Jørgensen's Inequality and Collars in n-dimensional Quaternionic Hyperbolic Space

Wensheng Cao *
Department of Mathematics,
Wuyi University, Jiangmen,
Guangdong 529020, P.R. China
e-mail: wenscao@yahoo.com.cn

John R. Parker
Department of Mathematical Sciences,
Durham University,
Durham DH1 3LE, England
email: j.r.parker@durham.ac.uk

Abstract

In this paper, we obtain analogues of Jørgensen's inequality for non-elementary groups of isometries of quaternionic hyperbolic n-space generated by two elements, one of which is loxodromic. Our result gives some improvement over earlier results of Kim [10] and Markham [15]. These results also apply to complex hyperbolic space and give improvements on results of Jiang, Kamiya and Parker [7].

As applications, we use the quaternionic version of Jørgensen's inequalities to construct embedded collars about short, simple, closed geodesics in quaternionic hyperbolic manifolds. We show that these canonical collars are disjoint from each other. Our results give some improvement over earlier results of Markham and Parker and answer an open question posed in [16].

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1 Introduction

Jørgensen's inequality [8] gives a necessary condition for a non-elementary two generator subgroup of $\mathrm{PSL}(2,\mathbb{C})$ to be discrete. As a quantitative version of Margulis' lemma, this inequality has been generalised in many ways. Viewing $\mathrm{PSL}(2,\mathbb{R})$, which is isomorphic to $\mathrm{PU}(1,1)$, as the holomorphic isometry group of complex hyperbolic 1-space, we can seek to generalise Jørgensen's inequality to $\mathrm{PU}(n,1)$ for $n>1$, the holomorphic isometry group of higher dimensional complex hyperbolic space. Examples of this are the stable basin theorem of Basmajian and Miner [1] (see also [20]) and the complex hyperbolic Jørgensen's inequality of Jiang, Kamiya and Parker [7].

Kellerhals has generalised Jørgensen's inequality to $\mathrm{PSp}(1,1)$. This group is the isometry group of quaternionic hyperbolic 1-space $\mathbf{H}^1_{\mathbb{H}}$, which is the same as real hyperbolic 4-space $\mathbf{H}^4_{\mathbb{R}}$. For more details of $\mathrm{PSp}(1,1)$, including a classification of the elements, see [3]. It is interesting to seek generalisations of Jørgensen's inequality to $\mathrm{PSp}(n,1)$ for $n>1$, that is to higher dimensional quaternionic hyperbolic isometries. The first steps in this programme were taken by Kim and Parker [11] who gave a quaternionic hyperbolic version of Basmajian and Miner's stable basin theorem. Subsequently, Markham [15] and Kim [10] independently gave versions of Jørgensen's inequality for $\mathrm{PSp}(2,1)$. Cao

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and Tan [4] obtained an analogue of Jørgensen's inequality for non-elementary groups of isometries of quaternionic hyperbolic n-space generated by two elements, one of which is elliptic.

In this paper we consider subgroups of $\mathrm{PSp}(n,1)$ with a loxodromic generator. Any loxodromic element g of $\mathrm{PSp}(n,1)$ can be conjugated in $\mathrm{Sp}(n,1)$ to the form:

$$\mathrm{diag}\Big(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_{n-1}, \lambda_n, \overline{\lambda}_n^{-1}\Big), \tag{1}$$

where $\lambda_i \in \mathbb{H}$ for $i = 1, \ldots, n$ and $\overline{\lambda}_n^{-1}$ are right eigenvalues of g with $|\lambda_i| = 1$ for $i = 1, \ldots, n-1$ and $|\lambda_n| > 1$. We want to consider loxodromic maps that are close to the identity. To make this precise, if $g \in \mathrm{Sp}(n,1)$ is a loxodromic map conjugate to (1), we define the following conjugacy invariants:

$$\delta(g) = \max\{|\lambda_i - 1| :\ i = 1, \cdots, n-1\}, \qquad M_g = 2\delta(g) + |\lambda_n - 1| + |\overline{\lambda}_n^{-1} - 1| \tag{2}$$

Observe that $M_g > 0$ and that the smaller M_g is the closer g is to the identity. Note that M_g is a natural generalisation of the invariant

$$M_g = 2|\lambda_1 - 1| + |\lambda_2 - 1| + |\overline{\lambda}_2^{-1} - 1|.$$

defined independently by Kim [10] and Markham [15] for $\mathrm{Sp}(2,1)$.

We consider groups generated by g and h that are close to each other. To make this precise, we use the cross ratio of the fixed points of the two loxodromic maps g and hgh^{-1}. We define the cross ratio in Section 2. The statement of our main theorem is:

Theorem 1.1. *Let g be a loxodromic element of $\mathrm{Sp}(n,1)$ with $M_g < 1$ and with fixed points u, $v \in \partial\mathbf{H}^n_{\mathbb{H}}$. Let h be any other element of $\mathrm{Sp}(n,1)$. If*

$$\big|[h(u), u, v, h(v)]\big|^{1/2}\big|[h(u), v, u, h(v)]\big|^{1/2} < \frac{1 - M_g}{M_g^2} \tag{3}$$

then the group $\langle g, h\rangle$ is either elementary or not discrete.

We remark that this theorem is also valid for $\mathrm{SU}(n,1)$ and is stronger than both Theorems 4.1 and 4.2 of [7]. This theorem has some useful corollaries which we gather into a single result:

Corollary 1.2. *Let g be a loxodromic element of $\mathrm{Sp}(n,1)$ with $M_g < 1$ and with fixed points u, $v \in \partial\mathbf{H}^n_{\mathbb{H}}$. Let h be any other element of $\mathrm{Sp}(n,1)$. Suppose that one of the following conditions holds:*

$$\big|[h(u), v, u, h(v)]\big|^{1/2} < \frac{1 - M_g}{M_g}, \tag{4}$$

$$\big|[h(u), u, v, h(v)]\big|^{1/2} < \frac{1 - M_g}{M_g}, \tag{5}$$

$$\big|[u, v, h(u), h(v)]\big|^{1/2} < 1 - M_g, \tag{6}$$

$$\big|[h(u), u, v, h(v)]\big| + \big|[h(u), v, u, h(v)]\big| < \frac{2(1 - M_g)}{M_g^2}. \tag{7}$$

Then the group $\langle g, h\rangle$ is either elementary or not discrete.

When $n = 2$ the statement of Corollary 1.2 with the conditions (4) and (5) was given independently by Kim, Theorem 3.1 of [10], and Markham Theorem 1.1 of [15] and for higher dimensions Cao gave these conditions in an earlier preprint [2]. These results are a direct generalisation of Theorem 4.1 of [7]. They all follow from Theorem 2.4 of Markham and Parker [17] and the observation (see the proof of Theorem 1.4 below) that for all $\mathbf{z} \in V_0$

$$\Big|\langle g(\mathbf{z}), \mathbf{z}\rangle\Big| \le M_g\big|\langle \mathbf{z}, \mathbf{u}\rangle\big|\,\big|\langle \mathbf{z}, \mathbf{v}\rangle\big|, \tag{8}$$

which, in terms of the Cygan metric, may be rewritten as equation (10) of [17] with $d_g = |\lambda_n|^{1/2}$ and $m_g = M_g^{1/2}$.

The statement of Corollary 1.2 with condition (7) is stronger than the corresponding results in dimension $n = 2$ given by Kim and Markham. Kim's criterion, Theorem 3.2 of [10], is $M_g \le \sqrt{2\sqrt{2}-1}-1$ and

$$\big|[h(u), u, v, h(v)]\big| + \big|[h(u), v, u, h(v)]\big| < \frac{2 - 2M_g - M_g^2 + \sqrt{4 - 8M_g - 8M_g^2 - 4M_g^3 - M_g^4}}{2M_g^2}.$$

Markham's criterion, Theorem 1.2 of [15], is $M_g \le \sqrt{2} - 1$ and

$$\big|[h(u), u, v, h(v)]\big| + \big|[h(u), v, u, h(v)]\big| < \frac{1 - M_g + \sqrt{1 - 2M_g - M_g^2}}{M_g^2},$$

which is a direct generalisation of Theorem 4.2 of [7]. It is easy to see that (when they are defined)

$$\begin{aligned} \frac{2(1-M_g)}{M_g^2} &> \frac{1 - M_g + \sqrt{1 - 2M_g - M_g^2}}{M_g^2} \\ &> \frac{2 - 2M_g - M_g^2 + \sqrt{4 - 8M_g - 8M_g^2 - 4M_g^3 - M_g^4}}{2M_g^2}. \end{aligned}$$

Therefore Kim and Markham's results follow from (7).

Meyerhoff [18] used Jørgensen's inequality to show that if a simple closed geodesic in a hyperbolic 3-manifold is sufficiently short, then there exists an embedded tubular neighbourhood of this geodesic, called a *collar*, whose width depends only on the length (or the complex length) of the closed geodesic. Moreover, he showed that these collars were disjoint from one another. In [13, 14] Kellerhals generalised Meyerhoff's results to real hyperbolic 4-space and 5-space with the aid of some properties of quaternions. Markham and Parker [16] used the complex and quaternionic hyperbolic Jørgensen's inequality obtained in [7, 15], to give analogues of Meyerhoff's (and Kellerhals') results for short, simple, closed geodesics in 2-dimensional complex and quaternionic hyperbolic manifolds. They showed that these canonical collars are disjoint from each other and from canonical cusps. For complex hyperbolic space, by using a lemma of Zagier they also gave an estimate based only on the length, and left the same question for the case of quaternionic space as an open question.

Let G be a discrete group of n-dimensional quaternionic hyperbolic isometries. Let $g \in G$ be loxodromic with axis the geodesic γ. The *tube* $T_r(\gamma)$ of radius r about γ is the collection of points a distance less than r from γ. It is clear that g maps $T_r(\gamma)$ to itself. The tube $T_r(\gamma)$ is *precisely invariant* under the subgroup $\langle g\rangle$ of G if $h\big(T_r(\gamma)\big)$ is disjoint from $T_r(\gamma)$ for all $h \in G - \langle g\rangle$. If $T_r(\gamma)$ is precisely invariant under G then $C_r(\gamma') = T_r(\gamma)/\langle g\rangle$ is an embedded tubular neighbourhood of the simple closed geodesic $\gamma' = \gamma/\langle g\rangle$. We call $C_r(\gamma')$ the *collar* of width r about γ'.

As applications of our quaternionic version Jørgensen's inequalities, we will give analogues of Markham and Parker's results for short, simple, closed geodesics in n-dimensional quaternionic hyperbolic manifolds.

Given a loxodromic map g with axis γ and satisfying $M_g < \sqrt{3} - 1$, we define a positive real number r by:

$$\cosh(2r) = \frac{2(1 - M_g)}{M_g^2}. \tag{9}$$

Then we call the tube $T_r(\gamma)$ with r given by (9) the *canonical tube* about γ. If $\gamma' = \gamma/\langle g\rangle$ then we call the collar $C_r(\gamma')$ with r given by (9) the *canonical collar* about γ'.

Theorem 1.3. *Let G be a discrete, non-elementary, torsion-free subgroup of* $\mathrm{Sp}(n, 1)$. *Let g be a loxodromic element of G with axis the geodesic γ. Suppose that $M_g < \sqrt{3} - 1$. Then the canonical tube $T_r(\gamma)$ whose width r is given by (9) is precisely invariant under $\langle g\rangle$ in G.*

In particular, the canonical collar $C_r(\gamma')$ of width r about $\gamma' = \gamma/\langle g\rangle$ is embedded in the manifold $\mathcal{M} = \mathbf{H}^n_{\mathbb{H}}/G$.

Furthermore, we have

Theorem 1.4. *Let $\mathcal{M}$ denote a quaternionic hyperbolic n-manifold. Then the canonical collars around distinct short, simple, closed geodesics in $\mathcal{M}$ are disjoint.*

By controlling the rotational part of loxodromic element, we obtain the radius of collars solely in terms of the length of the corresponding simple closed geodesic as the following, which answers the open problem posed in [16].

Theorem 1.5. *Let $N \geq 35$ be a positive integer. Let G be a discrete, torsion-free, non-elementary subgroup of* $\mathrm{Sp}(n, 1)$. *Let g be a loxodromic element of G with axis γ having the form (1) and let $l = 2\log|\lambda_n|$ be the length of the closed geodesic $\gamma/\langle g\rangle$ and suppose that*

$$R_N = 2\sqrt{\left(\cosh\frac{N^n l}{2} + 1\right)\left(\cosh\frac{N^n l}{2} - \cos\frac{2\pi}{N}\right)} + 2\sqrt{2\left(1 - \cos\frac{2\pi}{N}\right)} < \sqrt{3} - 1. \tag{10}$$

Define the positive number r by

$$\cosh(2r) = \frac{2(1 - R_N)}{R_N^2}.$$

Then the tube $T_r(\gamma)$ is precisely invariant under G.

Corollary 1.6. *Let G be a discrete, torsion-free, non-elementary subgroup of* $\mathrm{Sp}(2, 1)$. *Let g be a loxodromic element of G with axis γ having the form (1). Suppose that $l = 2\log|\lambda_2| < 0.00017681$. Let r be a positive number defined by*

$$\cosh(2r) = \frac{2(1 - R)}{R^2}$$

where

$$R = 2\sqrt{\left(\cosh\frac{1849\, l}{2} + 1\right)\left(\cosh\frac{1849\, l}{2} - \cos\frac{2\pi}{43}\right)} + 2\sqrt{2\left(1 - \cos\frac{2\pi}{43}\right)}. \tag{11}$$

Then the tube $T_r(\gamma)$ is precisely invariant under G.

The structure of the remainder of this paper is as follows. In Section 2, we give the necessary background material for quaternionic hyperbolic space. Section 3 contains the proof of Theorem 1.1 and Corollary 1.2. In Section 4, we use Theorem 1.1 to obtain the proof of Theorems 1.3 and 1.4. In Section 5, we give an example to illustrate the idea behind Theorem 1.5. Using the adapted *Pigeonhole Principle* (cf. [18]), we obtain the proof of Theorem 1.5 and Corollary 1.6.

2 Background

We begin with some background material on quaternionic hyperbolic geometry. Much of this can be found in [5, 6, 11, 19]. Let $\mathbb{H}^{n,1}$ be the quaternionic vector space of quaternionic dimension $n+1$ (so real dimension $4n+4$) with the quaternionic Hermitian form

$$\langle \mathbf{z},\, \mathbf{w}\rangle = \mathbf{w}^* J\mathbf{z} = \overline{w}_1 z_1 + \cdots + \overline{w}_{n-1} z_{n-1} - (\overline{w}_n z_{n+1} + \overline{w}_{n+1} z_n),$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors in $\mathbb{H}^{n,1}$ with entries $z_1, \cdots, z_{n+1}$ and $w_1, \cdots, w_{n+1}$ respectively, $\cdot^*$ denotes quaternionic Hermitian transpose and J is the Hermitian matrix

$$J = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We define a *unitary quaternionic transformation* (or symplectic transformation) g to be an automorphism of $\mathbb{H}^{n,1}$, that is, a linear bijection such that $\langle g(\mathbf{z}),\, g(\mathbf{w})\rangle = \langle \mathbf{z},\, \mathbf{w}\rangle$ for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{H}^{n,1}$. We denote the group of all unitary transformations by $\mathrm{Sp}(n,1)$.

Following Section 2 of [5], let

$$\begin{aligned} V_0 &= \Big\{\mathbf{z} \in \mathbb{H}^{n,1} - \{0\} : \langle \mathbf{z},\, \mathbf{z}\rangle = 0\Big\} \\ V_- &= \Big\{\mathbf{z} \in \mathbb{H}^{n,1} : \langle \mathbf{z},\, \mathbf{z}\rangle < 0\Big\}. \end{aligned}$$

It is obvious that V_0 and V_- are invariant under $\mathrm{Sp}(n,1)$.

We define an equivalence relation $\sim$ on $\mathbb{H}^{n,1}$ by $\mathbf{z} \sim \mathbf{w}$ if and only if there exists a non-zero quaternion λ so that $\mathbf{w} = \mathbf{z}\lambda$. Let $[\mathbf{z}]$ denote the equivalence class of $\mathbf{z}$. Let $\mathbb{P} : \mathbb{H}^{n,1} - \{0\} \longrightarrow \mathbb{H}\mathbb{P}^n$ be the *right projection* map given by $\mathbb{P} : \mathbf{z} \longmapsto [\mathbf{z}]$. If $z_{n+1} \neq 0$ then $\mathbb{P}$ is given by

$$\mathbb{P}(z_1,\, \ldots,\, z_n, z_{n+1})^t = (z_1 z_{n+1}^{-1}, \cdots, z_n z_{n+1}^{-1})^t \in \mathbb{H}^n.$$

We also define

$$\mathbb{P}(0,\, \ldots,\, 0,\, z_n,\, 0)^t = \infty. \tag{12}$$

Observe that

$$\langle \mathbf{z}\lambda, \mathbf{w}\mu\rangle = \overline{\mu}\mathbf{w}^* J\mathbf{z}\lambda = \overline{\mu}\langle \mathbf{z}, \mathbf{w}\rangle\lambda. \tag{13}$$

We define the Siegel domain model of *quaternionic hyperbolic n space* to be $\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_-)$ and its boundary to be $\partial\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_0)$. It is clear that $\infty \in \partial\mathbf{H}^n_{\mathbb{H}}$. Also for all $\mathbf{z} \in V_-$ we have $z_{n+1} \neq 0$ and so $\mathbb{P}$ is given by the formula above. Likewise for all $\mathbf{z} \in V_0$, either $z_{n+1} \neq 0$ or $\mathbb{P}(\mathbf{z}) = \infty$.

As in Chapter 19 of [19], the Bergman metric on $\mathbf{H}^n_{\mathbb{H}}$ is given by the distance formula

$$\cosh^2 \frac{\rho(z,w)}{2} = \frac{\langle \mathbf{z},\, \mathbf{w}\rangle\langle \mathbf{w},\, \mathbf{z}\rangle}{\langle \mathbf{z},\, \mathbf{z}\rangle\langle \mathbf{w},\, \mathbf{w}\rangle}, \quad \text{where } \; z, w \in \mathbf{H}^n_{\mathbb{H}}, \quad \mathbf{z} \in \mathbb{P}^{-1}(z), \mathbf{w} \in \mathbb{P}^{-1}(w).$$

This expression is independent of the choice of $\mathbf{z}$ and $\mathbf{w}$. Since $\mathrm{Sp}(n,1)$ preserves the form $\langle\cdot,\, \cdot\rangle$, it clearly preserves the right hand side of this expression. Therefore $g \in \mathrm{Sp}(n,1)$ acts on $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ as follows:

$$g(z) = \mathbb{P}g\mathbb{P}^{-1}(z).$$

This formula is well defined provided the action of $\mathrm{Sp}(n,1)$ is on the left and the action of projection $\mathbb{P}$ of $\mathrm{Sp}(n,1)$ is on the right. It is clear that multiples of g by a non-zero real number act in the same way. Since elements of $\mathrm{Sp}(n,1)$ have determinant ± 1 this real number can only be ± 1. Therefore we define $\mathrm{PSp}(n,1) = \mathrm{Sp}(n,1)/\{\pm I_{n+1}\}$. All elements of $\mathrm{PSp}(n,1)$ are isometries of $\mathbf{H}^n_{\mathbb{H}}$. We often find it convenient to work with matrices in $\mathrm{Sp}(n,1)$ rather than projective mappings in $\mathrm{PSp}(n,1)$ and we will pass between them without comment.

If $g \in \mathrm{Sp}(n,1)$, by definition, g preserves the Hermitian form. Hence

$$\mathbf{w}^* J\mathbf{z} = \langle \mathbf{z},\, \mathbf{w}\rangle = \langle g\mathbf{z},\, g\mathbf{w}\rangle = \mathbf{w}^* g^* Jg\mathbf{z}$$

for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{H}^{n,1}$. Letting $\mathbf{z}$ and $\mathbf{w}$ vary over a basis for $\mathbb{H}^{n,1}$, we see that $J = g^* J g$. From this we find $g^{-1} = J^{-1} g^* J$. That is:

$$g^{-1} = \begin{pmatrix} A^* & -\theta^* & -\eta^* \\ -\beta^* & \overline{d} & \overline{b} \\ -\alpha^* & \overline{c} & \overline{a} \end{pmatrix} \quad \text{for} \quad g = \begin{pmatrix} A & \alpha & \beta \\ \eta & a & b \\ \theta & c & d \end{pmatrix} \in \mathrm{Sp}(n,1). \tag{14}$$

Using the identities $gg^{-1} = g^{-1}g = I_{n+1}$ we obtain:

$$\begin{aligned}
AA^* - \alpha\beta^* - \beta\alpha^* &= I_{n-1}, & (15)\\
-A\theta^* + \alpha\overline{d} + \beta\overline{c} &= 0, & (16)\\
-A\eta^* + \alpha\overline{b} + \beta\overline{a} &= 0, & (17)\\
-\eta\theta^* + a\overline{d} + b\overline{c} &= 1, & (18)\\
-\eta\eta^* + a\overline{b} + b\overline{a} &= 0, & (19)\\
-\theta\theta^* + c\overline{d} + d\overline{c} &= 0, & (20)\\
A^*A - \theta^*\eta - \eta^*\theta &= I_{n-1}, & (21)\\
A^*\alpha - \theta^* a - \eta^* c &= 0, & (22)\\
A^*\beta - \theta^* b - \eta^* d &= 0, & (23)\\
-\beta^*\alpha + \overline{d}a + \overline{b}c &= 1, & (24)\\
-\beta^*\beta + \overline{d}b + \overline{b}d &= 0, & (25)\\
-\alpha^*\alpha + \overline{c}a + \overline{a}c &= 0. & (26)
\end{aligned}$$

Following Chen and Greenberg [5], we say that a non-trivial element g of $\mathrm{Sp}(n,1)$ is:

(i) *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbb{H}}$;

(ii) *parabolic* if it has exactly one fixed point which lies in $\partial\mathbf{H}^n_{\mathbb{H}}$;

(iii) *loxodromic* if it has exactly two fixed points which lie in $\partial\mathbf{H}^n_{\mathbb{H}}$.

A subgroup G of $\mathrm{Sp}(n,1)$ is called *elementary* if it has a finite orbit in $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$. If all of its orbits are infinite then G is *non-elementary*. In particular, G is non-elementary if it contains two non-elliptic elements of infinite order with distinct fixed points.

Let o be the origin in $\mathbb{H}^n$ and ∞ be as defined in (12). Both these points lie on $\partial\mathbf{H}^n_{\mathbb{H}}$. In what follows we make fixed choices of points in $\mathbb{H}^{n,1}$ that are preimages of these points. Namely

$$(0,\ldots,0,0,1)^t \in \mathbb{P}^{-1}(o) \subset V_0, \qquad (0,\ldots,0,1,0)^t \in \mathbb{P}^{-1}(\infty) \subset V_0.$$

Define the stabilisers of the points to be:

$$G_o = \{g \in \mathrm{Sp}(n,1) : g(o) = o\}, \quad G_\infty = \{g \in \mathrm{Sp}(n,1) : g(\infty) = \infty\}, \quad G_{o,\infty} = G_o \cap G_\infty.$$

Note that if g has the form (14) then if $g \in G_o$ we have $b = 0$ and if $g \in G_\infty$ we have $c = 0$.

Cross-ratios were generalised to complex hyperbolic space by Korányi and Reimann [12]. We will generalise this definition of complex cross-ratio to the non commutative quaternion ring.

Definition 2.1. *The quaternionic cross-ratio of four points z_1, z_2, w_1, w_2 in $\overline{\mathbf{H}}^n_{\mathbb{H}}$ is defined as:*

$$[z_1, z_2, w_1, w_2] = \langle \mathbf{w_1},\, \mathbf{z_1} \rangle \langle \mathbf{w_1},\, \mathbf{z_2} \rangle^{-1} \langle \mathbf{w_2},\, \mathbf{z_2} \rangle \langle \mathbf{w_2},\, \mathbf{z_1} \rangle^{-1}, \tag{27}$$

where $\mathbf{z}_i =\in \mathbb{P}^{-1}(z_i)$ and $\mathbf{w_i} \in \mathbb{P}^{-1}(w_i)$ for $i = 1,\, 2$.

Using (13) we see that

$$\begin{aligned}
[z_1\lambda_1, z_2\lambda_2, w_1\mu_1, w_2\mu_2] &= \langle \mathbf{w_1}\mu_1,\, \mathbf{z_1}\lambda_1 \rangle \langle \mathbf{w_1}\mu_1,\, \mathbf{z_2}\lambda_2 \rangle^{-1} \langle \mathbf{w_2}\mu_2,\, \mathbf{z_2}\lambda_2 \rangle \langle \mathbf{w_2}\mu_2,\, \mathbf{z_1}\lambda_2 \rangle^{-1} \\
&= \overline{\lambda}_1 \langle \mathbf{w_1},\, \mathbf{z_1} \rangle \mu_1 \mu_1^{-1} \langle \mathbf{w_1},\, \mathbf{z_2} \rangle^{-1} \overline{\lambda}_2^{-1} \overline{\lambda}_2 \langle \mathbf{w_2},\, \mathbf{z_2} \rangle \mu_2 \mu_2^{-1} \langle \mathbf{w_2},\, \mathbf{z_1} \rangle^{-1} \overline{\lambda}_1^{-1} \\
&= \overline{\lambda}_1 [z_1, z_2, w_1, w_2] \overline{\lambda}_1^{-1}.
\end{aligned}$$

The quaternionic cross-ratio $[z_1, z_2, w_1, w_2]$ depends on the choice of $\mathbf{z}_1 \in \mathbb{P}^{-1}(z_1)$. However, its absolute value

$$\big|[z_1, z_2, w_1, w_2]\big| = \frac{|\langle \mathbf{w_1},\, \mathbf{z_1} \rangle \langle \mathbf{w_2},\, \mathbf{z_2} \rangle|}{|\langle \mathbf{w_1},\, \mathbf{z_2} \rangle \langle \mathbf{w_2},\, \mathbf{z_1} \rangle|} \tag{28}$$

is independent of the preimage of z_i and w_i in $\mathbb{H}^{n,1}$. The following lemma is easy to prove.

Lemma 2.1. *Let $o, \infty \in \partial \mathbf{H}^n_{\mathbb{H}}$ stand for the images of $(0, \cdots, 0, 1)^t$ and $(0, \cdots, 0, 1, 0)^t \in \mathbb{H}^{n,1}$ under the projection map $\mathbb{P}$, respectively and let $h \in \mathrm{PSp}(n, 1)$ be given by (14). Then*

$$\begin{aligned}
\big|[h(\infty), o, \infty, h(o)]\big| &= |bc|, && (29)\\
\big|[h(\infty), \infty, o, h(o)]\big| &= |ad|, && (30)\\
\big|[\infty, o, h(\infty), h(o)]\big| &= \frac{|bc|}{|ad|}. && (31)
\end{aligned}$$

The following lemma is crucial for us to prove Theorem 1.1.

Lemma 2.2. *Let h be as in (14). Then*

$$\begin{aligned}
|\beta^*\alpha| &\leq 2|ad|^{1/2}|bc|^{1/2}, && (32)\\
|\eta\theta^*| &\leq 2|ad|^{1/2}|bc|^{1/2}, && (33)\\
|ad|^{1/2} &\leq |bc|^{1/2} + 1, && (34)\\
|bc|^{1/2} &\leq |ad|^{1/2} + 1, && (35)\\
1 &\leq |ad|^{1/2} + |bc|^{1/2}. && (36)
\end{aligned}$$

Proof. Using (25) and (26), we have

$$|\beta^*\alpha|^2 \leq |\beta^*\beta|\, |\alpha^*\alpha| = 2\Re(\overline{d}b)\, 2\Re(\overline{c}a) \leq 4|ad||bc|. \tag{37}$$

This gives (32). Similarly, using (19) and (20), we have

$$|\eta\theta^*|^2 \leq |\eta\eta^*|\, |\theta\theta^*| = 2\Re(a\overline{b})\, 2\Re(c\overline{d}) \leq 4|ad||bc|.$$

This gives (33).

Next, using (24) and (37), we have

$$\begin{aligned} 4\Re(\bar{d}b)\,\Re(\bar{c}a) &\geq |\beta^*\alpha|^2 \\ &= |\bar{d}a+\bar{b}c-1|^2 \\ &= 1+|ad|^2+|bc|^2-2\Re(\bar{d}a)-2\Re(\bar{b}c)+2\Re(\bar{d}a\bar{c}b). \end{aligned}$$

Thus

$$\begin{aligned} 1+|ad|^2+|bc|^2 &\leq 2\Re(\bar{d}a)+2\Re(\bar{b}c)-2\Re(\bar{d}a\bar{c}b)+4\Re(\bar{d}b)\,\Re(\bar{c}a) \\ &= 2\Re(\bar{d}a)+2\Re(\bar{b}c)+2\Re(b\bar{d}c\bar{a}) \\ &\leq 2|ad|+2|bc|+2|ad|\,|bc|. \end{aligned}$$

We can rearrange this expression to obtain

$$\bigl(1-|ad|-|bc|\bigr)^2 \leq 4|ad|\,|bc|.$$

Taking square roots gives

$$-2|ad|^{1/2}|bc|^{1/2} \leq 1-|ad|-|bc| \leq 2|ad|^{1/2}|bc|^{1/2}.$$

Rearranging gives

$$\bigl(|ad|^{1/2}-|bc|^{1/2}\bigr)^2 \leq 1 \leq \bigl(|ad|^{1/2}+|bc|^{1/2}\bigr)^2.$$

Taking square roots of both sides, including both choices of sign in the left hand inequality, gives (34), (35) and (36). □

3 The proof of Jørgensen's inequality

Proof of Theorem 1.1. Since (3) is invariant under conjugation, we may assume that g is of the form (1) and h is of the form (14). Using (29) and (30) our hypothesis (3) can be rewritten as

$$|ad|^{1/2}|bc|^{1/2} < \frac{1-M_g}{M_g^2}. \tag{38}$$

Let $h_0 = h$ and $h_{k+1} = h_k g h_k^{-1}$. We write

$$h_k = \begin{pmatrix} A_k & \alpha_k & \beta_k \\ \eta_k & a_k & b_k \\ \theta_k & c_k & d_k \end{pmatrix}.$$

Then

$$\begin{aligned} h_{k+1} &= \begin{pmatrix} A_{k+1} & \alpha_{k+1} & \beta_{k+1} \\ \eta_{k+1} & a_{k+1} & b_{k+1} \\ \theta_{k+1} & c_{k+1} & d_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} A_k & \alpha_k & \beta_k \\ \eta_k & a_k & b_k \\ \theta_k & c_k & d_k \end{pmatrix} \begin{pmatrix} L & 0 & 0 \\ 0 & \lambda_n & 0 \\ 0 & 0 & \bar{\lambda}_n^{-1} \end{pmatrix} \begin{pmatrix} A_k^* & -\theta_k^* & -\eta_k^* \\ -\beta_k^* & \overline{d_k} & \overline{b_k} \\ -\alpha_k^* & \overline{c_k} & \overline{a_k} \end{pmatrix}, \end{aligned}$$

where $L = diag(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_{n-1})$. Therefore

$$\begin{aligned} a_{k+1} &= -\eta_k L\theta_k^* + a_k\lambda_n\overline{d_k} + b_k\overline{\lambda}_n^{-1}\overline{c_k}, & (39)\\ b_{k+1} &= -\eta_k L\eta_k^* + a_k\lambda_n\overline{b_k} + b_k\overline{\lambda}_n^{-1}\overline{a_k}, & (40)\\ c_{k+1} &= -\theta_k L\theta_k^* + c_k\lambda_n\overline{d_k} + d_k\overline{\lambda}_n^{-1}\overline{c_k}, & (41)\\ d_{k+1} &= -\theta_k L\eta_k^* + c_k\lambda_n\overline{b_k} + d_k\overline{\lambda}_n^{-1}\overline{a_k}. & (42)\end{aligned}$$

Claim 1: We claim that if $|ad|^{1/2}|bc|^{1/2} < (1-M_g)/M_g^2$ then $|b_kc_k|$ tends to 0 as k tends to infinity.

By (19) and (40), we have

$$\begin{aligned} |b_{k+1}| &= |\eta_k(I_{n-1}-L)\eta_k^* + a_k(\lambda_n-1)\overline{b_k} + b_k(\overline{\lambda}_n^{-1}-1)\overline{a_k}|\\ &\le \delta(g)\eta_k\eta_k^* + (|\lambda_n-1| + |\overline{\lambda}_n^{-1}-1|)|b_ka_k|\\ &= \delta(g)2\Re(a_k\overline{b_k}) + (|\lambda_n-1| + |\overline{\lambda}_n^{-1}-1|)|b_ka_k|\\ &\le (2\delta(g) + |\lambda_n-1| + |\overline{\lambda}_n^{-1}-1|)|b_ka_k|\\ &= M_g|b_ka_k|.\end{aligned}$$

Similarly, by (20) and (41) we have

$$|c_{k+1}| = |\theta^{(k)}(I_{n-1}-L)\theta_k^* + c_k(\lambda_n-1)\overline{d_k} + d_k(\overline{\lambda}_n^{-1}-1)\overline{c_k}| \le M_g|c_kd_k|.$$

Therefore, for all $k\ge 0$ we have

$$|b_{k+1}c_{k+1}|^{1/2} \le M_g|a_kd_k|^{1/2}|b_kc_k|^{1/2}. \tag{43}$$

Using our hypothesis (38) with $k=0$ this immediately gives

$$|b_1c_1|^{1/2} \le M_g|a_0d_0|^{1/2}|b_0c_0|^{1/2} < \frac{1-M_g}{M_g}.$$

In particular,

$$M_g\big(1+|b_1c_1|^{1/2}\big) < 1.$$

From this point on the proof closely follows the proof of the similar result for complex hyperbolic space given by Jiang, Kamiya and Parker [7].

We claim that for $k\ge 1$ we have

$$|b_kc_k|^{1/2} \le \Big(M_g\big(1+|b_1c_1|^{1/2}\big)\Big)^{k-1}|b_1c_1|^{1/2}. \tag{44}$$

In particular,

$$|b_kc_k|^{1/2} \le |b_1c_1|^{1/2}.$$

Certainly (44) is true for $k=1$. Assume that (44) is true for some $k\ge 1$. Then, using (43) and (34), we have

$$\begin{aligned} |b_{k+1}c_{k+1}|^{1/2} &\le M_g|a_kd_k|^{1/2}|b_kc_k|^{1/2}\\ &\le M_g\big(1+|b_kc_k|^{1/2}\big)|b_kc_k|^{1/2}\\ &\le M_g\big(1+|b_1c_1|^{1/2}\big)|b_kc_k|^{1/2}\\ &\le M_g\big(1+|b_1c_1|^{1/2}\big)\Big(M_g\big(1+|b_1c_1|^{1/2}\big)\Big)^{k-1}|b_1c_1|^{1/2}\\ &= \Big(M_g\big(1+|b_1c_1|^{1/2}\big)\Big)^{k}|b_1c_1|^{1/2}.\end{aligned}$$

Then (44) is true for $k+1$. The result follows by induction.

Since $M_g\big(1+|b_1c_1|^{1/2}\big) < 1$, an immediate consequence of (44) is that

$$\lim_{k\to\infty} |b_k c_k|^{1/2} = 0. \tag{45}$$

This proves Claim 1.

Claim 2: If there exists some integer k such that

$$b_k c_k = 0, \tag{46}$$

then $\langle h, g\rangle$ is either elementary or not discrete.

If $b_k = 0$ then, by (25), we have $\beta_k = 0$ and $h_k(o) = o$. Similarly, if $c_k = 0$ then, by (26), we have $\alpha_k = 0$ and so $h_k(\infty) = \infty$. If $b_k c_k = 0$ but either b_k or c_k is non-zero then h_k fixes exactly one of o and ∞. Hence, $\langle g,\, h_k\rangle$ is not discrete by Theorem 3.1 of Kamiya [9]. This implies that $\langle g,\, h\rangle$ is not discrete.

Suppose then that $b_k = c_k = 0$ for some $k \geq 1$. Then h_k fixes both o and ∞. In particular,

$$o = h_k(o) = h_{k-1} g h_{k-1}^{-1}(o) \quad \text{and} \quad \infty = h_k(\infty) = h_{k-1} g h_{k-1}^{-1}(\infty).$$

This means that g fixes $h_{k-1}^{-1}(o)$ and $h_{k-1}^{-1}(\infty)$. If $k \geq 2$ then h_{k-1} is loxodromic and so cannot swap o and ∞. Thus $h_{k-1}(o) = o$ and $h_{k-1}(\infty) = \infty$. By induction, we find that g fixes $h_0^{-1}(o)$ and $h_0^{-1}(\infty)$. In other words $h_0 = h$ preserves the set $\{o,\, \infty\}$ and so $\langle g, h\rangle$ is elementary.

This proves Claim 2.

Claim 3: If

$$\lim_{k\to\infty} |b_k c_k| = 0 \quad \text{and} \quad b_k c_k \neq 0 \quad \text{for all } k \geq 1 \tag{47}$$

then $\langle h, g\rangle$ is not discrete.

Assume that (47) holds. Then from (34) we have

$$|a_k d_k|^{1/2} \leq |b_k c_k|^{1/2} + 1$$

and so $|a_k d_k|$ is bounded as k tends to infinity. Hence, from (32) and (33) we have

$$|\beta_k^* \alpha_k| \leq 2|a_k d_k|^{1/2} |b_k c_k|^{1/2} \quad \text{and} \quad |\eta_k \theta_k^*| \leq 2|a_k d_k|^{1/2} |b_k c_k|^{1/2}$$

and so

$$\lim_{k\to\infty} |\beta_k^* \alpha_k| = \lim_{k\to\infty} |\eta_k \theta_k^*| = 0.$$

Likewise,

$$\lim_{k\to\infty} \eta_k L \theta_k^* = \lim_{k\to\infty} \theta_k L \eta_k^* = 0.$$

From (24) we have

$$\lim_{k\to\infty} \overline{d}_k a_k = \lim_{k\to\infty} \big(1 + \beta_k^* \alpha_k - \overline{b}_k c_k\big) = 1.$$

Therefore, from (39) and (42) we have

$$\lim_{k\to\infty} |a_{k+1}| \;=\; \lim_{k\to\infty} \Big|-\eta_k L \theta_k^* + a_k \lambda_n \overline{d_k} + b_k \overline{\lambda_n}^{-1} \overline{c_k}\Big| \;=\; |\lambda_n|, \tag{48}$$

$$\lim_{k\to\infty} |d_{k+1}| \;=\; \lim_{k\to\infty} \Big|-\theta_k L \eta_k^* + c_k \lambda_n \overline{b_k} + d_k \overline{\lambda_n}^{-1} \overline{a_k}\Big| \;=\; |\lambda_n|^{-1}. \tag{49}$$

When proving Claim 1, we showed that

$$|b_{k+1}| \leq M_g |a_k|\,|b_k| \quad \text{and} \quad |c_{k+1}| \leq M_g |d_k|\,|c_k|.$$

Since $M_g < 1$ we can find K so that, using (48) and (49), for all $k \geq K$

$$M_g |a_k| < |\lambda_n| \quad \text{and} \quad M_g |d_k| < |\lambda_n|^{-1}.$$

Hence there exist constants κ_1 and κ_2 so that, for all $k \geq K$

$$M_g |a_k|\,|\lambda_n|^{-1} < \kappa_1 < 1 \quad \text{and} \quad M_g |d_k|\,|\lambda_n| < \kappa_2 < 1.$$

Therefore for $k \geq K$

$$\begin{aligned} |b_{k+1}|\,|\lambda_n|^{-k-1} &\leq \big(M_g |a_k|\,|\lambda_n|^{-1}\big)|b_k|\,|\lambda_n|^{-k} < \kappa_1 |b_k|\,|\lambda_n|^{-k} \leq \kappa_1^{k+1-K} |b_K|\,|\lambda_n|^{-K}, \\ |c_{k+1}|\,|\lambda_n|^{k+1} &\leq \big(M_g |d_K|\,|\lambda_n|\big)|c_k|\,|\lambda_n|^{k} < \kappa_2 |c_k|\,|\lambda_n|^{k} \leq \kappa_2^{k+1-K} |c_K|\,|\lambda_n|^{K}. \end{aligned}$$

Since K was chosen so that $\kappa_i < 1$ for $i = 1, 2$, we see that

$$\lim_{k\to\infty} |b_k|\,|\lambda_n|^{-k} = 0 \quad \text{and} \quad = \lim_{k\to\infty} |c_k|\,|\lambda_n|^{k} = 0. \tag{50}$$

Following Jørgensen, we now define the sequence $f_k = g^{-k} h_{2k} g^k$. As a matrix in $Sp(n,1)$ this is given by

$$f_k = \begin{pmatrix} L^{-k} A_{2k} L^k & L^{-k} \alpha_{2k} \lambda_n^k & L^{-k} \beta_{2k} \overline{\lambda}_n^{-k} \\ \lambda_n^{-k} \eta_{2k} L^k & \lambda_n^{-k} a_{2k} \lambda_n^k & \lambda_n^{-k} b_{2k} \overline{\lambda}_n^{-k} \\ \overline{\lambda}_n^{k} \theta_{2k} L^k & \overline{\lambda}_n^{k} c_{2k} \lambda_n^k & \overline{\lambda}_n^{k} d_{2k} \overline{\lambda}_n^{-k} \end{pmatrix}. \tag{51}$$

Using (48) and (49), we have

$$\lim_{k\to\infty} |\lambda_n^{-k} a_{2k} \lambda_n^k| = \lim_{k\to\infty} |a_{2k}| = |\lambda_n| \quad \text{and} \quad \lim_{k\to\infty} |\overline{\lambda}_n^{k} d_{2k} \overline{\lambda}_n^{-k}| = \lim_{k\to\infty} |d_{2k}| = |\lambda_n|^{-1}.$$

Similarly, using (50). we have

$$\lim_{k\to\infty} |\lambda_n^{-k} b_{2k} \overline{\lambda}_n^{-k}| = \lim_{k\to\infty} |b_{2k}|\,|\lambda|^{-2k} = 0 \quad \text{and} \quad \lim_{k\to\infty} |\overline{\lambda}_n^{k} c_{2k} \lambda_n^k| = \lim_{k\to\infty} |c_{2k}|\,|\lambda|^{2k} = 0.$$

Then, using (26), (25), (19) and (20) for the matrix f_k, we have

$$\begin{aligned} \lim_{k\to\infty} |L^{-k}\alpha_{2k}\lambda_n^k|^2 &\leq \lim_{k\to\infty} 2|\overline{\lambda}_n^{k} c_{2k}\lambda_n^k|\,|\lambda_n^{-k} a_{2k}\lambda_n^k| = 0, \\ \lim_{k\to\infty} |L^{-k}\beta_{2k}\overline{\lambda}_n^{-k}|^2 &\leq \lim_{k\to\infty} 2|\lambda_n^{-k} b_{2k}\overline{\lambda}_n^{-k}|\,|\overline{\lambda}_n^{k} d_{2k}\overline{\lambda}_n^{-k}| = 0, \\ \lim_{k\to\infty} |\lambda_n^{-k}\eta_{2k}L^k|^2 &\leq \lim_{k\to\infty} 2|\lambda_n^{-k} b_{2k}\overline{\lambda}_n^{-k}|\,|\lambda_n^{-k} a_{2k}\lambda_n^k| = 0, \\ \lim_{k\to\infty} |\overline{\lambda}_n^{k}\theta_{2k}L^k|^2 &\leq \lim_{k\to\infty} 2|\overline{\lambda}_n^{k} c_{2k}\lambda_n^k|\,|\overline{\lambda}_n^{k} d_{2k}\overline{\lambda}_n^{-k}| = 0. \end{aligned}$$

Finally, this means that $L^{-k}\alpha_{2k}\lambda_n^k$ and $L^{-k}\beta_{2k}\overline{\lambda}_n^{-k}$ both tend to the zero vector. Hence, using (15) on the matrix f_k, we see that

$$\begin{aligned} &\lim_{k\to\infty} (L^{-k}A_{2k}L^k)(L^{-k}A_{2k}L^k)^* \\ &= I_{n-1} + \lim_{k\to\infty} \Big((L^{-k}\alpha_{2k}\lambda_n^k)(L^{-k}\beta_{2k}\overline{\lambda}_n^{-k})^* + (L^{-k}\beta_{2k}\overline{\lambda}_n^{-k})(L^{-k}\alpha_{2k}\lambda_n^k)^* \Big) \\ &= I_{n-1}. \end{aligned}$$

Therefore $\{f_k \,:\, k \geq K\}$ lies in a compact subset of $\mathrm{Sp}(n,1)$ and so contains a convergent subsequence. This proves Claim 3, and hence completes the proof of Theorem 1.1.

□

Proof of Corollary 1.2. Without loss of generality, we assume $u = \infty$ and $v = o$, and g is of the form (1) and h is of the form (14). Using the identities (29), (30) and (31) from Lemma 2.1, the conditions (4), (5), (6) and (7) can be rewritten as

$$|bc|^{1/2} < \frac{1-M_g}{M_g}, \tag{52}$$

$$|ad|^{1/2} < \frac{1-M_g}{M_g}, \tag{53}$$

$$\frac{|bc|^{1/2}}{|ad|^{1/2}} < 1-M_g, \tag{54}$$

$$|ad|+|bc| < \frac{2(1-M_g)}{M_g^2}. \tag{55}$$

Our strategy will be to show that each of these conditions implies (38) and the result will then follow from Theorem 1.1.

Using (34) condition (52) implies

$$|ad|^{1/2}|bc|^{1/2} \leq \big(1+|bc|^{1/2}\big)|bc|^{1/2} < \left(1+\frac{1-M_g}{M_g}\right)\frac{1-M_g}{M_g} = \frac{1-M_g}{M_g^2}.$$

Similarly, using (35), condition (53) gives (38).

Using (34) condition (54) implies

$$M_g < 1-\left|\frac{bc}{ad}\right|^{1/2} \leq 1-\frac{|bc|^{1/2}}{|bc|^{1/2}+1} = \frac{1}{|bc|^{1/2}+1}.$$

Rearranging, this is equivalent to (52) and so the result follows from the earlier part of this proof.

Finally, condition (55) implies

$$|ad|^{1/2}|bc|^{1/2} \leq \frac{1}{2}\big(|ad|+|bc|\big) \leq \frac{1-M_g}{M_g^2}.$$

Therefore in each case $\langle h,g\rangle$ is either elementary or not discrete by Theorem 1.1. □

4 Collars in $\mathbf{H}^n_{\mathbb{H}}$

We need the following lemma, whose proof can be verified directly, to prove Theorem 1.3.

Lemma 4.1. *Let* $\mathbf{p}, \mathbf{q} \in V_0$ *be null vectors with* $\langle \mathbf{p}, \mathbf{q}\rangle = -1$. *For all real* t *let* $\gamma(t)$ *be the point in* $\mathbf{H}^n_{\mathbb{H}}$ *corresponding to the vector* $e^{\frac{t}{2}}\mathbf{p} + e^{-\frac{t}{2}}\mathbf{q}$ *in* $\mathbb{H}^{n,1}$. *Then* $\gamma = \{\gamma(t) | t \in \mathbb{R}\}$ *is the geodesic in* $\mathbf{H}^n_{\mathbb{H}}$ *with endpoints* $\mathbb{P}(\mathbf{p})$ *and* $\mathbb{P}(\mathbf{q})$ *parametrised by arc length* t.*HERE*

The following Proposition relates cross-ratios to the distance between geodesics. It will be crucial in our proofs of Theorems 1.3 and 1.4

Proposition 4.2. *Let γ_1 and γ_2 be geodesics in $\mathbf{H}^n_{\mathbb{H}}$ with endpoints u_1, v_1 and u_2, v_2 respectively. Then*

$$\cosh\big(\rho(\gamma_1,\gamma_2)\big) \geq \big|[v_2,u_1,v_1,u_2]\big| + \big|[v_2,v_1,u_1,u_2]\big|.$$

Proof. Without loss of generality, suppose that $u_1 = o$ and $v_1 = \infty$. Also, let $h \in \mathrm{PSp}(n,1)$ be a map so that $u_2 = h(o)$ and $v_2 = h(\infty)$. Suppose that $h \in G$ has the form (14), and so the cross-ratios are given by (29) and (30). Let p_t and q_s be two points on the geodesic γ_1 and $\gamma_2 = h(\gamma_1)$, respectively. Then, letting $\mathbf{0}$ denote the zero vector in $\mathbb{H}^{n-1}$, we can choose t, $s \in \mathbb{R}$ such that

$$\mathbf{p}_t = \begin{pmatrix} \mathbf{0} \\ e^{-t} \\ 1 \end{pmatrix} \in \mathbb{P}^{-1}(\gamma), \quad \mathbf{q}_s = h(\mathbf{p}_s) = \begin{pmatrix} A & \alpha & \beta \\ \eta & a & b \\ \theta & c & d \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ e^{-s} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha e^{-s}+\beta \\ a e^{-s}+b \\ c e^{-s}+d \end{pmatrix} \in \mathbb{P}^{-1}(\gamma).$$

Since $\langle \mathbf{p}_t,\, \mathbf{p}_t\rangle = -2e^{-t}$, $\langle \mathbf{q}_s,\, \mathbf{q}_s\rangle = \langle h(\mathbf{p}_s),\, h(\mathbf{p}_s)\rangle = \langle \mathbf{p}_s,\, \mathbf{p}_s\rangle = -2e^{-s}$ and

$$\langle \mathbf{p}_t,\, \mathbf{q}_s\rangle\langle \mathbf{q}_s,\, \mathbf{p}_t\rangle = \Big(\overline{a}e^{-s} + \overline{b} + (\overline{c}e^{-s}+\overline{d})e^{-t}\Big)\Big(ae^{-s}+b+(ce^{-s}+d)e^{-t}\Big),$$

we have

$$\begin{aligned}
\cosh\big(\rho(p_t,q_s)\big) &= 2\cosh^2\frac{\rho(p_t,q_s)}{2} - 1 \\
&= 2\,\frac{\langle \mathbf{p}_t,\, \mathbf{q}_s\rangle\langle \mathbf{q}_s,\, \mathbf{p}_t\rangle}{\langle \mathbf{p}_t,\, \mathbf{p}_t\rangle\langle \mathbf{q}_s,\, \mathbf{q}_s\rangle} - 1 \\
&= \frac{1}{2}\Big(|a|^2e^{t-s} + |d|^2e^{s-t} + |c|^2e^{-(s+t)} + |b|^2e^{s+t} + (\overline{d}a+\overline{b}c) + (\overline{a}d+\overline{c}b) \\
&\qquad +(\overline{a}b+\overline{b}a)e^t + (\overline{c}d+\overline{d}c)e^{-t} + (\overline{d}b+\overline{b}d)e^s + (\overline{c}a+\overline{a}c)e^{-s} - 2\Big).
\end{aligned}$$

By (15)-(26) and the property $a\overline{b}+b\overline{a} = \overline{b}a+\overline{a}b$ for $a, b \in \mathbb{H}$, we have

$$\begin{aligned}
\cosh\big(\rho(p_t,q_s)\big) &= \frac{1}{2}\Big(|a|^2e^{t-s} + |d|^2e^{s-t} + |c|^2e^{-(s+t)} + |b|^2e^{s+t} + \beta^*\alpha + \alpha^*\beta \\
&\qquad +\eta\eta^*e^t + \theta\theta^*e^{-t} + \beta^*\beta e^s + \alpha^*\alpha e^{-s}\Big) \\
&\geq |ad| + |bc| + \Re(\beta^*\alpha) + |\eta|\,|\theta| + |\beta^*\alpha| \\
&\geq |ad| + |bc| \\
&= \big|[v_2,u_1,v_1,u_2]\big| + \big|[v_2,v_1,u_1,u_2]\big|.
\end{aligned}$$

This is true for all points p_t and q_s and so it proves the proposition. □

Proof of Theorem 1.3. Without loss of generality, we suppose that g has the form (1) and so fixes o and ∞. If $h \in G$ maps γ to itself then it must map $T_r(\gamma)$ to itself.

Therefore we suppose that h does not map γ to itself. We must show that $T_r(\gamma)$ is disjoint from its image under h. We first use Proposition 4.2 to estimate the distance between γ and $h(\gamma)$ and then use condition (7) from Corollary 1.2 to conclude that, since G is discrete and non-elementary, we have

$$\begin{aligned}
\cosh\big(\gamma, h(\gamma)\big) &\geq \big|[h(\infty),o,\infty,h(o)]\big| + \big|[h(\infty),\infty,o,h(o)]\big| \\
&\geq \frac{2(1-M_g)}{M_g^2}.
\end{aligned}$$

This implies that $T_r(\gamma)$ is disjoint from its image under h. □

Proof of Theorem 1.4. Let $\mathcal{M} = \mathbf{H}^n_{\mathbb{H}}/G$ where G is a discrete, non-elementary, torsion-free subgroup of $\mathrm{Sp}(n,1)$. Let g and h be two loxodromic elements of G whose axes, γ_1 and γ_2, project to distinct short, simple, closed geodesics $\gamma_1' = \gamma_1/\langle g\rangle$. and $\gamma_2' = \gamma_2/\langle h\rangle$. Reordering if necessary, suppose that $M_h \le M_g$. Consider tubes $T_{r_1}(\gamma_1)$ and $T_{r_2}(\gamma_2)$ around γ_1 and γ_2 where

$$\cosh(2r_1) = \frac{2(1-M_g)}{M_g^2} \quad \text{and} \quad \cosh(2r_2) = \frac{2(1-M_h)}{M_h^2}.$$

We want to show that these tubes are disjoint. It suffices to show that $\rho(\gamma_1,\gamma_2) \ge r_1 + r_2$.

Without loss of generality, we suppose that g is of the form (1) and h has fixed points $p = (p_1,\cdots,p_n)^t \in \partial\mathbf{H}^n_{\mathbb{H}}$ and $q = (q_1,\cdots,q_n)^t \in \partial\mathbf{H}^n_{\mathbb{H}}$. That is

$$\sum_{i=1}^{n-1}|p_i|^2 = p_n + \overline{p}_n, \quad \sum_{i=1}^{n-1}|q_i|^2 = q_n + \overline{q}_n.$$

Let $\mathbf{p} = (p_1,\cdots,p_n,1)^t \in \mathbb{P}^{-1}(p)$ and $\mathbf{q} = (q_1,\cdots,q_n,1)^t \in \mathbb{P}^{-1}(q)$. Then by the definition of quaternionic cross-ratio, we have

$$\big|[o,q,p,\infty]\big| = \frac{|p_n|}{|\langle\mathbf{p},\mathbf{q}\rangle|}, \qquad \big|[o,p,q,\infty]\big| = \frac{|q_n|}{|\langle\mathbf{p},\mathbf{q}\rangle|}.$$

Direct computation implies that

$$\begin{aligned}
\big|\langle g(\mathbf{q}),\mathbf{q}\rangle\big| &= \left|-(\overline{q}_n\overline{\lambda}_n^{-1} + \lambda_n q_n) + \sum_{i=1}^{n-1}\overline{q}_i\lambda_i q_i\right| \\
&= \left|(1-\lambda_n)q_n + \overline{q}_n(1-\overline{\lambda}_n^{-1}) + \sum_{i=1}^{n-1}\overline{q}_i(\lambda_i - 1)q_i\right| \\
&\le |\lambda_n - 1||q_n| + |\overline{\lambda}_n^{-1} - 1||q_n| + \delta(g)\sum_{i=1}^{n-1}|q_i|^2 \\
&= |\lambda_n - 1||q_n| + |\overline{\lambda}_n^{-1} - 1||q_n| + \delta(g)(q_n + \overline{q}_n) \\
&\le M_g|q_n|.
\end{aligned}$$

Similarly, we have $|\langle g(\mathbf{p}),\mathbf{p}\rangle| \le M_g|p_n|$. We remark that these equations are special cases of (8). Therefore, we get

$$M_g^2\big|[o,p,q,\infty]\big|\,\big|[o,q,p,\infty]\big| = \frac{M_g|q_n|}{|\langle\mathbf{p},\mathbf{q}\rangle|}\,\frac{M_g|p_n|}{|\langle\mathbf{p},\mathbf{q}\rangle|} \ge \frac{|\langle g(\mathbf{q}),\mathbf{q}\rangle||\langle g(\mathbf{p}),\mathbf{p}\rangle|}{|\langle\mathbf{p},\mathbf{q}\rangle|^2} = \big|[g(p),q,p,g(q)]\big|. \tag{56}$$

Using Proposition 4.2, we get

$$\begin{aligned}
\cosh\big(\rho(\gamma_1,\gamma_2)\big) &\ge \big|[o,p,q,\infty]\big| + \big|[o,q,p,\infty]\big| \\
&\ge 2\big|[o,p,q,\infty]\big|^{1/2}\big|[o,q,p,\infty]\big|^{1/2} \\
&\ge \frac{2}{M_g}\big|[g(p),q,p,g(q)]\big|^{1/2}.
\end{aligned}$$

Using condition (4) with the roles of g and h interchanged we have $\left|[g(p),q,p,g(q)]\right|^{1/2} \geq (1-M_h)/M_h$. Using this and $M_g \geq M_h$ we have

$$\begin{aligned}
\cosh^2\big(\rho(\gamma_1,\gamma_2)\big) &\geq \left(\frac{2(1-M_h)}{M_g M_h}\right)^2 \\
&\geq \left(\frac{2(1-M_g)}{M_g^2}\right)\left(\frac{2(1-M_h)}{M_h^2}\right) \\
&= \cosh(2r_1)\cosh(2r_2) \\
&\geq \cosh^2(r_1+r_2).
\end{aligned}$$

Therefore $\rho(\gamma_1,\gamma_2) \geq r_1 + r_2$ as required. □

5 Collar width solely in terms of geodesic length

In the complex case Markham and Parker [16] used a lemma of Zagier to obtain the width of the tubular neighbourhood of the simple geodesic γ entirely in terms of its length. In this section, we will consider the counterpart in n-dimensional quaternionic hyperbolic manifold. First, we give an example to illustrate our idea.

Example 5.1. *Let G be a discrete, torsion-free, non-elementary subgroup of* $\mathrm{Sp}(2,1)$ *with*

$$g = \mathrm{diag}(e^{\mathbf{i}\beta}, e^{l/2+\mathbf{i}\alpha}, e^{-l/2+\mathbf{i}\alpha}) \in G.$$

(Here the imaginary units that generate $\mathbb{H}$ are denoted $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *in order to distinguish them from the indices denoted by i, j and k.) Define $f(k) = M_{g^k} = \left|e^{kl/2+\mathbf{i}k\alpha}-1\right| + \left|e^{-kl/2+\mathbf{i}k\alpha}-1\right| + 2\left|e^{\mathbf{i}k\beta}-1\right|$. Then we have*

$$f(k) = 2\sqrt{\left(\cosh\frac{kl}{2}+1\right)\left(\cosh\frac{kl}{2}-\cos(k\alpha)\right)} + 2\sqrt{2\Big(1-\cos(k\beta)\Big)}. \tag{57}$$

Consider the case where $l = 10^{-3}, \alpha = \frac{\pi}{3}, \beta = \frac{\pi}{4}$.

We see, for $k \in \mathbb{Z}$, that if k is not a multiple of 8 then $\cos(k\beta) \leq 1/\sqrt{2}$ and so

$$f(k) \geq 2\sqrt{2\Big(1-\cos(k\pi/4)\Big)} \geq 2\sqrt{2-\sqrt{2}} > 1.$$

Likewise, when k is not a multiple of 6 then $\cos(k\alpha) \leq 1/2$ and so

$$f(k) \geq 2\sqrt{\left(\cosh\frac{k}{2000}+1\right)\left(\cosh\frac{k}{2000}-\cos\frac{k\pi}{3}\right)} \geq 2\sqrt{2\left(1-\cos\frac{k\pi}{3}\right)} \geq 2\sqrt{2-1} = 2.$$

On the other hand, if k is a multiple of both 8 and 6, that is a multiple of 24, then

$$f(k) = 2\sqrt{\left(\cosh\frac{k}{2000}+1\right)\left(\cosh\frac{k}{2000}-1\right)} = 2\sinh\frac{k}{2000}.$$

Hence as k ranges over positive integers, the minimum value of $f(k)$ is attained for $k = 24$ and is approximately 0.024.

The above example shows that when l and k are small, then $\cos(k\alpha)$ and $\cos(k\beta)$ contribute the dominant part of the value $f(k)$. Although $f(k) \to \infty$ as $k \to \infty$, we sometimes can choose suitable k such that $k\alpha$ and $k\beta$ are close to multiples of 2π which may lead to $f(k) < \sqrt{3} - 1$. This observation gives an improvement of Theorem 1.3 by replacing M_g with a suitable M_{g^k}.

We now investigate how M_{g^k} varies with k. Let g be of the form (1). We can conjugate all its right eigenvalues to unique complex numbers with non-negative imaginary part, that is,

$$\begin{aligned} \lambda_i &= u_i e^{\mathbf{i}\beta_i} u_i^{-1}, \text{ for } 1 \leq i \leq n-1, \\ \lambda_n &= u_n |\lambda_n| e^{\mathbf{i}\beta_n} u_n^{-1}, \\ \overline{\lambda}_n^{-1} &= u_{n+1} |\lambda_n|^{-1} e^{\mathbf{i}\beta_n} u_{n+1}^{-1}, \end{aligned}$$

where $0 \leq \beta_i \leq \pi$ for $1 \leq i \leq n$. Recall that

$$\begin{aligned} M_g &= \left|\lambda_n - 1\right| + \left|\overline{\lambda}_n^{-1} - 1\right| + 2 \max_{1 \leq i \leq n-1} |\lambda_i - 1| \\ &= 2\sqrt{\left(\cosh\frac{l}{2} + 1\right)\left(\cosh\frac{l}{2} - \cos(\beta_n)\right)} + \max_{1 \leq i \leq n-1} 2\sqrt{2\Big(1 - \cos(\beta_i)\Big)}. \end{aligned}$$

Since the eigenvalues of M^k are λ_i^k, λ_n^k, $\overline{\lambda}_n^{-k}$ we have

$$M_{g^k} = 2\sqrt{\left(\cosh\frac{kl}{2} + 1\right)\left(\cosh\frac{kl}{2} - \cos(k\beta_n)\right)} + \max_{1 \leq i \leq n-1} 2\sqrt{2\Big(1 - \cos(k\beta_i)\Big)}. \tag{58}$$

Define T to be the minimum value of M_{g^k}. That is

$$T = \min_{1 \leq k < \infty} \left\{ 2\sqrt{\left(\cosh\frac{kl}{2} + 1\right)\left(\cosh\frac{kl}{2} - \cos(k\beta_n)\right)} + \max_{1 \leq i \leq n-1} 2\sqrt{2\Big(1 - \cos(k\beta_i)\Big)} \right\}. \tag{59}$$

Then by Theorem 1.3 we have the following corollary.

Corollary 5.1. *Let G be a discrete, non-elementary, torsion-free subgroup of $\mathrm{Sp}(n,1)$. Let g be a loxodromic element of G with axis the geodesic γ. Let T be given by (59) and suppose that $T < \sqrt{3} - 1$. Let r be positive real number defined by*

$$\cosh(2r) = \frac{2(1-T)}{T^2}.$$

Then the tube $T_r(\gamma)$ is precisely invariant under G.

In order to prove Theorem 1.5, we need the following adapted Pigeonhole Principle.

Lemma 5.2. *(cf. Pigeonhole Principle in [18]) Given $0 \leq \beta_1, \cdots, \beta_n < 2\pi$ and a positive integer $N \geq 2$, there exists $k \leq N^n$ such that*

$$\cos(k\beta_i) \geq \cos\frac{2\pi}{N},$$

for each $1 \leq i \leq n$.

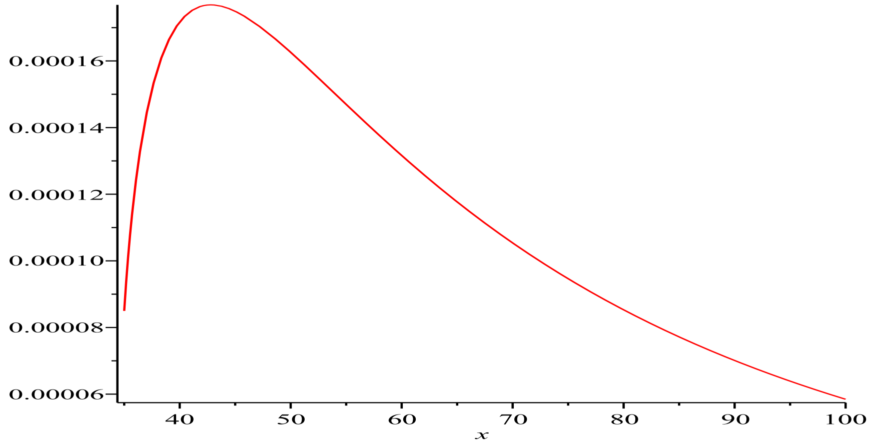

Figure 1: The graph of function $l = l(x)$ defined by (61) for $n = 2$.

Proof. Consider the solid n-cube $[0, 2\pi]^n$ in $\mathbb{R}^n$ and consider the points $z_k = (k\beta_1, k\beta_2 ..., k\beta_n)$ for each $1 \le k \le N^n$. There are N^n of them. For each $i \in \{1, \dots, n\}$ and each $k \in \{1, \dots, N^n\}$ let m_{ik} be an integer so that $k\beta_i - 2\pi m_{ik} \in [0, 2\pi)$. Let

$$\hat{z}_k = \big(k\beta_1 - 2\pi m_{1k}, \dots, k\beta_n - 2\pi m_{nk}\big) \in [0, 2\pi]^n.$$

Divide the n-cube into N^n cubes of side length $2\pi/N$ and consider which of these small cubes contain the points $\hat{z}_k$.

If, for some $j \in \{1, \dots, N^n\}$, the point $\hat{z}_j$ lies in the n-cube $[0, \frac{2\pi}{N}]^n$ then $\cos(j\beta_i) \ge \cos\frac{2\pi}{N}$ for each $i \in \{1, \dots, n\}$ and we have the result.

Suppose that none of the points $\hat{z}_k$ lie in the n-cube $[0, \frac{2\pi}{N}]^n$. Then there is at least one small n-cube with two points in it, say $\hat{z}_j$ and $\hat{z}_k$, where $j > k$. Then $\hat{z}_{j-k}$ is in the n-cube $I_1 \times I_2 \times \cdots \times I_n$, where each $I_i = [0, \frac{2\pi}{N}]$ or $[2\pi - \frac{2\pi}{N}, 2\pi]$. That is $\cos\big((j-k)\beta_i\big) \ge \cos\frac{2\pi}{N}$ for each $i \in \{1, \dots, n\}$. The proof is complete. □

Proof of Theorem 1.5. As in (58), we have

$$M_{g^k} = 2\sqrt{\left(\cosh\frac{kl}{2} + 1\right)\left(\cosh\frac{kl}{2} - \cos(k\beta_n)\right)} + \max_{1\le i\le n-1} 2\sqrt{2\Big(1 - \cos(k\beta_i)\Big)}.$$

By Lemma 5.2 for $N \ge 2$, there exists $k \le N^n$ such that

$$\cos(k\beta_i) \ge \cos\frac{2\pi}{N}$$

for each $1 \le i \le n$. Then there exists $k \le N^n$ such that

$$M_{g^k} \le 2\sqrt{\left(\cosh\frac{N^n l}{2} + 1\right)\left(\cosh\frac{N^n l}{2} - \cos\frac{2\pi}{N}\right)} + 2\sqrt{2\left(1 - \cos\frac{2\pi}{N}\right)}. \tag{60}$$

Define

$$h(x, l) = 2\sqrt{\left(\cosh\frac{x^n l}{2} + 1\right)\left(\cosh\frac{x^n l}{2} - \cos\frac{2\pi}{x}\right)} + 2\sqrt{2\left(1 - \cos\frac{2\pi}{x}\right)}.$$

Notice that $h(x,l)$ is an increasing function of l for fixed x. Therefore when $l>0$ we have

$$h(x,l) > h(x,0) = 4\sqrt{2\left(1-\cos\frac{2\pi}{x}\right)}.$$

When $x \geq 2$ the function $h(x,0)$ is a decreasing function of x. Define x_0 by $h(x_0,0)=\sqrt{3}-1$, that is

$$x_0 = \frac{2\pi}{\arccos\frac{14+\sqrt{3}}{16}} \approx 34.284.$$

Then if $2 \leq x < x_0$ and $l>0$ we have $h(x,l) > h(x,o) > h(x_0,0) = \sqrt{3}-1$. Hence, in order to have $h(x,l) \leq \sqrt{3}-1$ we must have $x \geq x_0$. For all $x \geq x_0$ the equation $h(x,l)=\sqrt{3}-1$ defines a function

$$l(x) = \frac{2}{x^n}\operatorname{arccosh}\left(\frac{\sqrt{13-2\sqrt{3}-6\cos\frac{2\pi}{x}+\cos^2\frac{2\pi}{x}-4(\sqrt{3}-1)\sqrt{2\left(1-\cos\frac{2\pi}{x}\right)}}-1+\cos\frac{2\pi}{x}}{2}\right). \tag{61}$$

Hence for all integers $N \geq 35$, we can find l satisfying the condition (10). Then our result follows from the application of Theorem 1.3. The proof is complete. □

With the aid of mathematical software, for case $n=2$, we find that when $N=43$, we get the maximal interval $0<l<l(43)\approx 0.00017681$ to apply our theorem. The graph of function $l(x)$ defined by (61) is given in Figure 1. This gives the proof of Corollary 1.6.

Given the rotational angles of loxodromic element, we may be able to use Corollary 5.1 to choose suitable N which may less than 35. For instance in Example 5.1, the optimum value occurs when $N=24$.

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