Unfaithful complex hyperbolic triangle groups

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Abstract

A complex hyperbolic triangle group is the group of complex hyperbolic isometries generated by complex involutions fixing three complex lines in complex hyperbolic space. Such a group is called symmetric if there is an isometry of order three that cyclically permutes the three complex lines. We consider symmetric triangle groups for which the product of each pair of involutions and the product of all three involutions are all non-loxodromic. We classify all such groups that are discrete.

1 Introduction

A complex hyperbolic triangle group is a group generated by three involutions that fix complex lines in complex hyperbolic space. The study of complex hyperbolic triangle groups was begun in [6] where ideal triangle groups were considered. Since then there have been many developments. There have been two strands to this work. First, following [6], discrete and faithful representations of triangle groups have been investigated; see [13] for example. On the other hand, there has been the study of discrete representations where certain group elements are required to be elliptic of finite order; see [3], [11], [15] for example. These representations are necessarily unfaithful. Schwartz has given an excellent survey [14] that outlines progress and gives a conjectural overview of what we might expect. A more recent survey is also contained in Pratoussevitch's paper [12]. Pratoussevitch also considers the case where the generators are complex reflections of higher order, which is related to earlier work of Mostow; see [2] for example. We will not be concerned with that here.

Our starting point is the following theorem, proved by Schwartz in Section 3.3 of [13]:

Theorem 1.1 (Schwartz [13]) *Let $\Delta = \langle I_1, I_2, I_3\rangle$ be the group of complex hyperbolic isometries generated by complex involutions I_j each fixing a complex line. If I_1I_2, I_2I_3 and I_3I_1 are all parabolic and $I_1I_2I_3$ is elliptic then Δ is not discrete.*

The main theorem of this paper is to consider $\Delta = \langle I_1, I_2, I_3\rangle$ where I_1I_2, I_2I_3 and I_3I_1 are all conjugate elliptic maps and $I_1I_2I_3$ is also elliptic. We classify all such Δ that are discrete and we find that there are remarkably few of them. The point is that I_1I_2 and $I_1I_2I_3$ should simultaneously have finite order. We now give a rough statement of our main theorem. For a more precise statement see Theorem 3.7 and Proposition 4.5.

Theorem 1.2 *Let $\Delta = \langle I_1, I_2, I_3\rangle$ be the group of complex hyperbolic isometries generated by complex involutions I_j each fixing a complex line. Suppose that I_1I_2, I_2I_3 and I_3I_1 are conjugate elliptic maps and that $I_1I_2I_3$ is also elliptic. If Δ is discrete then one of the following is true:*

(i) *Δ is a finite;*

(ii) *Δ is a normal subgroup of one of Livné's lattices;*

(iii) *Δ is Deraux's lattice;*

(iv) *Δ is the second group described in Section 4.2.*

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2 Parameters and traces

In this section we show how to parametrise symmetric complex hyperbolic triangle groups with a single complex parameter τ and we find the values of τ that correspond to such a group. We then describe how the properties of the group (for example the type of I_1I_2 and $I_1I_2I_3$) vary with τ. We will try to keep this account as self contained as possible. However, we shall assume a certain amount of background knowledge of complex hyperbolic geometry. For such background material on complex hyperbolic space see [5] and for material on complex hyperbolic triangle groups see [14] or [12].

2.1 Complex reflections

Let L_1 be a complex line in complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ and write I_1 for the complex reflection of order 2 fixing L_1. We may lift I_1 to a matrix in $\mathrm{SU}(2,1)$. If the polar vector of L_1 is $\mathbf{n}_1$ then I_1 is given by:

$$I_1(\mathbf{z}) = -\mathbf{z} + 2\frac{\langle \mathbf{z}, \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}\,\mathbf{n}_j. \tag{1}$$

In Section 3.3.2 of [5], Goldman uses the polar vectors of two complex lines to determine the geometry of their relative position.

Proposition 2.1 (Goldman [5]) *Suppose that L_1 and L_2 are complex lines in $\mathbf{H}^2_{\mathbb{C}}$ with polar vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Let*

$$N(L_1, L_2) = \frac{\langle \mathbf{n}_1, \mathbf{n}_2\rangle\langle \mathbf{n}_2, \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle\langle \mathbf{n}_2, \mathbf{n}_2\rangle}.$$

(i) *If $N(L_1, L_2) > 1$ then L_1 and L_2 are ultraparallel and the distance between them is 2δ where $N(L_1, L_2) = \cosh^2\delta$;*

(ii) *If $N(L_1, L_2) = 1$ then either L_1 and L_2 are asymptotic or $L_1 = L_2$;*

(iii) *If $N(L_1, L_2) < 1$ then L_1 and L_2 intersect with angle θ where $N(L_1, L_2) = \cos^2\theta$.*

We shall be interested groups $\Delta = \langle I_1, I_2, I_3\rangle$ generated by complex involutions fixing complex lines L_1, L_2 and L_3. We say that this configuration of lines is *symmetrical* if there is a map $J \in \mathrm{SU}(2,1)$ of order 3 so that $J(L_j) = L_{j+1}$, where the indices are taken mod 3. This implies that $I_2 = JI_1J^{-1}$ and $I_3 = JI_2J^{-1} = J^{-1}I_1J$ and so the group $\Delta = \langle I_1, I_2, I_3\rangle$ is a normal subgroup of $\Gamma = \langle I_1, J\rangle$ of index three. We say that Δ is a *symmetric triangle group*. Note that

$I_1 I_2 I_3 = I_1(JI_1J^{-1})(J^{-1}I_1J) = (I_1J)^3$. If L_1, L_2 and L_3 are symmetrical then, since J preserves the Hermitian form,

$$N(L_j, L_{j+1}) = \frac{\langle \mathbf{n}_j, \mathbf{n}_{j+1}\rangle\langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle\langle \mathbf{n}_{j+1}, \mathbf{n}_{j+1}\rangle} = \frac{\left|\langle J(\mathbf{n}_j), \mathbf{n}_j\rangle\right|^2}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle^2}.$$

2.2 The parameter space

Suppose we are given a symmetrical configuration of three complex lines L_1, L_2 and L_3 with polar vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. Because they are symmetrical $J(\mathbf{n}_j) = \mathbf{n}_{j+1}$ where $j = 1, 2, 3$ taken mod 3. Because J preserves the Hermitian form $\langle \mathbf{n}_j, \mathbf{n}_j\rangle$ is the same positive real number for each j. We normalise $\mathbf{n}_j$ so that this number is 2. Likewise $\langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle = \langle J(\mathbf{n}_j), \mathbf{n}_j\rangle$ is the same complex number for each j which we define to be τ. That is

$$\langle \mathbf{n}_1, \mathbf{n}_1\rangle = \langle \mathbf{n}_2, \mathbf{n}_2\rangle = \langle \mathbf{n}_3, \mathbf{n}_3\rangle = 2, \qquad \langle \mathbf{n}_2, \mathbf{n}_1\rangle = \langle \mathbf{n}_3, \mathbf{n}_2\rangle = \langle \mathbf{n}_1, \mathbf{n}_3\rangle = \tau. \tag{2}$$

All of this works for any Hermitian form of signature $(2,1)$. We now make a choice of vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. This determines a Hermitian form. We choose

$$\mathbf{n}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \mathbf{n}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \mathbf{n}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Together with (2), this means that, with this choice, the Hermitian form must be $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{w}^* H_\tau \mathbf{z}$ where

$$H_\tau = \begin{bmatrix}2 & \tau & \overline{\tau}\\ \overline{\tau} & 2 & \tau\\ \tau & \overline{\tau} & 2\end{bmatrix}. \tag{3}$$

We can immediately write down J and, using (1), the involutions I_j. They are

$$J = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \qquad I_1 = \begin{bmatrix}1 & \tau & \overline{\tau}\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}, \qquad I_2 = \begin{bmatrix}-1 & 0 & 0\\ \overline{\tau} & 1 & \tau\\ 0 & 0 & -1\end{bmatrix}, \qquad I_3 = \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ \tau & \overline{\tau} & 1\end{bmatrix}.$$

From this it is clear that the groups $\Gamma = \langle I_1,\, J\rangle$ and $\Delta = \langle I_1,\, I_2,\, I_3\rangle$ are completely determined up to conjugation by the parameter τ. However, not all values of τ correspond to complex hyperbolic triangle groups. It may be that the Hermitian matrix H_τ does not have signature $(2,1)$. We now determine this by finding the eigenvalues of H_τ. In this lemma and throughout the paper we write $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$.

Lemma 2.2 *Let H_τ be given by (3) and write $\tau = t + is$. The eigenvalues of H_τ are*

$$2 + \tau + \overline{\tau} = 2 + 2t, \qquad 2 + \tau\overline{\omega} + \overline{\tau}\omega = 2 - t + \sqrt{3}s, \qquad 2 + \tau\omega + \overline{\tau\omega} = 2 - t - \sqrt{3}s.$$

Proof: We could solve the characteristic polynomial of H_τ. However, it is easier to observe that eigenvectors for H_τ are

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad \begin{bmatrix}1\\ \overline{\omega}\\ \omega\end{bmatrix}, \qquad \begin{bmatrix}1\\ \omega\\ \overline{\omega}\end{bmatrix}.$$

□

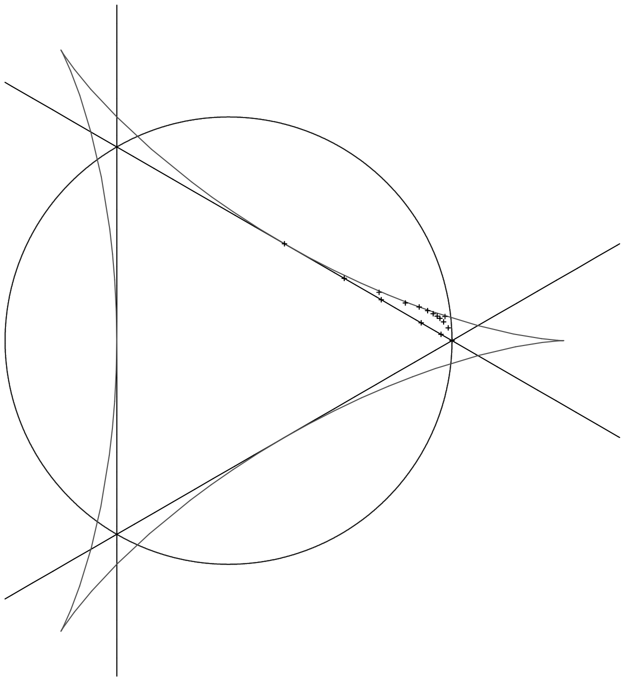

Figure 1: *The parameter space. We have drawn the points corresponding to the groups listed in Theorem 1.2. For more details of this part of the picture, see Figure 2.*

Corollary 2.3 *The matrix H_τ has signature $(2,1)$ if and only if*

$$6|\tau|^2 - \tau^3 - \overline{\tau}^3 - 8 > 0. \tag{4}$$

PROOF: It is easy to check (for example by adding them) that all three eigenvalues cannot be negative. Thus H_τ has signature $(2,1)$ if and only if its determinant is negative. That is

$$0 > (2+\tau+\overline{\tau})(2+\tau\overline{\omega}+\overline{\tau}\omega)(2+\tau\omega+\overline{\tau\omega}) = 8+\tau^3+\overline{\tau}^3-6|\tau|^2.$$

□

The locus where each eigenvalue is zero corresponds to a line in $\mathbb{C}$. In Figure 1 we have drawn these three lines. They divide $\mathbb{C}$ into seven components which fall into three types:

- The central triangle. Here all three eigenvalues are positive and H_τ is positive definite.
- Three infinite components that each share a common edge with the central triangle. Here two eigenvalues are positive and one negative. Hence H_τ has signature $(2,1)$. This is our parameter space.
- Three infinite components that each only abut the central triangle in a point. Here one eigenvalue is positive and two are negative. These correspond to groups of complex hyperbolic isometries generated by three complex involutions that each fix a point.

Therefore the parameter space we are interested in, that is those values of τ satisfying (4), has three components related by multiplication by powers of $\omega = e^{2\pi i/3}$. This corresponds to that fact that $J \in \mathrm{SU}(2,1)$ is only defined up to multiplication by a cube root of unity. Hence τ is only defined up to a cube root of unity. If we factor out by this equivalence, our parameter space becomes one of these components. Also, complex conjugating τ (that is sending J to its inverse) swaps the roles of I_2 and I_3. Our parameter space also respects this symmetry. Hence we may restrict our attention to those τ whose argument lies in $[0, \pi/3]$. In fact, this description can be shown to be a reformulation in terms of τ of Pratoussevitch's result, Proposition 1 of [12], for our special case. We will give details of how to pass from τ to Pratoussevitch's parameters in the next section.

2.3 Traces

We have seen that if τ satisfies (4) then τ corresponds to a group $\Gamma = \langle I_1, J\rangle$, in $\mathrm{SU}(2,1)$, with a symmetric triangle group $\Delta = \langle I_1, I_2, I_3\rangle$ as an index three normal subgroup. In this section we write down the traces of certain elements of Γ in terms of τ. This should be compared to Theorem 9 of [12] where Pratoussevitch gives (in our language) formulae for the traces of elements in Δ that are integer polynomials in $|\tau|^2$, τ^3 and $\overline{\tau}^3$. We could write down the traces directly from our expressions for J, I_1, I_2 and I_3. We choose to give a more general argument as this is more illuminating and is independent of our choice of Hermitian form.

Lemma 2.4 *Let A be any element of* $\mathrm{SU}(2,1)$. *Then*

$$\mathrm{tr}(I_1 A) = -\mathrm{tr}\,(A) + 2\frac{\langle A(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}.$$

PROOF: Using (1) we see that

$$I_1 A(\mathbf{z}) = -A(\mathbf{z}) + 2\frac{\langle A(\mathbf{z}), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}\mathbf{n}_1.$$

We must find the trace of the matrix corresponding to the linear map $T : \mathbf{z} \longmapsto \langle A(\mathbf{z}), \mathbf{n}_1\rangle \mathbf{n}_1$. Writing $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{w}^* H \mathbf{z}$ and using the fact that this is a complex scalar, we have

$$\langle A(\mathbf{z}), \mathbf{n}_1\rangle \mathbf{n}_1 = \mathbf{n}_1 \mathbf{n}_1^* H A \mathbf{z} = \big(\mathbf{n}_1 (A^* H \mathbf{n}_1)^*\big)\mathbf{z}.$$

Hence the matrix of T is $\mathbf{n}_1(A^* H \mathbf{n}_1)^*$. Now if a matrix can be written in the form $\mathbf{u}\mathbf{v}^*$ for column vectors $\mathbf{u}$ and $\mathbf{v}$, then its trace is just $\mathbf{v}^*\mathbf{u}$. Thus

$$\mathrm{tr}\big(\mathbf{n}_1 (A^* H \mathbf{n}_1)^*\big) = (A^* H \mathbf{n}_1)^* \mathbf{n}_1 = \mathbf{n}_1^* H A \mathbf{n}_1 = \langle A(\mathbf{n}_1), \mathbf{n}_1\rangle.$$

Hence

$$\mathrm{tr}(I_1 A) = -\mathrm{tr}\,(A) + 2\frac{\mathrm{tr}\big(\mathbf{n}_1 (A^* H \mathbf{n}_1)^*\big)}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle} = -\mathrm{tr}\,(A) + 2\frac{\langle A(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}.$$

□

Corollary 2.5 *Let I_1 be a complex involution fixing a complex line L_1 with polar vector $\mathbf{n}_1$. Let $J \in \mathrm{SU}(2,1)$ be a regular elliptic map of order 3. Then*

$$\mathrm{tr}(I_1 J) = 2\frac{\langle J(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle} = \tau.$$

PROOF: This follows directly from the previous lemmas using the fact that, since J is regular elliptic of order three, its trace is zero. □

We remark that our justification for naming our parameter τ is that it is the trace of $I_1 J$. The following theorem is just a restatement of Theorem 6.2.4 of [5]

Proposition 2.6 *Let τ be given by (2). Then I_1J is regular elliptic if and only if*
$$|\tau|^4 - 4\tau^3 - 4\overline{\tau}^3 + 18|\tau|^2 - 27 < 0.$$

The curve given by the equality in Proposition 2.6 is a deltoid; see Figure 1. Groups for which I_1J is regular elliptic correspond to points in the interior of this deltoid. Points on the deltoid correspond to points where I_1J is either a complex reflection or is parabolic. Since $I_1I_2I_3 = (I_1J)^3$ we can determine the type of $I_1I_2I_3$ from I_1J. (Note that it may be that I_1J is regular elliptic and that $I_1I_2I_3$ is a complex reflection.)

We now consider I_1I_2, and hence by symmetry I_2I_3 and I_3I_1 as well.

Proposition 2.7 *Let L_1 be a complex line in $\mathbf{H}^2_{\mathbb{C}}$ with polar vector $\mathbf{n}_1$. Let $J \in \mathrm{SU}(2,1)$ be a regular elliptic map of order 3 and write $L_2 = J(L_1)$. Let I_1 and I_2 denote the complex involutions fixing L_1 and L_2. Then*
$$\mathrm{tr}(I_1I_2) = |\tau|^2 - 1$$
where τ is given by (2).

PROOF: Let $\mathbf{z}_2$ and $\mathbf{z}_3$ be any distinct vectors on L_1 (for example, we could choose $\mathbf{z}_2 = L_1 \cap L_3$ and $\mathbf{z}_3 = L_1 \cap L_2$). Then $\{\mathbf{n}_1, \mathbf{z}_2, \mathbf{z}_3\}$ is a basis for $\mathbb{C}^{2,1}$. We write the polar vector to $J(L_1) = L_2$ as
$$J(\mathbf{n}_1) = \mathbf{n}_2 = \alpha\mathbf{n}_1 + \beta\mathbf{z}_2 + \gamma\mathbf{z}_3.$$
Since $\mathbf{n}_1$ is orthogonal to $\mathbf{z}_2$ and $\mathbf{z}_3$ then $\langle \mathbf{n}_2, \mathbf{n}_1 \rangle = \alpha\langle \mathbf{n}_1, \mathbf{n}_1 \rangle$ and so $\alpha = \tau/2$. Since $\mathbf{n}_1$ is a 1-eigenvector for I_1 and $\mathbf{z}_2$, $\mathbf{z}_3$ are both (-1)-eigenvectors we have
$$\mathbf{n}_2 = \frac{\tau}{2}\mathbf{n}_1 + \beta\mathbf{z}_2 + \gamma\mathbf{z}_3, \qquad I_1(\mathbf{n}_2) = \frac{\tau}{2}\mathbf{n}_1 - \beta\mathbf{z}_2 - \gamma\mathbf{z}_3.$$
Adding these two expressions we see that $I_1(\mathbf{n}_2) = \tau\mathbf{n}_1 - \mathbf{n}_2$. Therefore
$$\langle I_1(\mathbf{n}_2), \mathbf{n}_2 \rangle = \tau\langle \mathbf{n}_1, \mathbf{n}_2 \rangle - \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = \frac{|\tau|^2}{2}\langle \mathbf{n}_2, \mathbf{n}_2 \rangle - \langle \mathbf{n}_2, \mathbf{n}_2 \rangle.$$
Hence using Lemma 2.4 we have
$$\mathrm{tr}(I_2I_1) = -\mathrm{tr}(I_1) + 2\frac{\langle I_1(\mathbf{n}_2), \mathbf{n}_2 \rangle}{\langle \mathbf{n}_2, \mathbf{n}_2 \rangle} = 1 + |\tau|^2 - 2 = |\tau|^2 - 1.$$
□

An element of $\mathrm{SU}(2,1)$ with real trace is elliptic when its trace lies in $[-1, 3)$ (when the trace is -1 it may also be parabolic). Hence we see that I_1I_2 is elliptic if and only if $|\tau| < 2$ (one easily checks that if $\tau = 0$ then $I_1I_2 = I_3$). In Figure 1 we show how the circle $|\tau| = 2$ compares to the deltoid of Proposition 2.6. Furthermore, we can now see that Pratoussevitch's parameters [12] may be written in terms of τ as:
$$r_1 = r_2 = r_3 = |\tau|/2, \qquad \alpha = \arg(\tau^3).$$
Using Proposition 2.1 we can, in fact, relate the trace of I_1I_2 with the relative position of I_1 and I_2.

Corollary 2.8 *Let L_1 be a complex line in $\mathbf{H}^2_{\mathbb{C}}$ with polar vector $\mathbf{n}_1$. Let $J \in \mathrm{SU}(2,1)$ and write $L_2 = J(L_1)$. Let I_1 and I_2 denote the complex involutions fixing L_1 and L_2. Then:*

(i) If L_1 and L_2 are ultraparallel then $\mathrm{tr}(I_1I_2) = 2\cosh(2\delta) + 1$ where 2δ is the distance between them;

(ii) If L_1 and L_2 are asymptotic then $\mathrm{tr}(I_1I_2) = 3$;

(iii) If L_1 and L_2 intersect with angle θ then $\mathrm{tr}(I_1I_2) = 2\cos(\theta) + 1$.

3 When $I_1I_2I_3$ is elliptic and I_1I_2 is non-loxodromic

This section is the heart of the paper. We restrict our attention to those groups for which $I_1I_2I_3$ is elliptic of finite order and I_1I_2 is either elliptic of finite order or else parabolic. These are groups for which τ lies inside or on the deltoid and inside or on the circle in Figure 1. Since they have finite order, the eigenvalues of $I_1I_2I_3$ and I_1I_2 are all roots of unity. This fact leads to a linear equation in certain cosines of rational multiples of π. We find all solutions to this equation using a theorem of Conway and Jones [1]. We then go on find which of these solutions lie in parameter space. That is, which of the solutions lie outside the central triangle in 1. As we have already indicated, it suffices to consider those τ whose argument lies in $[0, \pi/3]$. Such values of τ lying outside the central triangle and yet inside both the deltoid and circle are shown in Figure 2.

3.1 The eigenvalue equation

We now investigate when both I_1I_2 and $I_1I_2I_3$ are elliptic of finite order. In fact our proof will be valid when I_1I_2 is parabolic and yields a new proof of Theorem 1.1. We know that, I_1J (and hence $I_1I_2I_3$) is elliptic of finite order if and only if

$$\tau = \mathrm{tr}(I_1J) = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}, \tag{5}$$

where α and β are rational multiples of π. Likewise for I_1I_2. In fact we know slightly more. Since the intersection of L_1 and L_2 is a (-1)-eigenvector for each of I_1 and I_2 it must be a $(+1)$-eigenvector for I_1I_2. Hence the eigenvectors of I_1I_2 are 1, $e^{i\theta}$ and $e^{-i\theta}$. That is

$$|\tau|^2 - 1 = \mathrm{tr}(I_1I_2) = 2\cos\theta + 1, \tag{6}$$

where θ is a rational multiple of π. From Corollary 2.8 (iii) we see that, geometrically, θ is just the angle between L_1 and L_2. If $\theta = 0$ then I_1I_2 then is parabolic (or the identity) and we shall include this case in our analysis.

We solve equations (5) and (6) by eliminating τ. That is, we seek θ, α, β rational multiples of π so that

$$2\cos\theta + 2 = |\tau|^2 = 3 + 2\cos(\alpha - \beta) + 2\cos(\alpha + 2\beta) + 2\cos(-2\alpha - \beta).$$

Rearranging, this becomes

$$\frac{1}{2} = \cos(\theta) - \cos(\alpha - \beta) - \cos(\alpha + 2\beta) - \cos(-2\alpha - \beta). \tag{7}$$

The theorem that we use to find all solutions to (7) is the following wonderful theorem due to Conway and Jones, Theorem 7 of [1]:

Theorem 3.1 (Conway and Jones [1]) *Suppose that we are given at most four distinct rational multiples of π lying strictly between 0 and $\pi/2$ for which some rational linear combination of their cosines is rational, but no proper subsum has this property. Then this linear combination is* *proportional to one of the following:*

$$
\begin{aligned}
\frac{1}{2} &= \cos\left(\frac{\pi}{3}\right), \\
0 &= -\cos(\phi) + \cos\left(\phi - \frac{\pi}{3}\right) + \cos\left(\phi + \frac{\pi}{3}\right), \qquad \textit{where } 0 < \phi < \frac{\pi}{6}, \\
\frac{1}{2} &= \cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right), \\
\frac{1}{2} &= \cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right), \\
\frac{1}{2} &= \cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{15}\right) + \cos\left(\frac{4\pi}{15}\right), \\
\frac{1}{2} &= -\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{15}\right) - \cos\left(\frac{7\pi}{15}\right), \\
\frac{1}{2} &= \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) - \cos\left(\frac{\pi}{21}\right) + \cos\left(\frac{8\pi}{21}\right), \\
\frac{1}{2} &= \cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{21}\right) - \cos\left(\frac{5\pi}{21}\right), \\
\frac{1}{2} &= -\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{4\pi}{21}\right) + \cos\left(\frac{10\pi}{21}\right), \\
\frac{1}{2} &= -\cos\left(\frac{\pi}{15}\right) + \cos\left(\frac{2\pi}{15}\right) + \cos\left(\frac{4\pi}{15}\right) - \cos\left(\frac{7\pi}{15}\right).
\end{aligned}
$$

We claim that we may find all solutions to equation (7) by inspection from this theorem. In order to see this, observe that by sending ϕ to $\pi - \phi$ we change the sign of $\cos\phi$. Thus for each equation in Theorem 3.1 we arrange for all the cosines to have the same sign, but to lie in $(0, \pi)$. We then look for three of the angles that (after possibly changing their sign) add up to a multiple of 2π. These angles are $\alpha - \beta$, $\alpha + 2\beta$ and $-2\alpha - \beta$. The trickiest cases are those where there are fewer than four angles listed in Theorem 3.1. One then has to add multiples of $\cos(0) = 1$, $\cos(\pi/3) = 1/2$ or $\cos(\pi/2) = 0$ to reconstruct equation (7). Hence, by inspection, we find the only candidates for $\alpha - \beta$, $\alpha + 2\beta$, $-2\alpha - \beta$ solving equation (7) and we list them in the following table along with θ (we have ordered them so that $\alpha - \beta < \alpha + 2\beta < 2\alpha + \beta$). From this we find α, β and $\alpha + \beta$.

θ	$\alpha - \beta$	$\alpha + 2\beta$	$2\alpha + \beta$	α	β	$\alpha + \beta$
ϕ	$\pi/3 - \phi$	$\pi/3 + \phi$	$2\pi/3$	$\pi/3 - \phi/3$	$2\phi/3$	$\pi/3 + \phi/3$
$2\pi/3$	$\pi/2 - \phi$	$\pi/2 + \phi$	π	$\pi/2 - \phi/3$	$2\phi/3$	$\pi/2 + \phi/3$
$\pi/3$	$\pi/4$	$\pi/2$	$3\pi/4$	$\pi/3$	$\pi/12$	$5\pi/12$
$\pi/5$	$3\pi/10$	$2\pi/5$	$7\pi/10$	$\pi/3$	$\pi/30$	$11\pi/30$
$3\pi/5$	$\pi/10$	$4\pi/5$	$9\pi/10$	$\pi/3$	$7\pi/30$	$17\pi/30$
$\pi/2$	$2\pi/7$	$4\pi/7$	$6\pi/7$	$8\pi/21$	$2\pi/21$	$10\pi/21$
$\pi/2$	$\pi/15$	$11\pi/15$	$4\pi/5$	$13\pi/45$	$2\pi/9$	$23\pi/45$
$\pi/2$	$2\pi/5$	$7\pi/15$	$13\pi/15$	$19\pi/45$	$\pi/45$	$4\pi/9$
$\pi/7$	$\pi/21$	$4\pi/7$	$13\pi/21$	$2\pi/9$	$11\pi/63$	$25\pi/63$
$3\pi/7$	$2\pi/7$	$11\pi/21$	$17\pi/21$	$23\pi/63$	$5\pi/63$	$4\pi/9$
$5\pi/7$	$5\pi/21$	$19\pi/21$	$8\pi/7$	$29\pi/63$	$2\pi/9$	$43\pi/63$

We then write down $\tau = \mathrm{tr}(I_1 J) = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}$ and $\mathrm{tr}\,(I_1 I_2 I_3) = e^{3i\alpha} + e^{3i\beta} + e^{-3i\alpha - 3i\beta}$ using this table. As indicated earlier, then $\tau\omega^j$ $\overline{\tau}\omega^j$ correspond to the same group as τ. Hence we restrict our attention to those τ whose argument lies in $[0, \pi/3]$. For the first case, we could again take $0 < \phi < \pi/6$, but it is clear that we may increase the range of ϕ and still have the same result, including the case where $\phi = 0$ when (7) becomes $1/2 = 1 - 1/2 - 1/2 + 1/2$. Therefore, it is no loss of generality to suppose that ϕ is any angle (although we then loose the size ordering of the angles). Likewise for the second line in the table, which arises from the degenerate solution $1/2 = -1/2 + \sin\phi - \sin\phi + 1$ to (7). From this table we immediately obtain:

Proposition 3.2 *Suppose that I_1I_2 and $I_1I_2I_3$ are both elliptic of finite order (or possibly I_1I_2 is parabolic) then one of the following is true:*

(i) $\tau = e^{2i\phi/3} + e^{-i\phi/3} = e^{i\phi/6}2\cos(\phi/2)$, *for some angle* $\phi \in [0, 2\pi]$*;*

(ii) $\tau = e^{2i\phi/3}$ *for some angle* $\phi \in [0, \pi/2]$*;*

(iii) $\tau = e^{2\pi i/3}\big(e^{-2\pi i/7} + e^{-4\pi i/7} + e^{-8\pi i/7}\big)$*;*

(iv) $\tau = e^{\pi i/3} + e^{-i\pi/6}2\cos\phi$ *for* $\phi = \pi/4,\ \pi/5$ *or* $2\pi/5$*;*

(v) $\tau = e^{2\pi i/9} + e^{-\pi i/9}2\cos\phi$ *for* $\phi = 2\pi/5,\ 2\pi/7$ *or* $4\pi/7$*;*

(vi) $\tau = \overline{\omega}(e^{2\pi i/9} + e^{-\pi i/9}2\cos\phi)$ *for* $\phi = 4\pi/5$ *or* $6\pi/7$*,*

Notice that the only solution with $\theta = 0$ involves setting $\phi = 0$ in part (i). This is the same as setting $\tau = 2$ and corresponds to the group where $I_1 = I_2 = I_3$. This gives a new proof of Schwartz' theorem, Theorem 1.1. We also want to point out that the first two cases of the above theorem includes various elementary groups:

- Putting $\phi = 0$ in (i) we obtain $\tau = 2$ which yields the elementary group where $I_1 = I_2 = I_3$.
- Putting $\phi = \pi/2$ in (i) gives $\tau = e^{\pi i/3}(1 - i)$.
- Putting $\phi = \pi/2$ in (ii) we obtain $\tau = e^{\pi i/3}$.

3.2 Solutions in parameter space

We now consider the values of τ found in Proposition 3.2 and we check which of them satisfy the conditions of Corollary 2.3. Note that since $|\tau| \le 2$ the only possibilities are that H_τ has signature $(2,1)$ or $(3,0)$. We state our results in terms of the signature of H_τ. First, for τ given in part (ii) H_τ has signature $(3,0)$ unless $\tau = e^{i\pi/3}$, when H_τ is degenerate. We now consider the other cases one by one.

Lemma 3.3 *If* $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ *then* H_τ *has signature* $(2,1)$ *if and only if* $0 < \cos\phi < 1$ *and signature* $(3,0)$ *if and only if* $-1 \le \cos\phi < 0$*.*

PROOF: We have $|\tau|^2 = 2 + 2\cos\phi$ and $\tau^3 = e^{2i\phi} + 3e^{i\phi} + 3 + e^{-i\phi}$. Therefore

$$6|\tau|^2 - \tau^3 - \overline{\tau}^3 - 8 = 6(2 + 2\cos\phi) - 2(2\cos^2\phi + 2\cos\phi + 2) - 8 = 4\cos\phi - 4\cos^2\phi.$$

This is positive when $0 \le \cos\phi \le 1$ and negative when $-1 \le \cos\phi < 0$. □

Lemma 3.4 *If* $\tau = e^{2\pi i/3}\big(e^{-2\pi i/7} + e^{-4\pi i/7} + e^{-8\pi i/7}\big)$ *then* H_τ *has signature* $(3,0)$*.*

PROOF: We can rewrite τ as

$$t+is=\tau=\frac{-1+i\sqrt{3}}{2}\cdot\frac{-1-i\sqrt{7}}{2}=\frac{1+\sqrt{21}+i\sqrt{7}-i\sqrt{3}}{4}.$$

Thus from Lemma 2.2, the eigenvalues of H_τ are

$$2+2t=\frac{5+\sqrt{21}}{2},\qquad 2-t+\sqrt{3}s=1,\qquad 2-t-\sqrt{3}s=\frac{5-\sqrt{21}}{2}.$$

These are all positive. □

Lemma 3.5 *Suppose that $\tau=e^{\pi i/3}+e^{-\pi i/6}2\cos\phi$. Then H_τ is degenerate, that is it has determinant zero.*

PROOF: We have $|\tau|^2=1+4\cos^2\phi$ and $\tau^3=-1+12\cos^2\phi+6i\cos\phi-8i\cos^3\phi$. Thus

$$6|\tau|^2-\tau^3-\overline{\tau}^3-8=6(1+4\cos^2\phi)-(-2+24\cos^2\phi)-8=0.$$

□

Lemma 3.6 *Suppose that either $\tau=e^{2\pi i/9}+e^{-\pi i/9}2\cos\phi$ or $\tau=\overline{\omega}(e^{2\pi i/9}+e^{-\pi i/9}2\cos\phi)$ Then H_τ has signature $(2,1)$ if $\cos(3\phi)<-1/2$ and signature $(3,0)$ if and only if $\cos(3\phi)>-1/2$.*

PROOF: In either case we have

$$|\tau|^2=1+2\cos\phi+4\cos^2\phi,\qquad \tau^3=e^{2\pi i/3}+e^{\pi i/3}6\cos\phi+12\cos^2\phi+e^{-\pi i/3}8\cos^3\phi.$$

Thus

$$\begin{aligned}6|\tau|^2-\tau^3-\overline{\tau}^3-8 &= 6(1+2\cos\phi+4\cos^2\phi)-(-1+6\cos\phi+24\cos^2\phi+8\cos^3\phi)-8\\ &= -1+6\cos\phi-8\cos^3\phi\\ &= -1-2\cos(3\phi).\end{aligned}$$

This is positive if and only if $\cos(3\phi)<-1/2$ and negative if and only if $\cos(2\phi)>-1/2$. □

We have shown that the values of τ given in parts (v), (vii) and (viii) of Proposition 3.2 correspond to values of τ for which H_τ has signature $(3,0)$. Thus they do not correspond to groups in $\mathrm{SU}(2,1)$. We now summarise the results of this section.

Theorem 3.7 *Let I_1, $I_2=JI_1J^{-1}$ and $I_3=J^{-1}I_1J$ be involutions in $\mathrm{SU}(2,1)$ each fixing a complex line, where $J\in\mathrm{SU}(2,1)$ has order three. Suppose that I_1I_2 and $I_1I_2I_3=(I_1J)^3$ both have finite order. Then one of the following is true:*

(i) $\tau=e^{\pi i/3}+e^{-\pi i/6}2\cos(2\pi/n)=e^{i\pi/3}(1-2i\cos(2\pi/n))$ *where* $n=4,\ 5,\ 6,\ 8,\ 10$ *or* 12.

(ii) $\tau=e^{2i\phi/3}+e^{-i\phi/3}$ *where ϕ is a rational multiple of π in $(0,\pi/2)$;*

(iii) $\tau=e^{2\pi i/9}+e^{-\pi i/9}2\cos(2\pi/5)$;

(iv) $\tau=e^{2\pi i/9}+e^{-\pi i/9}2\cos(2\pi/7)$.

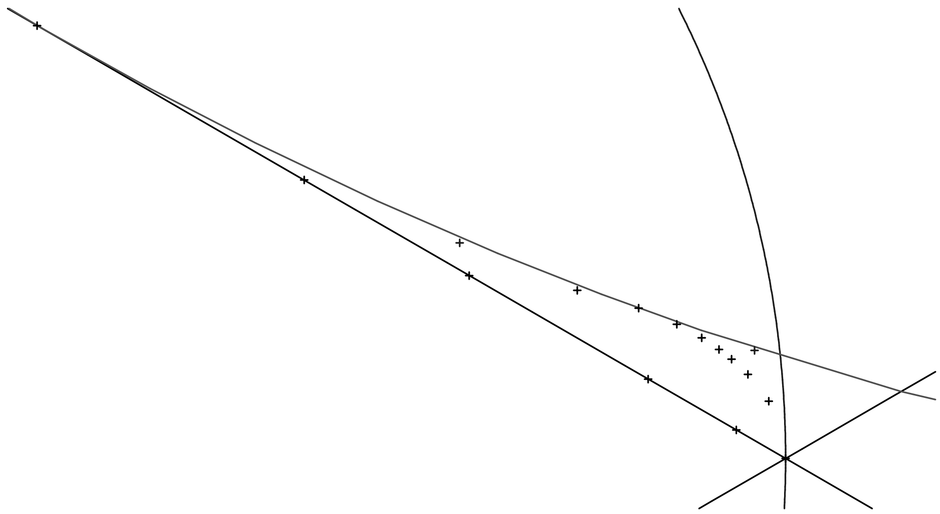

Figure 2: *An enlarged view of part of Figure 1 showing the values of τ in Theorem 3.7. Of the groups from Theorem 3.7 (ii) we have only plotted those that are discrete, as enumerated in Proposition 4.5.*

4 The discrete groups

In this section we analyse all the groups from Theorem 3.7. We show that those groups listed in parts (i), (iii) and (iv) are all discrete and that finitely many of those given in (ii) are discrete. These groups are all plotted in Figure 2.

4.1 The elementary groups

In this section we consider what happens when $\tau = e^{\pi i/3} + e^{-\pi i/6} 2\cos\phi = e^{i\pi/3}(1 - 2i\cos\phi)$ where $\phi = 2\pi/n$ for $n = 4$, 5, 6, 8, 10 or 12. These are the groups from Proposition 3.2 (iv) together with the case where $\phi = \pi/2$, which is the only interesting case from Proposition 3.2 (ii), and the cases $\phi = \pi/3$ and $\phi = \pi/6$, which correspond to putting $\phi = \pi/2$ and $\phi = 0$ respectively in (i).

In each case H_τ is degenerate. This is because L_1, L_2 and L_3 have a common point of intersection and so $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ span a complex 2-plane in $\mathbb{C}^3$. We must therefore use a different Hermitian form. We shall use the standard diagonal form $H = \mathrm{diag}(1, 1, -1)$ (see Chapter 3 of [5]) and normalise so that $L_1 \cap L_2 \cap L_3$ is the origin in the ball model.

In each case $\mathrm{tr}(I_j I_{j+1}) = 4\cos^2\phi = 1 + 2\cos(\theta)$. We could read off the values of θ from the table given earlier or we can calculate them directly. They are:

ϕ	$4\cos^2\phi$	$2\cos\theta$	θ
$\pi/2$	0	-1	$2\pi/3$
$2\pi/5$	$(3-\sqrt{5})/2$	$(1-\sqrt{5})/2$	$3\pi/5$
$\pi/3$	1	0	$\pi/2$
$\pi/4$	2	1	$\pi/3$
$\pi/5$	$(3+\sqrt{5})/2$	$(1+\sqrt{5})/2$	$\pi/5$
$\pi/6$	3	2	0

We now write down generators I_1 and J for these groups. Let $\psi = \arg(1 + 2i\cos\phi)$ and so $\tau = 2\cos(\theta/2)e^{\pi i/3 - i\psi}$.

$$I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad J = \begin{bmatrix} \cos(\theta/2)e^{\pi i/3 - i\psi} & -\sin(\theta/2)e^{\pi i/3 + i\psi} & 0 \\ \sin(\theta/2)e^{\pi i/3 - i\psi} & \cos(\theta/2)e^{\pi i/3 + i\psi} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{bmatrix}.$$

It is easy to see that $\langle I_1, J\rangle$ is a finite central extension of a tetrahedral, octahedral or icosahedral group in each case except the last where it is an abelian group of order six.

4.2 The groups with $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos(\phi)$.

In this section we consider the symmetric triangle groups corresponding to $\tau = e^{2\pi i/9}\big(1 - 2\omega\cos\phi\big)$ where $\omega = e^{2\pi i/3}$ is a cube root of unity and $\phi = 2\pi/5$ or $2\pi/7$. We remark that when $\phi = 2\pi/6$ we obtain $\tau = e^{2\pi i/9}\big(1 - 2\omega\cos\phi\big) = e^{2\pi i/9} + e^{-\pi i/3}$ which is one of the groups from Theorem 3.7 (ii) and is treated in Section 4.3. We want to eliminate the $e^{2\pi i/9}$ from our matrix entries. Therefore we apply the matrix $C = \mathrm{diag}(e^{-2\pi i/9},\, 1,\, e^{2\pi i/9})$ to the $\mathbf{n}_j$, and hence to the whole set-up. The images of the polar vectors under C are:

$$\mathbf{n}_1 = \begin{bmatrix} e^{-2\pi i/9} \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_3 = \begin{bmatrix} 0 \\ 0 \\ e^{2\pi i/9} \end{bmatrix}.$$

Applying C means conjugating H_τ. We call the conjugated matrix H_ϕ. It is

$$H_\phi = CH_\tau C^{-1} \begin{bmatrix} 2 & 1 - 2\omega\cos\phi & \overline{\omega} - 2\omega\cos\phi \\ 1 - 2\overline{\omega}\cos\phi & 2 & 1 - 2\omega\cos\phi \\ \omega - 2\overline{\omega}\cos\phi & 1 - 2\overline{\omega}\cos\phi & 2 \end{bmatrix}.$$

Obviously, since H_τ and H_ϕ are conjugate, they have the same eigenvalues. Using Lemma 3.6 we immediately have

Lemma 4.1 *When $\phi = 2\pi/5$ or $\phi = 2\pi/7$ the matrix H_ϕ has signature $(2,1)$. When $\phi = 4\pi/5$, $4\pi/7$ or $6\pi/7$ the matrix H_ϕ has signature $(3,0)$.*

After conjugating by C, the complex involutions are given by

$$I_1 = \begin{bmatrix} 1 & 1 - 2\omega\cos\phi & \overline{\omega} - 2\omega\cos\phi \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{8}$$

$$I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 - 2\overline{\omega}\cos\phi & 1 & 1 - 2\omega\cos\phi \\ 0 & 0 & -1 \end{bmatrix}, \tag{9}$$

$$I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \omega - 2\overline{\omega}\cos\phi & 1 - 2\overline{\omega}\cos\phi & 1 \end{bmatrix}. \tag{10}$$

The entries of I_1, I_2 and I_3 all have determinant one and lie in the ring of $\mathbb{Z}[\omega, 2\cos\phi]$. It is standard to write $\mathrm{SU}(H;\mathcal{O})$ for the group of unimodular matrices preserving the Hermitian form H whose entries lie in the ring $\mathcal{O}$. Thus:

Lemma 4.2 *The group $\Delta = \langle I_1, I_2, I_3\rangle$ generated by the matrices given in (8), (9) and (10) is a subgroup of $\mathrm{SU}\big(H_\phi; \mathbb{Z}[\omega, 2\cos\phi]\big)$.*

Since ω and $2\cos(2\pi/n)$ are both algebraic integers, we see that every element of the ring $\mathbb{Z}\big[\omega,\, 2\cos(2\pi/n)\big]$ is an algebraic integer in $\mathbb{Q}\big(\omega,\, 2\cos(2\pi/n)\big)$. The field $\mathbb{Q}\big(\omega,\, 2\cos(2\pi/n)\big)$ is a totally imaginary quadratic extension of the totally real number field $\mathbb{Q}\big(2\cos(2\pi/n)\big)$. The following result (in the case $\phi = 2\pi/5$) essentially identical to that given by Deraux in Corollary 2.6 of [3] and is similar to the proof of Corollary 1.4 of [15].

Proposition 4.3 *For $\phi = 2\pi/5$ or $\phi = 2\pi/7$ the group $\mathrm{SU}\big(H_\phi; \mathbb{Z}[\omega, 2\cos\phi]\big)$ is arithmetic and hence discrete. In particular, Δ is discrete.*

PROOF: We give the proof in the case of $n = 7$. The proof for $n = 5$ is almost the same; see also Corollary 2.6 of [3].

The field $\mathbb{Q}\big(\omega, 2\cos(2\pi/7)\big)$ is a totally imaginary quadratic extension of the totally real number field $\mathbb{Q}\big(2\cos(2\pi/7)\big)$. Let $\mathbb{Q}(c')$ be the totally real number field obtained from any non-trivial Galois conjugate of c' of $2\cos(2\pi/7)$ and let $\mathbb{Q}(\omega', c')$ be a compatible quadratic extension corresponding to $\mathbb{Q}(\omega, c)$. The only Galois conjugates of $2\cos(2\pi/7)$ are $c' = 2\cos(4\pi/7)$ and $c'' = 2\cos(6\pi/7)$. Likewise, ω' is either ω or $\overline{\omega}$.

Let $H_{4\pi/7}$ and $H_{6\pi/7}$ be the Hermitian forms obtained by applying these Galois conjugations to $H_{2\pi/7}$. From Lemma 4.1 we see that $H_{4\pi/7}$ and $H_{6\pi/7}$ are positive definite. Therefore the corresponding groups $\mathrm{SU}\big(H_{4\pi/7}; \mathbb{Z}\big[\omega, 2\cos(4\pi/7)\big]\big)$ and $\mathrm{SU}\big(H_{6\pi/7}; \mathbb{Z}\big[\omega, 2\cos(6\pi/7)\big]\big)$ are compact.

Let $x \in \mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]$. Then x is an algebraic integer in $\mathbb{Q}\big(\omega, 2\cos(2\pi/7)\big)$. Let x' and x'' be its Galois conjugates in $\mathbb{Q}\big(\omega, 2\cos(4\pi/7)\big)$ and $\mathbb{Q}\big(\omega, 2\cos(6\pi/7)\big)$. The map $x \longmapsto (x, x', x'')$ maps $\mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]$ to a discrete subset of $\mathbb{C}^3$. Hence

$$\mathrm{SU}\big(H_{2\pi/7}; \mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]\big) \times \mathrm{SU}\big(H_{4\pi/7}; \mathbb{Z}\big[\omega, 2\cos(4\pi/7)\big]\big) \times \mathrm{SU}\big(H_{6\pi/7}; \mathbb{Z}\big[\omega, 2\cos(6\pi/7)\big]\big)$$

is discrete. Since $\mathrm{SU}\big(H_{4\pi/7}; \mathbb{Z}\big[\omega, 2\cos(4\pi/7)\big]\big)$ and $\mathrm{SU}\big(H_{6\pi/7}; \mathbb{Z}\big[\omega, 2\cos(6\pi/7)\big]\big)$ are compact, the image of projection onto the first factor, namely $\mathrm{SU}\big(H_{2\pi/7}; \mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]\big)$ is also discrete. □

Note that the groups we eliminated from Proposition 3.2 (v) and (vi) using Lemma 3.6 are just the Galois conjugates of the two groups we are considering.

Proposition 4.4 *The group with $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos(2\pi/5)$ is Deraux's lattice.*

PROOF: We calculate that $|\tau|^2 = 1 + 2\cos(2\pi/5) + 4\cos^2(2\pi/5) = 2$. Hence $I_1 I_2$ has order 4. The eigenvalues of $I_1 I_2 I_3$ are $(e^{2\pi i/9})^3 = \omega$, $(e^{-\pi i/9 \pm 2\pi i/5})^3 = \omega e^{\pm i\pi/5}$. These are the same as the eigenvalues of $I_1 I_2 I_3$ found by Deraux in equation (2.15) of [3]. Thus the groups are the same. □

We now briefly discuss the group with $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos(2\pi/7)$. This does not seem to have previously appeared in the literature. It is easy to show that $\mathrm{tr}(I_1 I_2) = 1 + 2\cos(\pi/7)$ and so $I_1 I_2$ has order 14. Furthermore, $\mathrm{tr}(I_1 I_2 I_1 I_3) = 2\cos(\pi/7) + 2 > 3$ and so $I_1 I_2 I_1 I_3$ is loxodromic. Since $I_1 I_2 I_3$ is elliptic, this means that, in the language of [14], Δ is of Type B; see also Proposition 7.5 of [11] for other discrete, unfaithful groups of Type B. It is not clear whether or not Δ is the whole of the lattice $\mathrm{SU}\big(H_7; \mathbb{Z}\big[\omega, 2cos(2\pi/7)\big]\big)$. The fact that $I_1 I_2 I_1 I_3$ is loxodromic would indicate that, in fact, Δ may not be a lattice. This group merits further investigation.

4.3 Subgroups of Livné's lattices

We consider the groups for which $\tau = e^{2i\phi/3} + e^{-i\phi/3} = e^{i\phi/6} 2\cos(\phi/2)$ for some angle $\phi \in (0, \pi/2)$. Note that we have already treated the cases of $\phi = 0$ and $\phi = \pi/2$ in Section 4.1. The main result of this section is

Proposition 4.5 *Let $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ for some angle $\phi \in (0, \pi/2)$. Then the group $\langle I_1, I_1, I_3 \rangle$ is discrete if and only if $\phi = 2\pi/n$ where $n =$ 5, 6, 7, 8, 9, 10, 12 or 18.*

The discreteness part of this result follows from the following proposition:

Proposition 4.6 (Corollary 7.4 of [11]) *Let $\tau = e^{2i\phi/3} + e^{-i\phi/3}$. When $\phi = 2\pi/n$ for $n = 5$, 6, 7, 8, 9, 10, 12 or 18 the group generated by the matrices $\Delta = \langle I_1, I_2, I_3\rangle$ is discrete. Moreover, when $n = 5$ this group is a cocompact lattice, when $n = 6$, 7, 8, 9, 10, 12 or 18 it is geometrically infinite.*

This result is proved by demonstrating that, for such ϕ, the group Δ is a normal subgroup of one of the lattices first described by Livné [9]. To determine whether or not they are discrete one could relate such a group to one of Mostow's ball 5-tuples and then use his discreteness criterion ΣINT [10]; see also Theorem 16.1 of [2]. Alternatively, for $n = 5$, 6, 7, 8, 10, 12, 18 we could show that Δ is arithmetic and hence discrete using a similar argument to Proposition 4.3. (The case of $n = 6$ is particularly easy as the entries of I_1, I_2, I_3 all lie in the Eisenstein integers $\mathbb{Z}[\omega]$; see [4]. Since $\mathbb{Z}[\omega]$ is a discrete subring of $\mathbb{C}$ discreteness follows immediately.) In doing this we use Lemma 3.3 to show that H_τ has signature $(2, 1)$ if and only if $0 < \cos\phi < 1$. This argument does not work when $n = 9$. The Galois conjugates of $\cos(2\pi/9)$ are $\cos(4\pi/9)$ and $\cos(6\pi/9)$. The first of these is positive and so the corresponding Hermitian form has signature $(2, 1)$. In fact when $n = 9$ the group Δ is non-arithmetic. Hence we must use a geometrical characterisation of discreteness, which was done in [11] where fundamental domains for Livné's lattices were constructed.

We now prove the non-discreteness part. We could again use Mostow's ΣINT condition. A more direct approach would be to use the complex hyperbolic Jørgensen's inequality [7] to show that when ϕ is not one of the angles listed above then Δ is not discrete. We choose to give a simple proof using plane hyperbolic geometry and Knapp's theorem [8].

In the following discussion we exclude the case of $\phi = 2\pi/6$, which we have already treated. We can see from the table given earlier that the eigenvalues of I_1J are $e^{i\alpha} = e^{i(\pi-\phi)/3}$, $e^{i\beta} = e^{2i\phi/3}$ and $e^{-i(\alpha+\beta)} = e^{i(-\pi-\phi)/3}$. Hence the eigenvalues of $I_1I_2I_3 = (I_1J)^3$ are $e^{3i\alpha} = -e^{-i\phi}$, $e^{3i\beta} = e^{2i\phi}$ and $e^{-3i(\alpha+\beta)} = -e^{-i\phi}$. Therefore in this case $I_1I_2I_3$ has a repeated eigenvalue and so is a complex reflection with rotation angle $\pi - 3\phi$; it cannot be parabolic as it is the cube of an elliptic map. (Note that when $\phi = 2\pi/6$ then I_1J has a repeated eigenvalue and is parabolic, as is $I_1I_2I_3$.) By examining the eigenvectors, one can show that when $\phi < 2\pi/6$ then $I_1I_2I_3$ is complex reflection in a complex line and when $\phi > 2\pi/6$ then $I_1I_2I_3$ is complex reflection in a point. We will give the details in the former case. The latter case is almost identical. We leave its details to the reader.

Lemma 4.7 *Let $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ with $0 < \phi < \pi/3$. Let L_1 and L_{123} be the complex lines fixed by I_1 and $I_1I_2I_3$ respectively. These complex lines are ultraparallel and their common orthogonal $L^\perp$ is preserved by the group $\langle I_1, I_1I_2I_3\rangle$. This group acts on $L^\perp$ as the index 2 holomorphic subgroup of the group generated by reflections in the sides of a hyperbolic triangle with angles $\pi/2$, $\phi/2$, $(\pi - 3\phi)/2$.*

Proof: Since I_1 and $I_1I_2I_3$ are complex reflections, they preserves all complex lines orthogonal to L_1 and L_{123}, respectively. Suppose that L_1 and L_{123} are not ultraparallel. Let their intersection be $\mathbf{z}$. Then $\mathbf{z}$ is fixed by I_1 and by $I_1I_2I_3$. Hence it is also fixed by I_2I_3 and so must be $\mathbf{z}_1$. If I_1 fixed this point then the whole group would be fix it and so be elementary. This is a contradiction.

Hence L_1 and L_{123} are ultraparallel. Their common orthogonal is preserved by I_1 and $I_1I_2I_3$ which act as rotations through angles π and $\pi - 3\phi$ respectively. Moreover $L^\perp$ is preserved by I_2I_3 and so its polar vector must be an eigenvector of I_2I_3. The eigenvalues of I_2I_3 corresponding to positive vectors are $e^{i\phi}$ and $e^{-i\phi}$. Hence I_2I_3 acts on $L^\perp$ as a rotation through ϕ. □

We can now use plane hyperbolic geometry to complete the proof of Proposition 4.5.

Proposition 4.8 *Let $\tau = e^{2i\phi/3} + e^{-i\phi/3}$. The group $\langle I_1, I_1I_2I_3\rangle$ is discrete if and only if $\phi = 2\pi/n$ for $n = 5$, 6, 7, 8, 9, 10, 12 or 18.*

Proof: We consider the subgroup $\langle I_1,\, I_1I_2I_3\rangle$ and its action on $L^\perp$. This group is generated by two elliptic maps whose product is also elliptic. In Theorem 2.3 of [8], Knapp has characterised when such a group is discrete. Our case is particularly easy because I_1 has order 2 and so one of the angles in our triangle is a right angle. Hence if $\langle I_1,\, I_1I_2I_3\rangle$ is discrete then we are in Case I or Case IV of Knapp's theorem. In other words either $\phi/2 = \pi/n$ and $(\pi - 3\phi)/2 = \pi/d$ or else one of $\phi/2$ and $(\pi - 3\phi)/2$ equals π/p and the other equals $2\pi/p$ for some odd integer p. In fact, if we solve for p in the last case we see that either $p = 10$ or $p = 14$, neither of which is odd. Therefore we must have $\phi = 2\pi/n$ and $\pi - 3\phi = 2\pi/d$. The values of n in the proposition are precisely those (with $n \geq 5$) for which $2n/(n-6)$ is an integer d. □

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