Pseudo-Anosov Diffeomorphisms of the Twice Punctured Torus

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Introduction.

The group of isotopy classes of diffeomorphisms from a surface of finite type to itself (otherwise known as the mapping class group) is a familiar object. There are two fundamental theorems which enable one to describe elements of this group. The first theorem is due to Nielsen [9], [10], [11] and says (in the later reformulation of Thurston [13]) that surface diffeomorphisms may be classified as (i) periodic, (ii) reducible or (iii) pseudo-Anosov. The second theorem is due to Dehn [3] and says that the mapping class group is finitely generated by elementary diffeomorphisms called Dehn twists.

There are various algorithms for deciding whether a given diffeomorphism is periodic, reducible or pseudo-Anosov. In particular, there are algorithms due to Bestvina and Handel [1] and Hamidi-Tehrani and Chen [4]. The latter algorithm uses the piecewise linear action of the mapping class group on the piecewise linear structure of projective measured lamination space given by the π_1-train tracks of Birman and Series [2]. This piecewise linear structure and piecewise linear action was worked out in detail for the twice punctured torus by Parker and Series [12]. The purpose of this note is to use this description to give an algorithm which takes a particular diffeomorphism specified as a word in a given set of Dehn twist generators and decides whether or not it is pseudo-Anosov. Thus it can be thought of as a realisation of part of the Hamidi-Tehrani and Chen algorithm in this case.

It is known that the mapping torus of a pseudo-Anosov diffeomorphism is a hyperbolic 3-manifold. However there are very few descriptions in the literature of concrete examples of such manifolds. An application of our method is that we can construct many examples of hyperbolic 3-manifolds which fibre over the circle with fibre the twice punctured torus. In particular, we can construct the Whitehead link complement in this way (see [7]) using one of the simplest pseudo-Anosov diffeomorphisms. We work this example out in detail in the last section.

The results of this paper arose out of discussions during the memorial conference for Wilhelm Killing held in Braniewo (Poland) from 31st August to 2nd September 1998. They were continued during visits of CM to Durham and JRP to Bielefeld. The results were presented by JRP at the workshop on Algebra and Topology held at Pusan National University (Korea) from 14th to 26th August 2000. We would like to thank the organisers of both conferences and both universities, including SFB 343 in Bielefeld, for their support and hospitality.

1. Projective measured lamination space.

Let Σ be the twice punctured torus. In what follows we do not use the conventional Thurston theory of train tracks but a variant due to Birman and Series [2]. See [5] and [12] for more details.

A **simple loop** on Σ is a closed curve with no self intersections. A simple loop is **boundary parallel** or **peripheral** if it is homotopic to a loop around a puncture. A **multiple simple loop** is a collection of pairwise disjoint simple loops none of which is either homotopically trivial or boundary parallel. It is easy to see that for the twice punctured torus any multiple simple loop contains loops from at most 2 homotopy classes. Thus a multiple simple loop γ on Σ can be written as $m_1\gamma_1 + m_2\gamma_2$ where m_1 and m_2 are non-negative integers and γ_1 and γ_2 are distinct non-trivial homotopy classes of non-peripheral disjoint simple closed curves on Σ.

We fix a hyperbolic structure on Σ by specifying a fundamental polygon for the action of $\pi_1(\Sigma)$ on $\mathbb{H}^2$. The fundamental domain R that we choose to work with has six vertices, all of which project to punctures of Σ (see [5], [12]). We label these $v_1, \ldots, v_6$ in clockwise order. The side pairings will be S_1 identifying v_1v_2 to v_4v_3, S_2 identifying v_6v_1 with v_5v_4 and T identifying v_5v_6 with v_3v_2. We assume that S_1, S_2 and T match the endpoints of the respective sides. Clearly v_1 and v_4 project to one of the punctures and the other four vertices project to the other. The maps S_1, S_2 and T correspond to homotopy classes of simple closed curves that generate the fundamental group $\pi_1(\Sigma)$.

Let $\overline{R}$ be the closure of R in $\mathbb{H}^2$. A π_1**-train track** τ is a collection of pairwise disjoint arcs, called **strands**, $\alpha_j\colon [0,1] \longrightarrow \overline{R}$ so that

(i) $\alpha_j(0) \in v_av_b$ and $\alpha_j(1) \in v_cv_d$,

(ii) $\alpha_j(\lambda) \in R^\circ$ for $\lambda \in (0,1)$,

(iii) at most one strand joins each pair of sides.

(iv) no strand goes from one side to itself.

An arc of τ is called a **corner arc** if it joins adjacent sides of R (that is, it joins v_av_b and v_bv_c). Each corner arc faces a particular vertex of R (in our example v_b) and for each vertex cycle in the side pairing of R we have the corresponding **corner cycle** consisting of all corner arcs corresponding to the same puncture.

A **weighting** w on a π_1-train track τ is an assignment of a non-negative number $w(\alpha_j)$ to each arc α_j of τ. A weighting w on a π_1-train track τ is called a **proper weighting** if it satisfies the following two conditions:

(i) For each side pairing $\mu_k\colon \sigma_k \longrightarrow \sigma_{k'}$, the sum of the weights of arcs with endpoints on σ_k is the same as the sum of the weights of arcs with endpoints on $\sigma_{k'}$.

(ii) At least one arc in each corner cycle must have weight zero.

By a theorem of Birman and Series [2] the collection of all proper weightings on π_1-train tracks may be identified with the space $\mathcal{ML}(\Sigma)$ of measured laminations on Σ. Moreover, proper integral weightings on π_1-train tracks may be identified with multiple simple loops on Σ.

A π_1-train track τ is said to be **recurrent** if there exists a proper weighting w so that $w(\alpha_j)$ is non-zero for all branches α_j of τ. Such a π_1-train track τ is said to be **maximal** if there does not exist a recurrent π_1-train track τ' so that τ is properly contained in τ' in the obvious sense. We call the collection of all proper weightings $\Delta(\tau)$ on a maximal recurrent π_1-train track τ a **maximal cell**. It follows from Thurston's theory, or as can be verified along the lines given in [12], that if τ is a maximal recurrent train track on the twice punctured torus then the dimension of $\Delta(\tau)$ is 4. It was shown in [12] that there are 28 maximal recurrent π_1-train tracks on Σ. We denote these by $\tau_1, \ldots, \tau_{28}$. Each maximal cell, $\Delta(\tau_j)$ (or to make the notation easier Δ_j) is the positive linear span of four out of eleven elementary recurrent π_1 train train tracks. This gives a piecewise linear structure on $\mathcal{ML}(\Sigma)$. The details of this are given below, following [12].

The space $\Delta_j = \Delta(\tau_j)$ may be projectivised in a natural way to obtain $\mathbb{P}\Delta_j$. The space $\mathcal{PML} = \mathcal{PML}(\Sigma)$ is the union over all 28 of the τ_j of the corresponding cones $\mathbb{P}\Delta_j$, which we call π_1-cones, glued along their lower dimensional common simplices. Using the Birman-Series identification, the space $\mathcal{PML}(\Sigma)$ can be naturally identified with the space of projective measured laminations on Σ. The space $\mathcal{PML}$ is the Thurston boundary of the Teichmüller space of Σ and is a 3-sphere. This can be checked directly using the piecewise linear structure given above (see [12]).

We now introduce the elementary recurrent π_1-train tracks that will form the basis of $\mathcal{ML}(\Sigma)$. Fig. 1 is a schematic picture of the eleven recurrent π_1-train tracks as they appear on the fundamental domain R. The end of a strand on one side of R is glued by a side-pairing transformation to the corresponding end of the paired side. Thus shortest words representing these loops can be either computed directly or read off using the method of cutting sequences, see [2] or [5]. For example, in the loop $\mathbf{e}_1^1$ there are three strands. The end of the strand on v_4v_3 is glued to the end on v_1v_2; the end on v_2v_3 is glued to the end on v_5v_6, and the end on v_5v_4 is glued to the end on v_1v_6. Thus the cutting sequence is S_1TS_2,

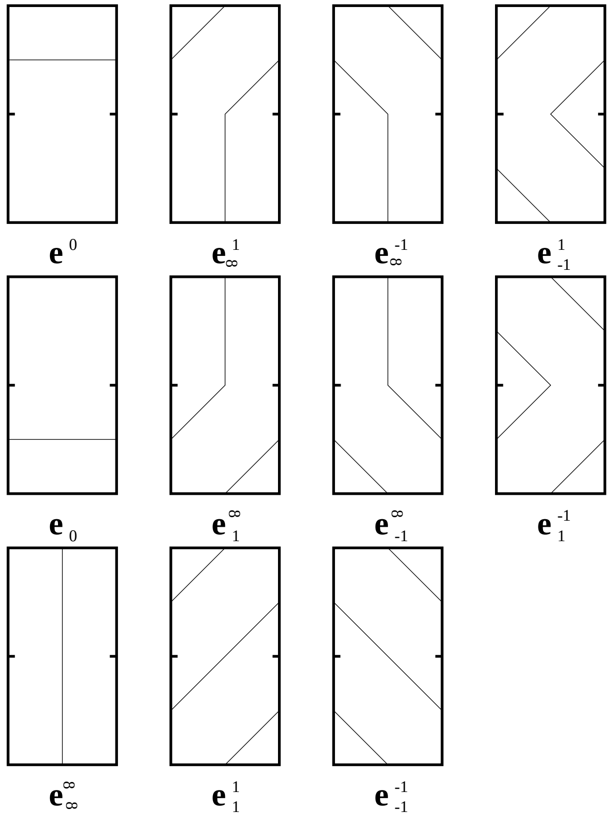

Fig. 1. The elementary recurrent π_1-train tracks.

which as one may easily verify represents this loop in $\pi_1(\Sigma)$. Since we are only interested in the unoriented loop up to free homotopy, any cyclic permutation of this sequence or its inverse would work just as well. The full list of cutting sequences for loops is

$$\begin{array}{llll} \mathbf{e}^0 = S_1, & \mathbf{e}^1_\infty = S_1T, & \mathbf{e}^{-1}_\infty = {S_1}^{-1}T, & \mathbf{e}^1_{-1} = S_1T{S_2}^{-1}, \\ \mathbf{e}_0 = S_2, & \mathbf{e}^\infty_1 = TS_2, & \mathbf{e}^\infty_{-1} = T{S_2}^{-1}, & \mathbf{e}^{-1}_1 = {S_1}^{-1}TS_2, \\ \mathbf{e}^\infty_\infty = T, & \mathbf{e}^1_1 = S_1TS_2, & \mathbf{e}^{-1}_{-1} = {S_1}^{-1}T{S_2}^{-1}. & \end{array}$$

The reason for our notation is the following. If we split R into two boxes, the upper one with vertices v_1, v_2, v_3, v_4 and the lower with vertices v_1, v_4, v_5, v_6 (see [5]) then $\mathbf{e}^i_j$ has gradient i in the upper box and j in the lower box. Where there is no superscript (subscript) then the relevant loop has no arcs in the upper (respectively lower) box. This idea is developed further in [5].

We now define 28 cells Δ_j in $\mathcal{ML}(\Sigma)$. As we shall indicate (see also [12]), these cells are maximal, meeting only on lower dimensional faces, and their union is $\mathcal{ML}(\Sigma)$.

$$\Delta_1 = \mathrm{sp}^+\{\mathbf{e}^0, \mathbf{e}_0, \mathbf{e}^1_1, \mathbf{e}^1_{-1}\}, \quad \Delta_2 = \mathrm{sp}^+\{\mathbf{e}^0, \mathbf{e}^1_\infty, \mathbf{e}^1_1, \mathbf{e}^1_{-1}\},$$

$$\Delta_3 = \mathrm{sp}^+\{\mathbf{e}^\infty_1, \mathbf{e}_0, \mathbf{e}^1_1, \mathbf{e}^1_{-1}\}, \quad \Delta_4 = \mathrm{sp}^+\{\mathbf{e}^\infty_1, \mathbf{e}^1_\infty, \mathbf{e}^1_1, \mathbf{e}^1_{-1}\},$$

$$\Delta_5 = \mathrm{sp}^+\{\mathbf{e}^\infty_1, \mathbf{e}_0, \mathbf{e}^\infty_{-1}, \mathbf{e}^1_{-1}\}, \quad \Delta_6 = \mathrm{sp}^+\{\mathbf{e}^\infty_1, \mathbf{e}^\infty_\infty, \mathbf{e}^\infty_{-1}, \mathbf{e}^1_{-1}\},$$

$$\Delta_7 = \mathrm{sp}^+\{\mathbf{e}^\infty_\infty, \mathbf{e}^\infty_1, \mathbf{e}^1_\infty, \mathbf{e}^1_{-1}\}.$$

$$\Delta_8 = \mathrm{sp}^+\{\mathbf{e}^0, \mathbf{e}_0, \mathbf{e}_{-1}^{-1}, \mathbf{e}_1^{-1}\}, \quad \Delta_9 = \mathrm{sp}^+\{\mathbf{e}^0, \mathbf{e}_\infty^{-1}, \mathbf{e}_{-1}^{-1}, \mathbf{e}_1^{-1}\},$$
$$\Delta_{10} = \mathrm{sp}^+\{\mathbf{e}_{-1}^\infty, \mathbf{e}_0, \mathbf{e}_{-1}^{-1}, \mathbf{e}_1^{-1}\}, \quad \Delta_{11} = \mathrm{sp}^+\{\mathbf{e}_{-1}^\infty, \mathbf{e}_\infty^{-1}, \mathbf{e}_{-1}^{-1}, \mathbf{e}_1^{-1}\},$$
$$\Delta_{12} = \mathrm{sp}^+\{\mathbf{e}_{-1}^\infty, \mathbf{e}_0, \mathbf{e}_1^\infty, \mathbf{e}_1^{-1}\}, \quad \Delta_{13} = \mathrm{sp}^+\{\mathbf{e}_{-1}^\infty, \mathbf{e}_\infty^\infty, \mathbf{e}_1^\infty, \mathbf{e}_1^{-1}\},$$
$$\Delta_{14} = \mathrm{sp}^+\{\mathbf{e}_\infty^\infty, \mathbf{e}_{-1}^\infty, \mathbf{e}_\infty^{-1}, \mathbf{e}_1^{-1}\}.$$

$$\Delta_{15} = \mathrm{sp}^+\{\mathbf{e}_0, \mathbf{e}^0, \mathbf{e}_1^1, \mathbf{e}_1^{-1}\}, \quad \Delta_{16} = \mathrm{sp}^+\{\mathbf{e}_0, \mathbf{e}_1^\infty, \mathbf{e}_1^1, \mathbf{e}_1^{-1}\},$$
$$\Delta_{17} = \mathrm{sp}^+\{\mathbf{e}_\infty^1, \mathbf{e}^0, \mathbf{e}_1^1, \mathbf{e}_1^{-1}\}, \quad \Delta_{18} = \mathrm{sp}^+\{\mathbf{e}_\infty^1, \mathbf{e}_1^\infty, \mathbf{e}_1^1, \mathbf{e}_1^{-1}\},$$
$$\Delta_{19} = \mathrm{sp}^+\{\mathbf{e}_\infty^1, \mathbf{e}^0, \mathbf{e}_\infty^{-1}, \mathbf{e}_1^{-1}\}, \quad \Delta_{20} = \mathrm{sp}^+\{\mathbf{e}_\infty^1, \mathbf{e}_\infty^\infty, \mathbf{e}_\infty^{-1}, \mathbf{e}_1^{-1}\},$$
$$\Delta_{21} = \mathrm{sp}^+\{\mathbf{e}_\infty^\infty, \mathbf{e}_\infty^1, \mathbf{e}_1^\infty, \mathbf{e}_1^{-1}\}.$$

$$\Delta_{22} = \mathrm{sp}^+\{\mathbf{e}_0, \mathbf{e}^0, \mathbf{e}_{-1}^{-1}, \mathbf{e}_{-1}^1\}, \quad \Delta_{23} = \mathrm{sp}^+\{\mathbf{e}_0, \mathbf{e}_{-1}^\infty, \mathbf{e}_{-1}^{-1}, \mathbf{e}_{-1}^1\},$$
$$\Delta_{24} = \mathrm{sp}^+\{\mathbf{e}_\infty^{-1}, \mathbf{e}^0, \mathbf{e}_{-1}^{-1}, \mathbf{e}_{-1}^1\}, \quad \Delta_{25} = \mathrm{sp}^+\{\mathbf{e}_\infty^{-1}, \mathbf{e}_{-1}^\infty, \mathbf{e}_{-1}^{-1}, \mathbf{e}_{-1}^1\},$$
$$\Delta_{26} = \mathrm{sp}^+\{\mathbf{e}_\infty^{-1}, \mathbf{e}^0, \mathbf{e}_\infty^1, \mathbf{e}_{-1}^1\}, \quad \Delta_{27} = \mathrm{sp}^+\{\mathbf{e}_\infty^{-1}, \mathbf{e}_\infty^\infty, \mathbf{e}_\infty^1, \mathbf{e}_{-1}^1\},$$
$$\Delta_{28} = \mathrm{sp}^+\{\mathbf{e}_\infty^\infty, \mathbf{e}_\infty^{-1}, \mathbf{e}_{-1}^\infty, \mathbf{e}_{-1}^1\}.$$

The statement that Δ_j is a cell should be interpreted in the following way. One needs to check that the four irreducible loops defining Δ_j are all supported on a common π_1-train track τ_j. This is immediate since one checks that, in each case, all four loops can be drawn in R in such a way that they intersect only on the boundary ∂R. The arcs may be homotoped so that their endpoints are at the midpoints of the sides of R. Since the midpoints are identified by the side pairings, this exactly gives a π_1-train track in the sense of [2]. The cell Δ_j consists of all proper weightings on the π_1-train track τ_j.

Notation. *When we want to speak of a point of one of these cells we write it as an ordered quadruple* (a, b, c, d) *to represent* $a\mathbf{e}_i + b\mathbf{e}_j + c\mathbf{e}_k + d\mathbf{e}_k \in \mathrm{sp}^+\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l\}$ *where the irreducible loops are taken in the order given above. When we want to refer to a general point of* $\mathcal{ML}(\Sigma)$ *we refer to it as* $(j; a, b, c, d)$ *which means* $(a, b, c, d) \in \Delta_j$.

For example $(1; a, b, c, d)$ means $(a, b, c, d) \in \Delta_1$ or equivalently $a\mathbf{e}^0 + b\mathbf{e}_0 + c\mathbf{e}_1^1 + d\mathbf{e}_{-1}^1$.

We will view these coordinates projectively. That is, we do not distinguish between $(j; a, b, c, d)$ and $(j; \lambda a, \lambda b, \lambda c, \lambda d)$ where λ is any positive real number. We say that the coordinates (a, b, c, d) are **rationally dependent** if there exists such a positive constant λ so that λa, λb, λc, λd are all rational numbers. This is the same as saying that the ratios of non-zero elements of the set $\{a, b, c, d\}$ are all rational. If (a, b, c, d) are rationally dependent then it is clear that we may choose λ so that λa, λb, λc, λd are all integers. Hence a point of $\mathcal{ML}(\Sigma)$ with rationally dependent coordinates corresponds to a multiple simple loop. The converse is not true, that is it may be that a point does not have rationally dependent coordinates but the underlying lamination is supported on the homotopy classes of two disjoint simple closed curves. For example, consider $\mathbf{e}^0 + \sqrt{2}\mathbf{e}_0$.

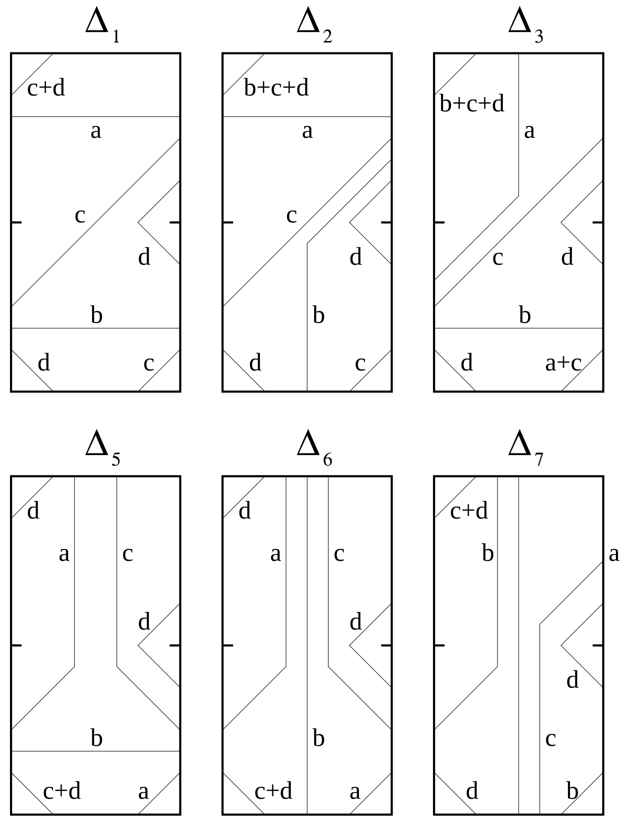

Fig. 2. Generic points in the maximal cells $\Delta_1, \ldots, \Delta_7$.

We now indicate how the lower dimensional facets in the boundaries of the maximal cells Δ_j for $j = 1, \ldots, 28$ fit together in such a way that the resulting cell complex is homeomorphic to a 3−sphere. The 1-skeleton of this cell complex is the suspension of a certain triangulation of the 2-sphere. This triangulation is shown in Fig. 3. Observe that there are fourteen maximal cells containing the irreducible loop $\mathbf{e}^1_{-1}$ and fourteen containing $\mathbf{e}^{-1}_1$. Moreover, these two irreducible loops never occur together in one of the cells (or else there would be loops around both punctures). Thus each maximal cell is a cone with apex $\mathbf{e}^1_{-1}$ or $\mathbf{e}^{-1}_1$ over the cell spanned by the other three irreducible loops. One can verify that there are fourteen possibilities for these cells spanned by three loops and that each one arises. Moreover, these fourteen cells may be glued together to form a triangulation of the 2−sphere as indicated in Fig. 3. Thus the fourteen maximal cells involving $\mathbf{e}^1_{-1}$ form a cone over the 2−sphere, that is a 3−ball. Similarly the other fourteen maximal cells also give a 3−ball. When the boundaries of these two balls are glued together in the obvious manner they form a 3−sphere.

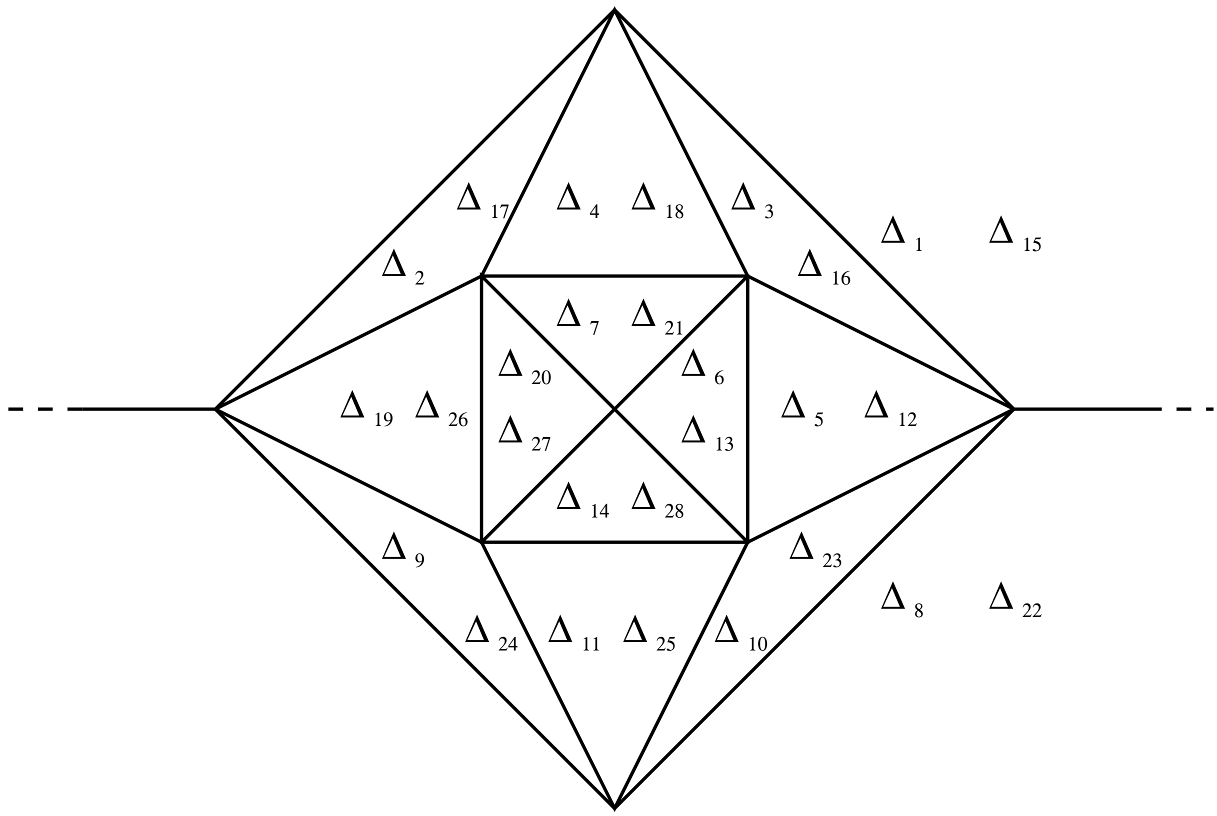

Fig. 3. How the maximal cells fit together.

Dehn twist generators for $\mathcal{MCG}(\Sigma)$

Let the Dehn twists about $\mathbf{e}^{\infty}_{\infty}$, $\mathbf{e}^0$ and $\mathbf{e}_0$ be denoted by δ_0, δ_1 and δ_2 respectively. These three Dehn twists generate the orientation preserving mapping class group $\mathcal{MCG}(\Sigma)$ [12]. We now give the action of these Dehn twists on the cell structure of $\mathcal{ML}(\Sigma)$ given in the previous section. Again this follows [12].

We begin by introducing some symmetries which fix the punctures. These will simplify matters. The symmetry group will be isomorphic to Klein's four group and we describe its non-trivial elements by their action on the vertices of R_2:

- ι_1 interchanges the pairs (v_1, v_4), (v_2, v_3), (v_5, v_6);
- ι_2 interchanges the pairs (v_1, v_4), (v_2, v_5), (v_3, v_6);
- ι_3 fixes v_1, v_4 and interchanges the pairs (v_2, v_6), (v_3, v_5).

When necessary we shall denote the identity by ι_0. It is clear that ι_1 and ι_3 are orientation reversing homeomorphisms of Σ and that ι_2 is orientation preserving. We easily see that ι_j has the following effect on the eleven irreducible loops:

$$\begin{aligned}
\iota_1\colon \mathbf{e}^i_j &\longmapsto \mathbf{e}^{-i}_{-j}, \quad \mathbf{e}^0 \longmapsto \mathbf{e}^0, \quad \mathbf{e}_0 \longmapsto \mathbf{e}_0,\\
\iota_2\colon \mathbf{e}^i_j &\longmapsto \mathbf{e}^j_i, \quad \mathbf{e}^0 \longmapsto \mathbf{e}_0, \quad \mathbf{e}_0 \longmapsto \mathbf{e}^0,\\
\iota_3\colon \mathbf{e}^i_j &\longmapsto \mathbf{e}^{-j}_{-i}, \quad \mathbf{e}^0 \longmapsto \mathbf{e}_0, \quad \mathbf{e}_0 \longmapsto \mathbf{e}^0
\end{aligned}$$

where $i, j \in \{\infty, \pm 1\}$ and $-\infty = \infty$.

Thus these actions clearly respect the cell structure of $\mathcal{ML}(\Sigma)$. In particular, the maximal cells $\Delta_8, \ldots, \Delta_{28}$ can be expressed as $\Delta_{j+7k} = \iota_k(\Delta_j)$ for $j = 1, \ldots, 7$ and $k = 1, 2, 3$.

The symmetries ι_k conjugate the Dehn twists δ_j to one another. It is easy to check that

$$\begin{array}{ccc} \iota_1\delta_0\iota_1 = {\delta_0}^{-1}, & \iota_1\delta_1\iota_1 = {\delta_1}^{-1}, & \iota_1\delta_2\iota_1 = {\delta_2}^{-1}, \\ \iota_2\delta_0\iota_2 = \delta_0, & \iota_2\delta_1\iota_2 = \delta_2, & \iota_2\delta_2\iota_2 = \delta_1, \\ \iota_3\delta_0\iota_3 = {\delta_0}^{-1}, & \iota_3\delta_1\iota_3 = {\delta_2}^{-1}, & \iota_3\delta_2\iota_3 = {\delta_1}^{-1}. \end{array}$$

We can also express ι_2 in terms of the δ_j as

$$\iota_2 = \delta_2\delta_0\delta_2\delta_1\delta_0\delta_2 = (\delta_0\delta_1\delta_2)^2.$$

Using general results of Birman-Series [2] and Hamidi-Tehrani-Chen [4] we know that the Dehn twist generators of $\mathcal{MCG}(\Sigma)$ act piecewise linearly on π_1-train tracks (see also the final paragraph of [13]). These piecewise linear maps were found explicitly in [12]. We now summarise these results for the action of ${\delta_k}^{\pm 1}$ on Δ_j for $j = 1, \ldots, 7$ and $k = 0, 1, 2$. For the action on $\Delta_8, \ldots, \Delta_{28}$ we need to apply the symmetries.

$$\delta_0(1; a, b, c, d) = (7; c, b + c, a + c, d)$$

$$\delta_0(2; a, b, c, d) = (7; b + c, c, a + b + c, d)$$

$$\delta_0(3; a, b, c, d) = (7; a + c, a + b + c, c, d)$$

$$\delta_0(4; a, b, c, d) = (7; a + b + c, a + c, b + c, d)$$

$$\delta_0(5; a, b, c, d) = \begin{cases} (5; a + b, c - a, a, d) & \text{if } a \le d \\ (6; a + b, a - c, c, d) & \text{if } d \le a. \end{cases}$$

$$\delta_0(6; a, b, c, d) = \begin{cases} (5; a, c - a - b, a + b, d) & \text{if } a + b \le c \\ (6; a, a + b - c, c, d) & \text{if } c \le a + b. \end{cases}$$

$$\delta_0(7; a, b, c, d) = (7; a + b + c, b, c, d).$$

$$\delta_0{}^{-1}(1;a,b,c,d) = (25;a,b,c,d)$$

$$\delta_0{}^{-1}(2;a,b,c,d) = (24;a,b,c,d)$$

$$\delta_0{}^{-1}(3;a,b,c,d) = (23;a,b,c,d)$$

$$\delta_0{}^{-1}(4;a,b,c,d) = (22;a,b,c,d)$$

$$\delta_0{}^{-1}(5;a,b,c,d) = \begin{cases} (5;c,a-c,b+c,d) & \text{if } c \le a \\ (6;a,c-a,b+c,d) & \text{if } a \le c. \end{cases}$$

$$\delta_0{}^{-1}(6;a,b,c,d) = \begin{cases} (5;b+c,a-b-c,c,d) & \text{if } b+c \le a \\ (6;a,b+c-a,c,d) & \text{if } a \le b+c. \end{cases}$$

$$\delta_0{}^{-1}(7;a,b,c,d) = \begin{cases} (7;a-b-c,b,c,d) & \text{if } b+c \le a \\ (4;a-c,a-b,b+c-a,d) & \text{if } b,c \le a \le b+c \\ (3;a-c,b-a,c,d) & \text{if } c \le a \le b \\ (2;c-a,a-b,b,d) & \text{if } b \le a \le c \\ (1;c-a,b-a,a,d) & \text{if } a \le b,c. \end{cases}$$

$$\delta_1(1;a,b,c,d) = \begin{cases} (1;a-c-d,b,c,d) & \text{if } c+d \le a \\ (3;c+d-a,b,a-d,d) & \text{if } d \le a \le c+d \\ (5;c,b,d-a,a) \in \Delta_5 & \text{if } a \le d. \end{cases}$$

$$\delta_1(2;a,b,c,d) = \begin{cases} (2;a-b-c-d,b,c,d) & \text{if } b+c+d \le a \\ (4;b+c+d-a,b,a-b-d,d) & \text{if } b+d \le a \le b+c+d \\ (7;b+d-a,c,a-d,d) & \text{if } d \le a \le b+d \\ (6;c,b,d-a,a) & \text{if } a \le d. \end{cases}$$

$$\delta_1(3;a,b,c,d) = (12;d,b,c,a)$$

$$\delta_1(4;a,b,c,d) = (13;d,b,c,a)$$

$$\delta_1(5;a,b,c,d) = (10;d,b,c,a)$$

$$\delta_1(6;a,b,c,d) = (11;d,b,c,a)$$

$$\delta_1(7;a,b,c,d) = (14;c,d,a,b).$$

$$\delta_1{}^{-1}(1;a,b,c,d) = (1;a+c+d,b,c,d)$$

$$\delta_1{}^{-1}(2;a,b,c,d) = (2;a+b+c+d,b,c,d)$$

$$\delta_1{}^{-1}(3;a,b,c,d) = (1;c+d,b,a+c,d)$$

$$\delta_1{}^{-1}(4;a,b,c,d) = (2;b+c+d,b,a+c,d)$$

$$\delta_1{}^{-1}(5;a,b,c,d) = (1;d,b,a,c+d)$$

$$\delta_1{}^{-1}(6;a,b,c,d) = (2;d,b,a,c+d)$$

$$\delta_1{}^{-1}(7;a,b,c,d) = (2;c+d,a+c,b,d).$$

$$\delta_2(1;a,b,c,d)=\begin{cases}(1;a,b+d-c,c,d) & \text{if } c\le b+d\\ (2;a,c-b-d,b+d,d) & \text{if } b+d\le c.\end{cases}$$

$$\delta_2(2;a,b,c,d)=\begin{cases}(1;a,d-c,c,b+d) & \text{if } c\le d\\ (2;a,c-d,d,b+d) & \text{if } d\le c.\end{cases}$$

$$\delta_2(3;a,b,c,d)=\begin{cases}(3;a,b+d-a-c,c,d) & \text{if } a+c\le b+d\\ (4;a,a+c-b-d,b+d-a,d) & \text{if } a\le b+d\le a+c\\ (7;a-b-d,b+d,c,d) & \text{if } b+d\le a.\end{cases}$$

$$\delta_2(4;a,b,c,d)=\begin{cases}(3;a,d-a-c,c,b+d) & \text{if } a+c\le d\\ (4;a,a+c-d,d-a,b+d) & \text{if } a\le d\le a+c\\ (7;a-d,d,c,b+d) & \text{if } d\le a.\end{cases}$$

$$\delta_2(5;a,b,c,d)=\begin{cases}(5;a,b+c+d-a,c,d) & \text{if } a\le b+c+d\\ (6;b+c+d,a-b-c-d,c,d) & \text{if } b+c+d\le a.\end{cases}$$

$$\delta_2(6;a,b,c,d)=\begin{cases}(5;a,c+d-a,b+c,d) & \text{if } a\le c+d\\ (6;c+d,a-c-d,b+c,d) & \text{if } c+d\le a.\end{cases}$$

$$\delta_2(7;a,b,c,d)=\begin{cases}(5;b,d-b,a,c+d) & \text{if } b\le d\\ (6;d,b-d,a,c+d) & \text{if } d\le b.\end{cases}$$

$$\delta_2{}^{-1}(1;a,b,c,d)=\begin{cases}(1;a,b+c-d,c,d) & \text{if } d\le b+c\\ (2;a,d-b-c,c,b+c) & \text{if } b+c\le d.\end{cases}$$

$$\delta_2{}^{-1}(2;a,b,c,d)=\begin{cases}(1;a,c-d,b+c,d) & \text{if } d\le c\\ (2;a,d-c,b+c,c) & \text{if } c\le d.\end{cases}$$

$$\delta_2{}^{-1}(3;a,b,c,d)=\begin{cases}(3;a,a+b+c-d,c,d) & \text{if } d\le a+b+c\\ (4;a,d-a-b-c,c,a+b+c) & \text{if } a+b+d\le d.\end{cases}$$

$$\delta_2{}^{-1}(4;a,b,c,d)=\begin{cases}(3;a,a+c-d,b+c,d) & \text{if } d\le a+c\\ (4;a,d-a-c,b+c,a+c) & \text{if } a+c\le d.\end{cases}$$

$$\delta_2{}^{-1}(5;a,b,c,d)=\begin{cases}(5;a,a+b-c-d,c,d) & \text{if } c+d\le a+b\\ (6;a,c+d-a-b,a+b-d,d) & \text{if } d\le a+b\le c+d\\ (7;c,a,d-a-b,a+b) & \text{if } a+b\le d.\end{cases}$$

$$\delta_2{}^{-1}(6;a,b,c,d)=\begin{cases}(5;a+b,a-c-d,c,d) & \text{if } c+d\le a\\ (6;a+b,c+d-a,a-d,d) & \text{if } d\le a\le c+d\\ (7;c,a+b,d-a,a) & \text{if } a\le d.\end{cases}$$

$$\delta_2{}^{-1}(7;a,b,c,d)=\begin{cases}(3;a+b,b-d,c,d) & \text{if } d\le b\\ (4;a+b,d-b,c,b) & \text{if } b\le d.\end{cases}$$

Classification of diffeomorphisms

A diffeomorphism ϕ from Σ to itself is said to be **pseudo-Anosov** if there is a number $\lambda > 1$ and a pair of transverse measured foliations $\mathcal{F}$ and $\mathcal{F}'$ so that $\phi(\mathcal{F}) = \lambda\mathcal{F}$ and $\phi(\mathcal{F}') = 1/\lambda\mathcal{F}'$. A consequence of the correspondence between measured foliations and π_1-train tracks [2] is that there are properly weighted π_1-train tracks $(j; a, b, c, d)$ and $(j'; a', b', c', d')$ corresponding to $\mathcal{F}$ and $\mathcal{F}'$ so that $\phi(j; a, b, c, d) = (j; \lambda a, \lambda b, \lambda c, \lambda d)$ and $\phi(j'; a', b', c', d') = (j'; a'/\lambda, b'/\lambda, c'/\lambda, d'/\lambda)$ for the same value of λ. Moreover, the measured laminations $\mathcal{F}$ and $\mathcal{F}'$ are not multiple simple loops and therefore the coordinates $(j; a, b, c, d)$ and $(j'; a', b', c', d')$ are not rationally dependent. A consequence of piecewise linearity is that there is a closed subset U of Δ_j containing $(j; a, b, c, d)$ so that on U the diffeomorphism ϕ is given by a matrix A with eigenvector (a, b, c, d) with eigenvalue λ. Likewise there is $U' \subset \Delta_{j'}$ containing $(j'; a', b', c', d')$ so that ϕ' is given by a matrix A' on U' and A' has an eigenvector (a', b', c', d') with eigenvalue $1/\lambda$.

The basis of this section is the following classification of diffeomorphisms of Σ to itself (see [9], [10], [11] and [13]).

Theorem. *Let ϕ be a diffeomorphism of the twice punctured torus Σ to itself. Then ϕ is isotopic to a diffeomorphism ϕ' for which one one of the following holds:*

(i) ϕ' has finite order,

(ii) ϕ' is pseudo-Anosov,

(iii) ϕ' fixes a non-trivial, non-peripheral simple closed curve γ on Σ and is the identity or pseudo-Anosov on each component of the complement of a tubular neighbourhood of γ,

(iv) ϕ' fixes a multiple simple loop on Σ which consists of two disjoint non-trivial, non-peripheral simple closed curves.

We now investigate these possibilities a little more closely.

Proposition. *Any periodic orientation preserving automorphism of the twice punctured torus has order at most 12.*

Proof. This follows from the possible torsion in $\mathcal{MCG}(\Sigma)$ and may be deduced from the presentation of $\mathcal{MCG}(\Sigma)$ given in [12] Theorem 3.2.1. Alternatively, a diffeomorphism of finite order gives rise to an isometry of a suitable geometrical structure on Σ. The group of isometries of a twice punctured torus has order at most 12. □

Proposition. *Suppose that ϕ leaves simple closed curves γ_1 and γ_2 invariant. Then ϕ^2 is isotopic to a word in the Dehn twists about γ_1 and γ_2. In this case each of the curves γ_1 and γ_2 is both an attractive as well as a repulsive fixed point.*

Proof. The diffeomorphism ϕ either interchanges γ_1 and γ_2 or maps them to themselves. Thus ϕ^2 leaves γ_1 invariant and γ_2 invariant. Each component of the complement of a tubular neighbourhood of $\gamma_1 \cup \gamma_2$ is homotopic to a three holed sphere (pair of pants). The diffeomorphism ϕ^2 leaves invariant each boundary component of these three holed spheres. Thus ϕ^2 is homotopic to the identity on each component of the complement of a tubular neighbourhood of γ_1 and γ_2. The only possibility on these tubular neighbourhoods is that ϕ^2 is a power of a Dehn twist. The result follows. □

Proposition. *Let ϕ be reducible with one invariant simple closed curve γ. Suppose that ϕ is pseudo-Anosov on one component of the complement of a tubular neighbourhood of γ. Then:*

(i) ϕ has three fixed points in $\mathcal{PML}$: an attractive fixed point x_∞, a repulsive fixed point $x_{-\infty}$ and an indifferent fixed point x_0 (which corresponds to γ),

(ii) either x_∞ and $x_{-\infty}$ both lie the same maximal cell Δ_j or $x_\infty \in \Delta_j$, $x_{-\infty} \in \Delta_k$ and $x_0 \in \Delta_j \cap \Delta_k$,

(iii) ϕ acts linearly on the $\mathrm{sp}^+\{x_\infty, x_0\}$ *and* $\mathrm{sp}^+\{x_{-\infty}, x_0\}$.

Proof. By construction ϕ has three fixed points. One corresponding to the simple closed curve γ which we call x_0 and two corresponding to the fixed points of the pseudo-Anosov on the complement of a tubular neighbourhood of γ. One of these x_∞ is attractive and the other $x_{-\infty}$ is repulsive. In order to see that x_0 is indifferent, we observe that ϕ is either the identity or a power of a Dehn twist on a tubular neighbourhood of γ. In both cases any matrix A_0 representing ϕ near x_0 has x_0 as an eigenvector with eigenvalue 1. Any convergence to x_0 is dominated by the exponential convergence to x_∞. This gives (i).

By construction, the simple closed curve γ corresponding to x_0 is disjoint from both the laminations corresponding to x_∞ and $x_{-\infty}$. Thus x_0 and x_∞ lie in the same maximal cell and also x_0 and $x_{-\infty}$ lie in the same maximal cell. Thus either all three of them lie in the same maximal cell or else x_0 lies on the common boundary of two maximal cells, one containing x_∞ and the other containing $x_{-\infty}$. This gives (ii).

Finally, consider $\mathrm{sp}^+\{x_\infty, x_0\}$. This consists of all π_1-train tracks which are supported on x_∞ and x_0 with different relative weightings. The image of x_∞ and x_0 under any of the elementary Dehn twists ${\delta_j}^{\pm 1}$ are disjoint and so lie in the same maximal cell. By inspection we see that this means ${\delta_j}^{\pm 1}$ acts linearly on $\mathrm{sp}^+\{x_\infty, x_0\}$. Since ϕ is a word in the elementary Dehn twists, applying this argument repeatedly shows that ϕ also acts linearly on $\mathrm{sp}^+\{x_\infty, x_0\}$. Similarly for $\{x_{-\infty}, x_0\}$. □

Finally, analysis of all the other cases gives the following result.

Proposition. *The diffeomorphism ϕ is pseudo-Anosov if and only if it has exactly two fixed points neither of which is a multiple simple loop.*

The procedure

Let ϕ be a diffeomorphism given as a word in the elementary Dehn twists δ_0, δ_1, δ_2 and their inverses. By composing the piecewise linear actions of the elementary Dehn twists we can produce a piecewise linear action of ϕ on $\mathcal{ML}(\Sigma)$. In this section we give the procedure for determining whether or not ϕ is pseudo-Anosov.

Start with $x \in \mathcal{ML}(\Sigma)$ with integer coordinates. By definition x corresponds to a multiple simple loop and so cannot be a fixed point of a pseudo-Anosov diffeomorphism. Now iterate ϕ to obtain a sequence of points $x_n = \phi^n(x)$. The points x_n all lie in $\mathcal{ML}(\Sigma)$. We are only interested in their image under projectivisation. Therefore it can be useful to normalise these points in a suitable way.

If ϕ were periodic then after n (which is at most 12) iterations we will have $\phi^n(x) = x$. If $x_n = x$ for some $1 \leq n \leq 12$ then ϕ^n is not pseudo-Anosov and so ϕ is not pseudo-Anosov. If $x_n \neq x$ for all $n = 1, \ldots, 12$ then ϕ cannot be periodic and we see numerical convergence towards a fixed point. After a certain number of iterations either all the x_n lie in the same maximal cell or else there is a repeating cycle of $m \leq 5$ maximal cells. In the latter case replace ϕ with ϕ^m. In either case we find a sequence of points lying in the same maximal cell whose coordinates appear to converge numerically to a point x_∞. An approximation $\tilde{x}_\infty$ of x_∞ with finitely many decimals is called an **approximate fixed point**.

Our procedure relies on the following proposition:

Proposition. *Suppose that the approximate fixed point $\tilde{x}_\infty$ agrees with the true fixed point x_∞ up to N decimal places. We can choose N so that there is a small closed subset U containing $\tilde{x}_\infty$ and x_∞ with the property that ϕ acts linearly on U. This choice of N only depends on the length of the word ϕ in the generating Dehn twists.*

Proof. The diffeomorphism ϕ acts piecewise linearly on $\mathcal{ML}(\Sigma)$. There are finitely many closed subsets of $\mathcal{ML}(\Sigma)$ on which ϕ acts linearly. The number of these subsets is bounded by a function of the length of the word ϕ in the elementary Dehn twists. If we have convergence of the sequence x_n to a fixed point x_∞ then for all $n \geq N$ the points x_n will lie in a closed subset containing x_∞ on which ϕ acts linearly. By making our numerical convergence sufficiently accurate the point $\tilde{x}_\infty$ will lie in this subset. $\square$

Let $\tilde{x}_\infty \in \Delta_j$ be an approximate fixed point. There is a closed subset U of Δ_j on which ϕ acts linearly for which $\tilde{x}_\infty \in U$ and so that $\phi(U) \subset \Delta_j$.

We now work out the linear map A of ϕ on U. We can solve the eigenvalue problem for A. Observe that the characteristic polynomial of A has degree 4 and integral coefficients. We can use the standard formulae for solutions of quartic polynomials to obtain the eigenvalues of A in terms of radicals. From this it is clear whether or not these eigenvalues are rational or not. A (non-repeated) eigenvector corresponding to a rational eigenvalue must have rationally dependent entries and so corresponds to a multiple simple loop. This cannot be a fixed point of a pseudo-Anosov diffeomorphism. Similarly an eigenvector corresponding to an irrational eigenvalue must have rationally independent entries but, as we remarked earlier, this does not determine whether or not such an eigenvector corresponds to a multiple simple loop.

Thus we may assume that we have found a fixed point x_∞ of ϕ with an irrational eigenvalue λ. Otherwise we will have already determined that ϕ is not pseudo-Anosov.

Now apply the same procedure to ϕ^{-1} to obtain a fixed point $x_{-\infty}$. If $x_\infty = x_{-\infty}$ then ϕ is not pseudo-Anosov.

Thus we have ϕ having an attractive fixed point x_∞ and a repulsive fixed point $x_{-\infty}$. There are two possibilities. First, that ϕ is indeed pseudo-Anosov and secondly that ϕ fixes a simple closed curve γ and acts as a pseudo-Anosov on (one component of) the complement. In order to distinguish between them we need to decide whether or not ϕ has an indifferent fixed point x_0 corresponding to γ.

We know that, if x_0 exists, then it intersects neither x_∞ nor $x_{-\infty}$. Thus either all three lie in the same maximal cell or else x_∞ and $x_{-\infty}$ lie in neighbouring maximal cells and x_0 lies on their common intersection. This means that if the cells containing x_∞ and $x_{-\infty}$ are disjoint then there can be no point x_0 and so ϕ is pseudo-Anosov.

If the cells containing x_∞ and $x_{-\infty}$ have a non-empty intersection (that is they are the same or have a common lower dimensional face) then we need to investigate more closely. Suppose that ϕ has an additional fixed point x_0. We know that ϕ acts linearly on $\mathrm{sp}^+\{x_\infty, x_0\}$. This means that, if A is the matrix of ϕ near x_∞, then A will have an eigenvector corresponding to x_0. Therefore, if A only has one eigenvector with non-negative real entries this must correspond to x_∞ and so ϕ does not have an indifferent fixed point x_0. Thus ϕ is pseudo-Anosov. On the other hand, if A has a second eigenvector with non-negative entries this may be an additional fixed point. (It may be the case that, although this is fixed by A, this point is outside the set where ϕ is given by A.) In order to check whether this is the case we must check that this point is fixed by ϕ. If it is then ϕ is not pseudo-Anosov. If the only eigenvector of A with non-negative entries that is fixed by ϕ is x_∞ then ϕ is pseudo-Anosov.

In each case we have determined whether or not ϕ is pseudo-Anosov.

Example. In [7] Menzel showed that the Whitehead link complement fibres over the circle with fibre the twice punctured torus. Menzel gives the corresponding pseudo-Anosov diffeomorphism ϕ in terms of the presentation for $\mathcal{MCG}(\Sigma)$ given by Magnus in section 5 of [6]. Namely, in Magnus' notation $\phi = r^{-1}sr^{-1}\tau$ [7]. In section 3.2 of [12] Parker and Series show that one may write Magnus' generators in terms of δ_k as follows: One may pass from our presentation to Magnus' presentation via the substitution

$$r=\delta_1,\quad s=\delta_1\delta_0\delta_1,\quad \rho=\delta_1\delta_0\delta_1{\delta_2}^{-1}{\delta_0}^{-1}{\delta_1}^{-1},$$
$$\sigma={\delta_1}^{-1}{\delta_0}^{-1}{\delta_1}^{-1}{\delta_1}^{-1}{\delta_0}^{-1}{\delta_1}^{-1},\quad \tau=\delta_1{\delta_2}^{-1}.$$

Hence, we see that $\phi=\delta_0\delta_1{\delta_2}^{-1}$.

We now show how to use the above procedure to show that ϕ is indeed pseudo-Anosov. We did this using Mathematica routines written by Menzel [8]. Iterating ϕ using `iterate.m` we see convergence from the starting point $(1;1,1,1,1)$ to an approximate fixed point $\tilde{x}_\infty=(21;1,2.29663,5.83909,5.27451)$ (here we have chosen the first coordinate to be 1). Using `findmatrix.m` we find that near $\tilde{x}_\infty$ the diffeomorphism ϕ is given by the matrix

$$\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&1&1&1\\1&0&1&1\end{pmatrix}.$$

There is only one non-negative eigenvector of this matrix. This has eigenvalue

$$\frac{1}{2}+\frac{\sqrt{3}}{2}+\sqrt{\frac{\sqrt{3}}{2}}.$$

An eigenvector for this eigenvalue corresponds to the true fixed point x_∞. Up to scalar multiples, we have:

$$x_\infty=\left(21;\sqrt{2}(\sqrt{3}-1),3^{1/4}(\sqrt{3}-1)+\sqrt{2},3^{1/4}(\sqrt{3}+1)+\sqrt{6},2(3^{1/4}+\sqrt{2})\right).$$

Now consider the inverse of this diffeomorphism, namely $\delta_2{\delta_1}^{-1}{\delta_0}^{-1}$. Using `iterate.m` we see convergence to an approximate fixed point $\tilde{x}_{-\infty}=(22;1,0.267949,0.20665,1.08998)$. Using `findmatrix.m` we find that near $\tilde{x}_{-\infty}$ the diffeomorphism ϕ^{-1} is given by the matrix

$$\begin{pmatrix}1&0&1&1\\0&-1&-1&1\\0&1&1&0\\1&0&2&1\end{pmatrix}.$$

This has one non-negative eigenvector with eigenvalue

$$\frac{1}{2}+\frac{\sqrt{3}}{2}+\sqrt{\frac{\sqrt{3}}{2}}.$$

Finding an eigenvector for this eigenvalue gives the true fixed point

$$y_{-\infty}=\left(22;\sqrt{2}+3^{1/4}(\sqrt{3}+1),\sqrt{2}(2-\sqrt{3})+3^{1/4}(\sqrt{3}-1),\sqrt{2}(\sqrt{3}-1),2(3^{1/4}+\sqrt{2})\right).$$

We remark that the maximal cells Δ_{21} and Δ_{22} are disjoint. This guarantees that the diffeomorphism ϕ is pseudo-Anosov.

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