-4.309016995	3	4	5	4	4	10	lox
-4.309016995	4	4	10	3	5	5	lox
-4.309016995	4	5	10	3	4	10	lox
-4.309016995	5	10	10	4	4	lox	lox
-4.623489802	6	7	7	7	7	14	lox
-4.623489802	7	7	14	6	lox	lox	lox
-4.809016997	5	5	6	6	10	10	lox
-4.809016997	5	6	10	5	10	lox	lox
-4.809016997	5	10	10	6	6	lox	lox
-4.809016995	4	4	5	6	10	10	lox
-4.809016995	4	4	6	5	10	10	lox
-4.809016995	4	5	10	4	6	lox	lox
-4.809016995	4	6	10	4	5	lox	lox
-4.809016995	4	5	5	6	6	∞	lox
-4.809016995	5	5	∞	4	lox	lox	lox
-4.809016995	4	6	6	5	5	lox	lox
-4.809016995	4	5	6	5	6	lox	lox
$-7/2 - \cos(2\pi/n)$	3	3	4	n	n	n	lox
$-7/2 - \cos(2\pi/n)$	3	3	n	4	4	4	lox
$-7/2 - \cos(2\pi/n)$	3	4	n	3	3	n	lox
$-7/2 - \cos(2\pi/n)$	4	n	n	3	3	lox	lox
$-4 - \cos(2\pi/n)$	4	4	4	n	n	n	lox
$-4 - \cos(2\pi/n)$	4	4	n	4	n	n	lox
$-4 - \cos(2\pi/n)$	4	n	n	4	4	lox	lox

Table 1. Possible discrete $(p, q, r; n)$ groups

23	31	12	1323	3212	2131	123	Discrete	$\mathbb{Z}[*]$	Cite
18	18	18	18	18	18	9	Dis	$\mathbb{Z}[\zeta_9]$	[7], [8]
3	3	5	5	5	5	15	Lat	$\mathbb{Z}[\zeta_5]$	
3	5	5	3	3	10	15	Lat	$\mathbb{Z}[\zeta_5]$	[13]
5	5	10	3	5	5	15	Lat	$\mathbb{Z}[\zeta_5]$	
3	3	4	7	7	7	42	Lat	$\mathbb{Z}[\zeta_7]$	
3	4	7	3	3	7	42	Lat	$\mathbb{Z}[\zeta_7]$	
3	3	7	4	4	4	42	Lat	$\mathbb{Z}[\zeta_7]$	[13]
4	7	7	3	3	14	42	Lat	$\mathbb{Z}[\zeta_7]$	
7	7	14	4	7	7	42	Lat	$\mathbb{Z}[\zeta_7]$	
4	4	4	5	5	5	10	Lat	$\mathbb{Z}(\sqrt{5}, \omega)$	
4	4	5	4	5	5	10	Lat	$\mathbb{Z}(\sqrt{5}, \omega)$	[8]
4	5	5	4	4	6	10	Lat	$\mathbb{Z}(\sqrt{5}, \omega)$	& [1]
5	5	6	4	5	5	10	Lat	$\mathbb{Z}(\sqrt{5}, \omega)$	
5	5	5	5	5	5	10	Lat	$\mathbb{Z}[\zeta_5]$	[7], [8]
4	4	4	6	6	6	lox	Dis	$\mathbb{Z}[i\sqrt{23}]$	
4	4	6	4	6	6	lox	Dis	$\mathbb{Z}[i\sqrt{23}]$	
4	6	6	4	4	∞	lox	Dis	$\mathbb{Z}[i\sqrt{23}]$	(2.1)
6	6	∞	4	∞	∞	lox	Dis	$\mathbb{Z}[i\sqrt{23}]$	
6	∞	∞	6	6	lox	lox	Dis	$\mathbb{Z}[i\sqrt{23}]$	
4	4	∞	3	6	6	lox	Dis	$\mathbb{Z}[(1+i\sqrt{15})/2]$	
4	6	∞	3	4	lox	lox	Dis	$\mathbb{Z}[(1+i\sqrt{15})/2]$	
3	4	6	4	4	∞	lox	Dis	$\mathbb{Z}[(1+i\sqrt{15})/2]$	[14]
3	4	4	6	6	∞	lox	Dis	$\mathbb{Z}[(1+i\sqrt{15})/2]$	
4	5	5	5	5	10	lox	Dis	$\mathbb{Z}((1+i), (1+\sqrt{5})/2)$	
5	5	10	4	10	10	lox	Dis	$\mathbb{Z}((1+i), (1+\sqrt{5})/2)$	(2.1)
5	10	10	5	5	lox	lox	Dis	$\mathbb{Z}((1+i), (1+\sqrt{5})/2)$	
4	6	6	6	6	lox	lox	Dis	$\mathbb{Z}[i]$	[4]
4	4	6	6	∞	∞	lox	Dis	$\mathbb{Z}[i\sqrt{2}]$	
4	6	∞	4	6	lox	lox	Dis	$\mathbb{Z}[i\sqrt{2}]$	(2.1)
3	3	∞	6	6	6	lox	Dis	$\mathbb{Z}[i\sqrt{3}]$	
3	6	∞	3	3	3	lox	Dis	$\mathbb{Z}[i\sqrt{3}]$	(2.1)
3	3	6	∞	∞	∞	lox	Dis	$\mathbb{Z}[i\sqrt{3}]$	
4	4	4	∞	∞	∞	lox	Dis	$\mathbb{Z}[i\sqrt{7}]$	
4	4	∞	4	∞	∞	lox	Dis	$\mathbb{Z}[i\sqrt{7}]$	[14]
6	6	6	∞	∞	∞	lox	Dis	$\mathbb{Z}[(1+i\sqrt{11})/2]$	
6	6	∞	6	lox	lox	lox	Dis	$\mathbb{Z}[(1+i\sqrt{11})/2]$	(2.1)
4	4	∞	∞	lox	lox	lox	Dis	$\mathbb{Z}[i\sqrt{3}]$	[14]

TABLE 2. Some known discrete $(p, q, r; n)$ groups

References

1. M. Deraux, *Deforming the $\mathbb{R}$-Fuchsian (4,4,4)-triangle groups into a lattice.* Topology, 45 (2006), 989-1020.
2. M. Deraux, E. Falbel, J. Paupert, *New constructions of fundamental polyhedra in complex hyperbolic space.* Acta Math 194 (2005) 155-201
3. E. Falbel, J.R. Parker, *The geometry of the Eisenstein-Picard group.* Duke Math. J., 131 (2006), 249-289
4. E. Falbel, G. Francis, J.R. Parker, *The geometry of the Gauss-Picard modular group.* Preprint `http://www.maths.dur.ac.uk/~dma0jrp/img/GaussPicard.pdf`
5. W.M. Goldman, *Complex hyperbolic geometry*, Oxford University Press, New York, 1999

6. S. Kamiya, J.R.Parker, J.M.Thompson, *Non-Discrete Complex Hyperbolic Triangle Groups of Type* $(n, n, \infty; k)$. To appear in Canadian Mathematical Bulletin
7. R.A. Livné, *On certain covers of the universal elliptic curves.* Ph.D. thesis, Havard University, 1981
8. J.R. Parker, *Unfaithful complex hyperbolic triangle groups I.* Pacific Journal of Mathematics, 238 (2008), 145-169
9. J.R. Parker, *Cone metrics on the sphere and Livné's lattices.* Acta Math., 196 (2006), 1-64
10. A. Pratoussevitch, *Traces in complex hyperbolic triangle groups.* Geometriae Dedicata 111 (2005), 159-185
11. R.E. Schwartz, *Complex hyperbolic triangle groups.* Proc. of the International Cong. of Math. Vol. II (Beijing, 2002) 339-349, Higher Ed. Press Beijing,2000
12. R.E. Schwartz, *Real hyperbolic on the outside, complex hyperbolic on the inside.* Invent. Math. 151 (2003), no. 2, 221–295.
13. J.M. Thompson, *Two new triangle group lattices in the complex hyperbolic plane.* Preprint 2009.
14. J. Wyss-Gallifent, *Complex Hyperbolic Triangle Groups.* Ph.D. thesis, University of Maryland, 2000

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