MATH 1012 Core A: Geometry

Problems

Michaelmas 2008

1 Geometry in $\mathbb{R}^2$

1. Find the following scalar products:

$$\text{(i)}\ \begin{pmatrix}3\\5\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix};\quad \text{(ii)}\ \begin{pmatrix}4\\0\end{pmatrix}\cdot\begin{pmatrix}3\\7\end{pmatrix};\quad \text{(iii)}\ \begin{pmatrix}-1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix};\quad \text{(iv)}\ \begin{pmatrix}-5\\6\end{pmatrix}\cdot\begin{pmatrix}8\\-1\end{pmatrix};$$
$$\text{(v)}\ \begin{pmatrix}2\\4\end{pmatrix}\cdot\begin{pmatrix}-6\\3\end{pmatrix};\quad \text{(vi)}\ \begin{pmatrix}1\\5\end{pmatrix}\cdot\begin{pmatrix}3\\2\end{pmatrix};\quad \text{(vii)}\ \begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix};\quad \text{(viii)}\ \begin{pmatrix}0\\0\end{pmatrix}\cdot\begin{pmatrix}2\\6\end{pmatrix}.$$

2. Find the lengths of the following vectors:

$$\text{(i)}\ \begin{pmatrix}3\\4\end{pmatrix};\qquad \text{(ii)}\ \begin{pmatrix}5\\-12\end{pmatrix};\qquad \text{(iii)}\ \begin{pmatrix}-1\\5\end{pmatrix};\qquad \text{(iv)}\ \begin{pmatrix}-5\\0\end{pmatrix};\qquad \text{(v)}\ \begin{pmatrix}\sqrt{5}\\2\end{pmatrix}.$$

3. Find the angles (in radians) between the following vectors:

$$\text{(i)}\ \begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix};\quad \text{(ii)}\ \begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}6\\3\end{pmatrix};\quad \text{(iii)}\ \begin{pmatrix}\sqrt{3}\\1\end{pmatrix},\begin{pmatrix}1\\\sqrt{3}\end{pmatrix};\quad \text{(iv)}\ \begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}5\\-5\end{pmatrix};$$
$$\text{(v)}\ \begin{pmatrix}2\\-4\end{pmatrix},\begin{pmatrix}-3\\6\end{pmatrix};\quad \text{(vi)}\ \begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}-\sqrt{3}\\3\end{pmatrix};\quad \text{(vii)}\ \begin{pmatrix}3\\-6\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}.$$

4. Find the polar coordinates of the following vectors:

$$\text{(i)}\ \begin{pmatrix}\sqrt{3}\\1\end{pmatrix};\qquad \text{(ii)}\ \begin{pmatrix}-1\\\sqrt{3}\end{pmatrix};\qquad \text{(iii)}\ \begin{pmatrix}-1\\0\end{pmatrix};\qquad \text{(iv)}\ \begin{pmatrix}0\\-1\end{pmatrix}.$$

5. Find the vectors corresponding to the polar coordinates:

$$\text{(i)}\ (10, \pi/3);\qquad \text{(ii)}\ (\sqrt{2}, 5\pi/4);\qquad \text{(iii)}\ (4, \pi/2).$$

6. Find the solution sets to the following (nonlinear!) equations or systems of equations (with x and y real):

   (i) $$y = x^2,$$ $$(x-1)^2 + y^2 = 1;$$

   (ii) $1 - 2x = \sqrt{x^2 + 3x + 7}$;

   (iii) $\sin x = 1/\sqrt{2}$;

   (iv) $y = x^2 + 1$;

   (v) $\cosh(x) = 3$.

7. Solve the following simultaneous equations:

   (i)
   $$\begin{aligned} 2x + 3y &= 1, \\ 5x + y &= 9; \end{aligned}$$

   (ii)
   $$\begin{aligned} 2x + 3y &= 2, \\ 4x + 6y &= 8; \end{aligned}$$

   (iii)
   $$\begin{aligned} 3x + y &= 1, \\ 2x - y &= 1; \end{aligned}$$

   (iv)
   $$\begin{aligned} 3x + y &= 1, \\ -6x - 2y &= -2. \end{aligned}$$

8. Find (some) a, b, c, d such that
   $$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

   Do there exist p, q, r, s such that
   $$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}?$$

9. Determine which of the following matrices are invertible and find the inverse when it exists.

$$\text{(i)}\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \quad \text{(ii)}\ \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}; \quad \text{(iii)}\ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}; \quad \text{(iv)}\ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}; \quad \text{(v)}\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$
$$\text{(vi)}\ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}; \quad \text{(vii)}\ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}.$$

10. Show that the quadrilaterals with the following vertices are parallelograms and find their areas:

$$\text{(i)}\ \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \qquad \text{(ii)}\ \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

11. Find the area of the triangles with vertices:

$$\text{(i)}\ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad \text{(ii)}\ \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

12. Let $f(x) = (x+1)(x-1)^2(3-x)$.
   (a) Find the tangent line to the curve $y = f(x)$ at $x = -1$, $x = 0$ and $x = 1$.
   (b) Sketch the curve $y = f(x)$ for $-2 \leq x \leq 4$.
   (c) Find the area below the curve $y = f(x)$ and above the x-axis.

13. Suppose that $\alpha(t)$ is the parametrised curve given by

$$\alpha(t) = \begin{pmatrix} \sin^2(t) \\ \sin(t)\cos(t) \end{pmatrix}$$

for $0 \leq t \leq \pi$.
   (a) Show that $\alpha(t)$ is parametrised by arc length.
   (b) Find the length of $\alpha(t)$.
   (c) Find the normal vector to $\alpha(t_0)$ at where $0 < t_0 < \pi$.
   (d) Do you recognise this curve?

14. Characterise the following conic, find its largest and smallest distance from the origin and use Maple to draw it:

$$1 = 2x^2 + 2xy + 2y^2.$$

15. For which values of θ do there exist solutions to the equation

$$1 = \begin{pmatrix} r\cos(\theta) & r\sin(\theta) \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}?$$

Find the smallest distance from the origin to the curve given by this equation.

16. $\star$ Let the matrix L be given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and, given a vector $\mathbf{v}$, define $\mathbf{v}'$ to be the matrix product $L\mathbf{v}$.

(a) What are the components of $\mathbf{v}'$ in general?

(b) Now suppose that $\theta = \pi/4$. Write down $L(\pi/4)$, and compute $\mathbf{v}'$ for the following vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad ; \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Draw your answers on a diagram.

(c) What is the transformation $L(\pi/4)$? And the general transformation?

17. $\star$ For the matrix L of question 16, write down L^T. What is $L^T L$?

Hence show that the dot product is invariant under the linear transformation

$$\begin{aligned} \mathbf{L} \quad &: \quad \mathbb{R}^2 \to \mathbb{R}^2 \\ &: \quad \mathbf{v} \mapsto L\mathbf{v} \end{aligned}$$

[Recall that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$]

18. $\star$ Show that the set

$$U(1) = \{L(\theta) \mid \theta \in [0, 2\pi)\}$$

Forms a group under matrix multiplication. [The set must be *closed*, multiplication must be *associative*, there must be an *identity* and an *inverse*.]

This is the group of rotations in the plane and is called $SO(2)$, or $U(1)$ (for special orthogonal transformations in two real dimensions, or unitary transformations in one complex dimension respectively).

19. $\star$ Sketch the curve with parametric equation

$$\alpha(t) = \begin{cases} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} & \text{for} \quad t \in [0, \pi/2] \\ \begin{pmatrix} 1 \\ \pi/2 - t \end{pmatrix} & \text{for} \quad t \in [\pi/2, 3\pi/2] \end{cases}$$

and show that it is smooth, and parametrized by arc length.

Find $\alpha''(t)$, and use it to write down a (continuous) unit normal vector to the curve, $\mathbf{N}(t)$.

20. $\star$ For a curve parametrized by arc length, the variation of the unit tangent vector $\mathbf{T} = \alpha'(s)$ along the curve gives its *curvature*, κ:

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

Find κ for the curve in question 19.

*Now suppose a curve is not parametrized by arc length, show that

$$\kappa = \frac{\left[|\alpha'|^2|\alpha''|^2 - (\alpha' \cdot \alpha'')^2\right]^{1/2}}{|\alpha'|^3} \tag{1}$$

21. $\star$ The quantity $\rho = \kappa^{-1}$ (for nonzero κ) is the radius of curvature of the curve α. Find the radius of curvature of the planar curves:

(a) $\alpha_1(t) = \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}$

(b) $\alpha_2(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ [Hint: use equation (1)]

22. $\star\star$ Suppose we define a new scalar product on $\mathbb{R}^2$, the 'circle product':

$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \circ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 - u_2 v_2$$

(a) Find the circle product of the following vectors:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \circ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix} \circ \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad .$$

Now suppose we define the circle length as

$$|\mathbf{u}|_\circ = |\mathbf{u} \circ \mathbf{u}|^{1/2}$$

(b) Find the circle lengths of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad .$$

Comment on the difference with the dot product.

(c) Now suppose $v \in [0, 1)$. Prove that

$$\left| \begin{pmatrix} \frac{x - vy}{\sqrt{1-v^2}} \\ \frac{y - vx}{\sqrt{1-v^2}} \end{pmatrix} \right|_\circ = \left| \begin{pmatrix} x \\ y \end{pmatrix} \right|_\circ$$

If you replace x by t, and send $v \to v/c$ does this look familiar? (It is a Lorentz transformation, the basic relation in Special Relativity.)

2 Complex Numbers

23. Find the real and imaginary parts of:

$$\text{(i) } (3+4i)(2+i), \quad \text{(ii) } \frac{3+4i}{2+5i}, \quad \text{(iii) } \frac{3+i}{2-3i}, \quad \text{(iv) } \frac{1+5i}{1-i} + \frac{1-5i}{1+i},$$

24. Find the real and imaginary parts of:

$$\text{(i)} \quad \frac{2\cos(\pi/3) + 2i\sin(\pi/3)}{\cos(\pi/6) + i\sin(\pi/6)}, \qquad \text{(ii)} \quad \frac{2\cos(\pi/3) + 2i\sin(\pi/3)}{\cos(\pi/6) - i\sin(\pi/6)},$$

$$\text{(iii)} \quad \frac{2\sin(\pi/3) + 2i\cos(\pi/3)}{\cos(\pi/6) + i\sin(\pi/6)}, \qquad \text{(iv)} \quad \frac{2\sin(\pi/3) + 2i\cos(\pi/3)}{\cos(\pi/6) - i\sin(\pi/6)}.$$

25. Find the modulus and argument of:

$$\text{(i) } 2+i2\sqrt{3}, \quad \text{(ii) } 2-i2\sqrt{3}, \quad \text{(iii) } -2+i2\sqrt{3}, \quad \text{(iv) } -2-i2\sqrt{3}, \quad \text{(v) } \frac{2-i2\sqrt{3}}{1+i}.$$

26. (a) If $z = x + iy$ satisfies $|z-4| + |z+4| = 10$ then show that $(x/5)^2 + (y/3)^2 = 1$.
    (b) Conversely, if x and y satisfy $(x/5)^2 + (y/3)^2 = 1$ then show that $z = x + iy$ satisfies $|z-4| + |z+4| = 10$.

27. Let E be the ellipse defined by $|z - z_0| + |z + z_0| = 2$ for some complex number with z_0 with $|z_0| < 1$. Writing $z = x + iy$ find the symmetric matrix A so that E is given as a central conic by the equation

$$\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

28. Find the set of complex solutions of the equation

$$|z+2| = |z-5i|.$$

    Explain why your answer is also the perpendicular bisector of the line segment joining -2 to $5i$.

29. Show that the real part of $(z + z^{-1})$ is positive if and only if the real part of z is positive. For which z is the imaginary part of $(z + z^{-1})$ positive?

30. Find all complex roots of each of the following equations

$$\text{(i) } z^2 = 2i, \quad \text{(ii) } z^3 = 2-2i, \quad \text{(iii) } z^4 = 2i\sqrt{3} - 2, \quad \text{(iv) } z^8 + 4z^4 + 16 = 0.$$

31. Use de Moivre's theorem to show that $\sin(5\theta) = \big(1 - 12\cos^2(\theta) + 16\cos^4(\theta)\big)\sin(\theta)$. Deduce that $\cos(\pi/5) = (1+\sqrt{5})/4$.

32. Use de Moivre's theorem to express $\cos(6\theta)$ as a polynomial in $\cos(\theta)$. Use this to find $\cos(n\pi/6)$ for $n = 0, 1, \ldots, 6$. You should treat odd and even n separately.

33. If n is a positive integer, find all complex roots of the equation

$$(z+1)^{2n+1} + (z-1)^{2n+1} = 0.$$

[*Hint:* First show that z must be purely imaginary and then write $z+1$ in polar coordinates.]

34. $\star$ We can construct functions of a complex variable:

$$F(z) = F(x+iy) = f(x,y) + ig(x,y)$$

Find f and g for the following $F(z)$:

$$(i)\ z \quad ; \qquad (ii)\ z^2 \quad ; \qquad (iii)\ z^n \quad ; \qquad (iv)\ 1/z \quad .$$

35. $\star\star$ For this question, you need to know about partial differentiation.

Suppose we define differentiation of the function F in the previous question in the obvious way: $\frac{d}{dz} z^n = n z^{n-1}$.

By looking at the functions of the previous question, (i), (ii), and (iv), see if you can spot two pairwise relations between the partial derivatives: $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial x}$, and $\frac{\partial g}{\partial y}$.

Hence show that f and g both satisfy the Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

3 Geometry in $\mathbb{R}^3$

36. Find the cross products of the following pairs of vectors:

$$(i)\ \begin{pmatrix}-3\\2\\0\end{pmatrix} \times \begin{pmatrix}1\\7\\0\end{pmatrix} \ ; \quad (ii)\ \begin{pmatrix}1\\2\\5\end{pmatrix} \times \begin{pmatrix}3\\-1\\7\end{pmatrix} \ ; \quad (iii)\ \begin{pmatrix}3\\2\\7\end{pmatrix} \times \begin{pmatrix}1\\1\\1\end{pmatrix} \ ; \quad (iv)\ \begin{pmatrix}8\\8\\1\end{pmatrix} \times \begin{pmatrix}5\\5\\2\end{pmatrix} \ .$$

37. Find the most general form for the vector $\mathbf{u}$ satisfying the equation

$$\mathbf{u} \times \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}2\\1\\1\end{pmatrix}$$

38. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ with $\mathbf{a} \neq \mathbf{0}$ show that the equation $\mathbf{a} \times \mathbf{u} = \mathbf{b}$ has a solution if and only if $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$ and find all the solutions in this case. [*Hint:* Before you start, ask yourself what sort of answer you expect to get and what the set of solutions represents geometrically.]

39. Show that there are exactly two unit vectors in $\mathbb{R}^3$ which make an angle of $\frac{\pi}{3}$ with both of the vectors

$$\begin{pmatrix}1\\2\\2\end{pmatrix}, \qquad \begin{pmatrix}2\\1\\-2\end{pmatrix}.$$

Find the angle these two unit vectors make with each other.

40. Find the equation (in the form $ax+by+cz=d$) of the planes in $\mathbb{R}^3$ which contain the triples of points:

$$\text{(i)} \quad \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\-1\end{pmatrix}; \qquad \text{(ii)} \quad \begin{pmatrix}-2\\3\\1\end{pmatrix}, \begin{pmatrix}2\\2\\3\end{pmatrix}, \begin{pmatrix}-4\\-1\\1\end{pmatrix}.$$

41. Given a line ℓ in $\mathbb{R}^3$ and a point $\mathbf{a}$ not on ℓ show that there is a unique plane Π containing $\mathbf{a}$ and ℓ. Find the equation of Π in the form $ax+by+cz=d$ when

$$\mathbf{a} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

and ℓ is the line

$$x-2 = \frac{y-1}{3} = \frac{z-2}{-2}.$$

42. If ℓ is a line in $\mathbb{R}^3$ and $\mathbf{a}$ is a point not on ℓ show that there is a unique plane Π through $\mathbf{a}$ which meets ℓ orthogonally. Find the equation of Π in the form $ax+by+cz=d$ when

$$\mathbf{a} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

and ℓ is the line

$$\frac{x-1}{2} = \frac{y+1}{2} = z.$$

43. Write down the equations for the line in $\mathbb{R}^3$ through the point

$$\mathbf{a} = \begin{pmatrix}1\\2\\4\end{pmatrix}$$

parallel to the line

$$x-1 = \frac{y+5}{2} = \frac{z}{2}.$$

Find the distance between these lines.

44. Find the distance between the lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$ when:

$$\begin{array}{llll} \text{(i)} & \ell_1 : \ \dfrac{x-2}{3} = \dfrac{y-5}{2} = \dfrac{z-1}{-1}, & \ell_2 : \ \dfrac{x-4}{-4} = \dfrac{y-5}{4} = z+2; \\ \text{(ii)} & \ell_1 : \ x = y = z, & \ell_2 : \ 2x-1 = y+1 = 2z+1. \end{array}$$

45. If ℓ_1, ℓ_2 are non-parallel lines in $\mathbb{R}^3$ show there is a unique line ℓ_3 which meets ℓ_1 and ℓ_2 orthogonally. Find ℓ_3 when ℓ_1, ℓ_2 are given by

$$\ell_1 : \quad x-2 = \frac{y-1}{3} = \frac{z-3}{-2}, \qquad \ell_2 : \quad x+1 = y+2 = z+3.$$

46. If ℓ is a line in $\mathbb{R}^3$ and $\mathbf{a}$ is a point not on ℓ show that there is a unique line ℓ' through $\mathbf{a}$ which meets ℓ orthogonally. Find ℓ' when

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

and ℓ is the line

$$x+2 = \frac{y-1}{2} = \frac{z+1}{2}.$$

47. Determine all lines which meet the lines

$$\ell_1 : \quad \frac{x-1}{2} = \frac{y}{2} = z-1, \qquad \ell_2 : \quad x-1 = \frac{y+1}{-2} = \frac{z}{2}$$

at an angle of $\frac{\pi}{3}$. What do you notice about the configuration formed by these lines?

48. Let ℓ be a line in $\mathbb{R}^3$ not passing through the origin. Show that there is a unique line through the origin that intersects ℓ orthogonally. What goes wrong if ℓ passes through the origin?

49. Find the scalar triple products of the following sets of vectors:

$$(i) \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right]; \ (ii) \left[\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \right]; \ (iii) \left[\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} \right].$$

50. Find the volume of the parallelepiped with vertices

$$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

51. Find the volume of the tetrahedron with vertices

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.$$

52. (i) Show that every pair of opposite edges of a regular tetrahedron is orthogonal.
    (ii) Show that if two pairs of opposite edges of a tetrahedron are orthogonal then the third pair of opposite edges are also orthogonal.
    (iii) Is it true that if every pair of opposite edges of a tetrahedron is orthogonal then the tetrahedron is regular? Give either a proof or a counterexample.

53. Compute the following determinants:

$$\text{(i)} \quad \begin{vmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \\ 1 & 3 & 2 \end{vmatrix}, \quad \text{(ii)} \quad \begin{vmatrix} 5 & 3 & 1 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix}, \quad \text{(iii)} \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{vmatrix}, \quad \text{(iv)} \quad \begin{vmatrix} -2 & 3 & -2 \\ 4 & 1 & -3 \\ -1 & 3 & -2 \end{vmatrix},$$

$$\text{(v)} \quad \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -2 \\ 2 & 2 & -1 \end{vmatrix}, \quad \text{(vi)} \quad \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}, \quad \text{(vii)} \quad \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}.$$

54. Use Maple to solve the equations

$$\begin{aligned} x - 2y + 3z &= v_1, \\ 3x - 2y + z &= v_2, \\ -x + 2y + 4z &= v_3 \end{aligned}$$

when

$$\text{(i)} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \text{(ii)} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Use Maple to produce a plot of the planes given by the equations in (i) and (ii) where the point of intersection can be clearly seen.

55. Solve the following systems of linear equations:

    (i)

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0, \\ 3x_1 - 2x_2 + x_3 &= 0, \\ x_1 - 3x_2 - 2x_3 &= 1; \end{aligned}$$

(ii)

$$\begin{aligned} x_1 + x_2 - x_3 &= 7, \\ 4x_1 - x_2 + 5x_3 &= 4, \\ 2x_1 + 2x_2 - 3x_3 &= 0; \end{aligned}$$

(iii)

$$\begin{aligned} x_1 + x_2 - x_3 &= 7, \\ 4x_1 - x_2 + 5x_3 &= 4, \\ 6x_1 + x_2 + 3x_3 &= 18; \end{aligned}$$

(iv)

$$\begin{aligned} 2x_1 - x_2 - x_3 &= 0, \\ x_1 + x_2 + 2x_3 &= 0, \\ 7x_1 + x_2 - 3x_3 &= 0, \\ 2x_2 - x_3 &= 0. \end{aligned}$$

56. Find the conditions on a, b, c such that the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= a, \\ x_1 - x_2 + 3x_3 &= b, \\ 3x_1 + 7x_2 - 5x_3 &= c \end{aligned}$$

is consistent and find the set of solutions in this case.

57. For which values of t does the system of linear equations

$$\begin{aligned} tx_1 + x_2 + x_3 &= 1, \\ x_1 + tx_2 + x_3 &= 1, \\ x_1 + x_2 + tx_3 &= 1 \end{aligned}$$

have (a) a unique solution; (b) infinitely many solutions; (c) no solution? Find the solutions in cases (a) and (b).

4 Curves and Surfaces in $\mathbb{R}^3$

58. Sketch the curves whose parametric equations are for $t \in \mathbb{R}$:

$$(i)\ \mathbf{r} = \begin{pmatrix} \sin t \\ 0 \\ 0 \end{pmatrix} \ ; \quad (ii)\ \mathbf{r} = \begin{pmatrix} \cos \pi t \\ 3\sin \pi t \\ t \end{pmatrix} \ ; \quad (iii)\ \mathbf{r} = \begin{pmatrix} \cos t \\ \sin t \\ e^t \end{pmatrix} \ ; \quad (iv)\ \mathbf{r} = \begin{pmatrix} t - \sin t \\ t + \sin t \\ t \end{pmatrix} .$$

59. Find the tangent vectors of each of the curves (i)-(iv) of question 58 at the point $t = 0$.

60. Show that the unit tangent to the curve

$$\mathbf{r} = \begin{pmatrix} t + \cos t \\ \sqrt{2}\sin t \\ t - \cos t \end{pmatrix} \qquad \text{is} \qquad \hat{\mathbf{T}} = \frac{\mathbf{T}}{|\mathbf{T}|} = \frac{1}{2}\begin{pmatrix} 1 - \sin t \\ \sqrt{2}\cos t \\ 1 + \sin t \end{pmatrix}$$

Sketch the curve.

61. Find the unit tangent vector (i.e. $\hat{\mathbf{T}} = \mathbf{T}/|\mathbf{T}|$) to the curve 58(iii): $\mathbf{r} = \begin{pmatrix} \cos t \\ \sin t \\ e^t \end{pmatrix}$, for arbitrary t. Does this have a limit as $t \to \infty$?

62. The parametric curves γ_1 and γ_2 are given by:

$$\gamma_1 : \quad \mathbf{r} = \begin{pmatrix} \lambda^2 + c \\ \lambda \\ 1 \end{pmatrix} \quad ; \quad \gamma_2 : \quad \mathbf{r} = \begin{pmatrix} 2\mu \\ \mu \\ \mu \end{pmatrix}$$

where $c \in \mathbb{R}$ is an arbitrary constant, and λ, μ are the parameters along γ_1 and γ_2 respectively. Find the value of c for which γ_1 and γ_2 intersect, and find the point of intersection. Sketch the curves in this case.

63. Show that the curves with parametric equations

$$\gamma_1 : \quad \mathbf{r} = \begin{pmatrix} \lambda \\ 0 \\ \lambda^2 \end{pmatrix} \quad ; \quad \gamma_2 : \quad \mathbf{r} = \begin{pmatrix} 1 + 2\mu \\ -\mu^2 \\ 1 - \mu \end{pmatrix}$$

intersect at right angles.

64. Show that the helix:

$$\mathbf{r}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos s \\ \sin s \\ s \end{pmatrix}$$

is parametrised by arc length, and compute the tangent vector for arbitrary s.

65. Sketch the following surfaces whose parametric equations are:

$$(i) \quad \mathbf{X} = \begin{pmatrix} \lambda\cos\phi \\ \lambda\sin\phi \\ 2 - \lambda^2 \end{pmatrix} \quad ; \quad (ii) \quad \mathbf{X} = \begin{pmatrix} \cosh\lambda\cos\phi \\ \cosh\lambda\sin\phi \\ \sinh\lambda \end{pmatrix} \quad ;$$

$$(iii) \quad \mathbf{X} = \begin{pmatrix} 10\sin\theta\cos\phi \\ 10\sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \quad ; \quad (iv) \quad \mathbf{X} = \begin{pmatrix} 10\cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix} \quad .$$

where $\lambda \in \mathbb{R}$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ as appropriate.

66. Compute the two tangent vectors to the surface (ii) of question 65, $\mathbf{X} = \begin{pmatrix} \cosh\lambda\cos\phi \\ \cosh\lambda\sin\phi \\ \sinh\lambda \end{pmatrix}$, at $\lambda = \phi = 0$. Hence find the equation of the tangent plane at this point. Sketch your answer.

67. Find the equation of the tangent plane of the surface (i) of question 65, $\mathbf{X} = \begin{pmatrix} \lambda\cos\phi \\ \lambda\sin\phi \\ 2-\lambda^2 \end{pmatrix}$, at a general point λ_0, ϕ_0. What happens if $\lambda = \phi = 0$?

68. Find the normal to the surface (i) of question 65, $\mathbf{X} = \begin{pmatrix} \lambda\cos\phi \\ \lambda\sin\phi \\ 2-\lambda^2 \end{pmatrix}$, at a general point λ_0, ϕ_0. Hence find the equation of the tangent plane in constraint form. Check your answer against qn 67.

69. Show that the two surfaces:

$$\mathcal{S}_1 = \begin{pmatrix} t\cos\phi \\ t\sin\phi \\ t \end{pmatrix} \quad ; \quad \mathcal{S}_2 = \begin{pmatrix} 1 \\ \lambda \\ \mu \end{pmatrix}$$

intersect in the hyperbola

$$\begin{pmatrix} 1 \\ \sinh\alpha \\ \cosh\alpha \end{pmatrix} \cup \begin{pmatrix} 1 \\ \sinh\alpha \\ -\cosh\alpha \end{pmatrix}$$

Sketch your answer.

70. Find the intersection of the two surfaces:

$$\mathcal{S}_1 = \begin{pmatrix} t\cos\phi \\ t\sin\phi \\ t \end{pmatrix} \quad ; \quad \mathcal{S}_2 = \begin{pmatrix} \mu \\ \lambda \\ 1+\mu \end{pmatrix}$$

in the form of a parametric curve. What shape is this curve?

71. Compute the gradient of the following scalar fields:

$$\begin{aligned}
&(i) \quad f(x,y,z) = x^2 + x^3 y + 3yz + 4 \ ; \\
&(ii) \quad f(x,y,z) = \sin xy + \sin xz \ ; \\
&(iii) \quad f(x,y,z) = x^2 + 3y^2 + 4xz \ ; \\
&(iv) \quad f(x,y,z) = x^2 + y^2 + 2xz + z^2 \ ; \\
&(v) \quad f(x,y,z) = z^2 - x^2 - y^2 \ .
\end{aligned}$$

72. For the surface defined by $f(x, y, z) = z^2 - x^2 - y^2 = -1$, find the normal at $x = y = z = 1$ using ∇f. Hence find the equation of the tangent plane at $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

    Rewrite this surface in parametric form.

73. Characterize the following quadrics:

$$(i)\ \ xz + y^2 = 1 \ \ ; \quad (ii)\ \ xz - y^2 = 1 \ \ ; \quad (iii)\ \ 7x^2 - 10xz + y^2 + 7z^2 = 1 \ \ .$$

5 Linear Maps

74. Identify which of the following are bases of $\mathbb{R}^3$:

$$(i) \left\{ \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix} \right\} \ ; \ (ii) \left\{ \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$$

$$(iii) \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix} \right\} \ ; \ (iv) \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

75. Show that

$$\left\{ \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}7\\5\\0\end{pmatrix}, \begin{pmatrix}3\\1\\-1\end{pmatrix} \right\}$$

    is a basis for $\mathbb{R}^3$. Find the coordinates of the vectors

$$\begin{pmatrix}7\\13\\0\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}1\\3\\2\end{pmatrix}$$

    with respect to this basis.

76. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors in $\mathbb{R}^3$ the show that $\{\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b}\}$ is a basis of $\mathbb{R}^3$ if and only if $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is.

77. Show that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors in $\mathbb{R}^3$ then $\{\mathbf{b} + \mathbf{c}, \mathbf{c} + \mathbf{a}, \mathbf{a} + \mathbf{b}\}$ is a basis of $\mathbb{R}^3$ if and only if $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is.

78. Compute each of the following matrix products:

$$(i) \begin{pmatrix}-3 & 1 & 0\\ 0 & 2 & 5\end{pmatrix} \begin{pmatrix}1 & 1 & 0\\ 0 & -2 & 1\\ -4 & 1 & -3\end{pmatrix} \ ; \ (ii) \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ 1 & 0 & 2\end{pmatrix} \begin{pmatrix}-1 & 1 & -1\\ 2 & -6 & -1\\ -7 & 0 & 1\end{pmatrix};$$

$$(iii)\begin{pmatrix}1\\-1\\2\end{pmatrix}\begin{pmatrix}3 & 0 & -1\end{pmatrix}\ ;\quad (iv)\begin{pmatrix}2 & 0\\1 & -3\\0 & 4\end{pmatrix}\begin{pmatrix}2 & 1 & 0\\-3 & 0 & 5\end{pmatrix}\ ;\quad (v)\begin{pmatrix}2 & 1 & 0\\-3 & 0 & 5\end{pmatrix}\begin{pmatrix}2 & 0\\1 & -3\\0 & 4\end{pmatrix}.$$

79. Compute

$$\begin{pmatrix}1 & a & c\\0 & 1 & b\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & z\\0 & 1 & y\\0 & 0 & 1\end{pmatrix}.$$

Hence show by induction that

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & n & n(n-1)/2\\0 & 1 & n\\0 & 0 & 1\end{pmatrix}.$$

80. Determine which of the following matrices are invertible and find the inverse when it exists.

$$\text{(i)}\begin{pmatrix}2 & 1 & 0\\3 & 2 & 1\\4 & 3 & 2\end{pmatrix};\quad \text{(ii)}\begin{pmatrix}2 & 3 & 4\\1 & 2 & 3\\0 & 1 & 2\end{pmatrix};\quad \text{(iii)}\begin{pmatrix}2 & 1 & 0\\1 & 2 & 1\\0 & 1 & 2\end{pmatrix};\quad \text{(iv)}\begin{pmatrix}1 & a & c\\0 & 1 & b\\0 & 0 & 1\end{pmatrix};$$

$$\text{(v)}\begin{pmatrix}2 & -1 & 2\\1 & -2 & -2\\2 & 2 & -1\end{pmatrix};\quad \text{(vi)}\begin{pmatrix}2 & -1 & -2\\2 & -2 & -2\\1 & 2 & -1\end{pmatrix};\quad \text{(vii)}\begin{pmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{pmatrix}.$$

81. Find the vectors obtained from

$$\begin{pmatrix}2\\1\\1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

by reflection in

(a) the plane $x_1 - 2x_2 + 2x_3 = 0$;

(b) the plane $x_1 - 2x_2 + 2x_3 = 1$.

[*Hint:* For (b) first apply a translation (to all the data) to get a plane through the origin, do the calculation for that plane, and then translate back again.]

82. Find the matrix A of the linear map of $\mathbb{R}^3$ to itself given by reflection in the plane $2x - y + 2z = 0$. Verify that $A^2 = I$.

83. Find the matrix A of the linear map of $\mathbb{R}^3$ to itself given by orthogonal projection onto the plane $x + y + z = 0$. Verify that $A^2 = A$.

84. Determine the matrices A, B of the linear map of $\mathbb{R}^3$ to itself given by rotation through $\pm\frac{\pi}{3}$ about the line $2x = y = z$. What do you notice about the way A is related to B? Check that A, B are orthogonal matrices and compute A^2 and B^2.

85. If $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the linear transformation defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ z+x \\ x+y \end{pmatrix},$$

find the image under T of each of the following:

(a) the line $x = y = z$;

(b) the plane $x + y + z = 0$.

Use your answer to describe T geometrically and find T^{-1}.