MATH 1012 Core A: Geometry

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4.4 Change of basis

WARNING! The matrix of a linear transformation depends on which basis we are using. The matrices we have found above are the matrices **with respect to the standard basis** of $\mathbb{R}^3$.

Theorem 4.4.1 *Let T be a linear map. Suppose that with respect to the standard basis T is given by the matrix A. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be any basis of $\mathbb{R}^3$ and let $P = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$. Then with respect to the basis S the transformation T is given by the matrix $B = P^{-1}AP$.*

Proof: We have already seen that if

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

then

$$\mathbf{x} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3$$

where

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P \begin{pmatrix} y_1 \\ y_2 \\ y_2 \end{pmatrix}.$$

Thus, with respect to the basis S the transformation T is given by

$$B \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = T \begin{pmatrix} y_1 \\ y_2 \\ y_2 \end{pmatrix} = T \left(P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) = P^{-1}T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P^{-1}A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P^{-1}AP \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

In the third line we used the linearity of T to take the matrix P^{-1} outside the brackets. □

Corollary 4.4.2 *Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $S' = \{\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3\}$ are any bases of $\mathbb{R}^3$. Let P be a matrix so that $\mathbf{v}'_j = P\mathbf{v}_j$. Let T be a linear map given with respect to the basis S be the matrix A and with respect to the basis S' by the matrix B. Then $B = P^{-1}AP$.*

Example 4.4.3 *(i) Suppose that the linear map T is given by the matrix A with respect to the standard basis. Find the matrix of T with respect to the basis S. Where*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 3 & 2 & 1 \end{pmatrix}. \quad \text{and} \quad S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Let

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{so } P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the matrix of T with respect to the basis S is

$$B = P^{-1}AP = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -2 & 0 \\ 3 & -1 & -1 \end{pmatrix}.$$

(ii) Find the matrix of the linear map T with respect to the basis S, where

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x - z \\ 2x + 3y - 2z \\ 2x + 2y - z \end{pmatrix} \quad \text{and} \quad S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

We write

$$A = \begin{pmatrix} 4 & 0 & -1 \\ 2 & 3 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Then with respect to the standard basis, $T(\mathbf{x}) = A\mathbf{x}$. Let

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad \text{and } P^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

With respect to S the linear map T is given by $T(\mathbf{x}) = B\mathbf{x}$ where

$$B = P^{-1}AP$$

We check that

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -1 \\ 2 & 3 & 2 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$T\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -1 \\ 2 & 3 & 2 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

$$T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -1 \\ 2 & 3 & 2 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$