MATH 1012 Core A: Geometry

Elementary Row Operations and Gauss-Jordan Elimination

Michaelmas 2004

The technique for solving systematically any system of linear equations is to write the system in the form of a matrix and then perform a process process on this matrix called **Gauss-Jordan elimination**. This reduces an $m \times n$ matrix to **reduced row echelon form** by the use of **elementary row operations**. The process is described in terms of a basic routine applied to a matrix which is then used repeatedly as described in the sections below.

Read chapter 1 of Allenby.

1 Augmented matrices and row operations

Suppose we are given a set of linear equations

$$\begin{aligned} ax + by + cz &= l, \\ dx + ey + fz &= m, \\ gx + hy + jz &= n. \end{aligned}$$

In order to solve these equations we can perform various operations. For example we can add any multiple of one row to any other row; we can multiply a row by a constant; we can interchange the orders of the equations.

We can formalise this by writing the equations as a matrix and then performing certain operations on the rows of the matrix. Because we want to do the operations on both the left and right hand sides of the equations we want to include both into our matrix, but we also want a way to remember which side of the equation is which. In order to achieve this, we write our equations as an **augmented matrix**. This consists of as many rows as we have equations. There is a vertical line down the middle with the coefficients on the left hand side to the left of the line and those to the right to the right of the line. So, for example, the augmented matrix corresponding to the equations above is

$$\left(\begin{array}{ccc|c} a & b & c & l \\ d & e & f & m \\ g & h & j & n \end{array}\right).$$

We may perform **elementary row operations (EROs)** on the augmented matrix. The significance of this is that if the matrix represents a system of linear equations, then an ERO gives an equivalent system, that is one with the *same solution set*.

The EROs are:

(i) P_{rs}: switch row r with row s;

(ii) $\mathrm{M}_r(\lambda)$: multiply (all the entries in) row r by the number λ (which must be nonzero).

(iii) $\mathrm{A}_{rs}(\lambda)$: add λ times (each entry in) row r to (the corresponding entry in) row s (with $s \neq r$).

2 Echelon form

A matrix of numbers is said to be in **row reduced echelon form (RREF)** if:

(i) The first (leftmost) non-zero entry in any non-zero row is a 1 (called its **leading** 1).

(ii) If a row has its leading 1 in the rth column then

(a) all the other entries in the rth column are 0; and

(b) the leading 1s of subsequent rows are in columns to the right of the rth column.

(iii) Any row(s) of zeros come after all the rows with non-zero entries.

In other words, the matrix looks like (with perhaps a block of 0s to the left and a block of 0s below):

$$\begin{pmatrix} 1 & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * \\ 0 & 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & \dots & * \end{pmatrix}$$

Why should this be useful? If the augmented matrix of a system of linear equations is in RREF then it is straightforward to write down its general solution. See below for an example.

3 Gauss-Jordan Elimination

There is an algorithm for reducing a matrix to RREF by a sequence of elementary row operations. This is called **Gauss-Jordan elimination**.

3.1 The basic routine

(i) Find the first non-zero column.

(ii) Go down the first non-zero column and find the first non-zero entry.

(iii) If this is in the i-th row with $i \neq 1$ apply the elementary row operation P_{1i} to get a non-zero entry, λ say, at the top of the first non-zero column.

(iv) Apply the elementary row operation $M_1(1/\lambda)$ to make the first non-zero entry of the first non-zero column a 1.

(v) Using elementary row operations of the form $A_{1i}(\mu)$ clear out all all other non-zero entries in the first non-zero column.

The resulting matrix is of the form:

$$\begin{pmatrix} 1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

or, if the first non-zero column is not the first one,

$$\begin{pmatrix} 0 & \dots & 0 & 1 & * & \dots & * \\ 0 & \dots & 0 & 0 & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & * & \dots & * \end{pmatrix}.$$

3.2 The algorithm

Let A be a given $m \times n$ matrix. The process of Gauss-Jordan elimination applied to A is described by repeatedly applying the Basic Routine in stages. (For simplicity of description we will assume that the first column of A is non-zero).

Stage 1 Apply the basic routine to the given $m \times n$ matrix A. This gives a new $m \times n$ matrix A_1 of the for

$$A_1 = \begin{pmatrix} 1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{pmatrix}.$$

Stage 2

(i) 1. Apply the basic routine to the submatrix of A_1 obtained by ignoring the first row. This gives a new $m \times n$ matrix $\tilde{A}_2$ of the form:

$$\tilde{A}_2 = \begin{pmatrix} 1 & * & \dots & * & * & * & \dots & * \\ 0 & 0 & \dots & 0 & 1 & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & * & \dots & * \end{pmatrix}.$$

(ii) Apply an elementary row operation of the form $A_{21}(\mu)$, if necessary, to $\tilde{A}_2$ to make the entry above the first 1 in the second row a zero. This gives a new $m \times n$ matrix A_2 of the form:

$$A_2 = \begin{pmatrix} 1 & * & \dots & * & 0 & * & \dots & * \\ 0 & 0 & \dots & 0 & 1 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & * & \dots & * \end{pmatrix}.$$

Now continue inductively. Assume stages 1 to $k-1$ have been completed.

Stage k

(i) Apply the basic routine to the submatrix of A_{k-1} obtained by ignoring the first $k-1$ rows. This gives a new $m \times n$ matrix $\tilde{A}_k$ of the form:

$$\tilde{A}_k = \begin{pmatrix} 1 & * & \dots & * & 0 & * & \dots & * & * & \dots & * \\ 0 & 0 & \dots & 0 & 1 & * & \dots & * & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & * & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & * & \dots & * \end{pmatrix}.$$

(ii) Apply elementary row operations of the form $A_{ki}(\mu)$, if necessary, to $\tilde{A}_k$ make all remaining non-zero entries in the column containing the first 1 in the k-th row equal to zero. This gives a new $m \times n$ matrix A_k of the form:

$$A_k = \begin{pmatrix} 1 & * & \dots & * & 0 & * & \dots & 0 & * & \dots & * \\ 0 & 0 & \dots & 0 & 1 & * & \dots & 0 & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & * & \dots & * \end{pmatrix}.$$

Continuing inductively in this way gives the matrix A_m which is the reduced row echelon form of A.

4 Examples

4.1 Solution of a system of linear equations

Solve the equations

$$\begin{aligned} x + 2y - 3z &= 0, \\ 2x + 3y + z &= 5, \\ -x - y + 2z &= 1. \end{aligned}$$

We write the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 3 & 1 & 5 \\ -1 & -1 & 2 & 1 \end{array}\right).$$

First we use the basic routine to get the second two entries in the first column to be 0. That is

$$\mathrm{A}_{12}(-2),\ \mathrm{A}_{13}(1) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 7 & 5 \\ 0 & 1 & -1 & 1 \end{array}\right).$$

Next we make the first non-zero entry in the second row into a 1. That is

$$\mathrm{M}_{2}(-1) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -7 & -5 \\ 0 & 1 & -1 & 1 \end{array}\right).$$

Next we make the first and last entries in the second column into a zero. That is

$$\mathrm{A}_{21}(-2),\ \mathrm{A}_{23}(-1) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 11 & 10 \\ 0 & 1 & -7 & -5 \\ 0 & 0 & 6 & 6 \end{array}\right).$$

Next we make the first non-zero entry in the third row into a 1. That is

$$\mathrm{M}_{3}(1/6) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 11 & 10 \\ 0 & 1 & -7 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Next we make the first two entries in the third column into a 0. That is

$$\mathrm{A}_{31}(-11),\ \mathrm{A}_{32}(7) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Thus our equations are equivalent to

$$\begin{aligned} x &= -1, \\ y &= 2, \\ z &= 1. \end{aligned}$$

4.2 Finding the inverse of a matrix

Given a $n \times n$ matrix A we form an $n \times (2n)$ augmented matrix with A to the left of the line and I to the right. By performing elementary row operations (if possible) we transform this into a matrix with I to the left of the line and some other matrix B to the right. This matrix B is the inverse of A.

For example: Find the inverse of the matrix from the previous example. That is

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 1 \\ -1 & -1 & 2 \end{pmatrix}.$$

We form a 3×6 augmented matrix and use the same sequence of EROs as before:

$$\begin{aligned}
& \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \\
\mathrm{A}12(-2),\ \mathrm{A}_{13}(1) \quad \rightarrow \quad & \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -1 & 7 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right) \\
\mathrm{M}_2(-1) \quad \rightarrow \quad & \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -7 & -2 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right) \\
\mathrm{A}_{21}(-2),\ \mathrm{A}_{23}(-1) \quad \rightarrow \quad & \left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -3 & 2 & 0 \\ 0 & 1 & -7 & 2 & -1 & 0 \\ 0 & 0 & 6 & -1 & 1 & 1 \end{array}\right)
\end{aligned}$$

$$
\begin{aligned}
\mathrm{M}_3(1/6) \quad &\rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -3 & 2 & 0 \\ 0 & 1 & -7 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1/6 & 1/6 & 1/6 \end{array}\right) \\
\mathrm{A}_{31}(-11),\ \mathrm{A}_{32}(7) \quad &\rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/6 & 1/6 & -11/6 \\ 0 & 1 & 0 & 5/6 & 1/6 & 7/6 \\ 0 & 0 & 1 & -1/6 & 1/6 & 1/6 \end{array}\right).
\end{aligned}
$$

Thus we see that

$$
\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 1 \\ -1 & -1 & 2 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} -7 & 1 & -11 \\ 5 & 1 & 7 \\ -1 & 1 & 1 \end{pmatrix}.
$$

We can easily check this fact!

5 Remarks

(i) For the geometrical interpretation of such systems of equations, see the pictures, for example, on pages 3–5 of Allenby.

(ii) For an engineering application, see page 13 of Allenby.

(iii) There are various pieces of software to help with Gauss-Jordan elimination, see the DUO pages.