Mémoire de M2

Dynamics on the space of 2-lattices of $\mathbb{R}^3$ through the study of random walks

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Contents

Introduction

Homogeneous dynamics is the branch of mathematics that studies the actions of subgroups of a Lie group on its homogeneous spaces. This theory gained greater popularity and attention in the 80's due to the fact that it was succesfully applied to solve old open problems in number theory. As an example, let us mention the celebrated solution by G. Margulis of the Oppenheim conjecture.

The main aim of this "mémoire" is to present a small part of the results of the a recent article in homogeneous dynamics entitled *Dynamics on the space of 2-lattices of 3-space*, by O. Sargent and U. Shapira. Having as motivation an open question in *diophantine approximation*, which in loose terms is the branch of number theory that studies how well real irrational numbers can be approximated by rational numbers, the authors are led to study actions of sub(semi)groups Γ of $SL(3,\mathbb{R})$ in the homogeneous space X of homothety classes of 2-lattices of $\mathbb{R}^3$. A 2-lattice of $\mathbb{R}^3$ is just a discrete subgroup of $\mathbb{R}^3$ of rank 2. They focus in two cases: when the Zariski closure of Γ in $SL(3,\mathbb{R})$ is $SO(2,1)$ (which is the relevant case for the motivating problem), and when it is $SL(3,\mathbb{R})$. Our exposition of their results deals only with the second case. An interesting feature of this article (present also for example in [5] and [1]) is that dynamical information, namely about the closures of the Γ-orbits in X, is deduced from the study of *random walks* on X arising in the following way: we consider a Borel probability measure μ_Γ on $SL(3,\mathbb{R})$ whose support generates the semigroup Γ, and we choose independently a sequence of random elements $\gamma_1, \gamma_2, \cdots$ of $SL(3,\mathbb{R})$ with law μ, and a random point x_0 in X with some law ν. By applying succesively the γ_n's to x_0 we obtain a sequence of random points

$$x_0, x_1 = \gamma_1 x_0, x_2 = \gamma_2 x_1, \dots.$$

The distribution ν of the initial point x_0 is said to be *μ-stationary* if all the random points $x_1, x_2, \dots$ have also law ν. The main idea is that, by understanding the space of μ_Γ-stationary probability measures on X (which is done using tools from ergodic theory and probability theory), we gain insight into the dynamics of Γ on X, specially when there exists only one μ-stationary probability measure.

For the sake of completeness, let us present briefly the number theory problem that motivates [7]. A real number is *algebraic* if it is the root of a non-nul polynomial $p(x)$ with integer coefficients, and the degree of an algebraic number is the minimal degree of such a $p(x)$. For example, algebraic numbers of degree 1 are precisely rational numbers. Algebraic numbers of degree 2 and 3 are also commonly called quadratic and cubic numbers, respectively. Recall that any irrational number $\alpha > 1$ can be expressed in a unique way as a *simple continued fraction*, i.e. an expression of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where the $a_n' s$ are positive integers. The number α is said to be *well-approximable* if the sequence (a_n) is unbounded, otherwise it is *badly-approximable*. A classic theorem of Lagrange tells us that an irrational number is quadratic if and only if its continued fraction development is eventually periodic, in particular all of these numbers are badly-approximable. The question is to determine whether algebraic real numbers of degree 3 are well-approximable.

We close this introduction by describing the structure of this work. It has two chapters: The first chapter is an exposition of the background material needed in order to follow the proofs of the results from [7] that we present in Chapter 2. We begin Chapter 1 by introducing formally in Section 1.1 what is, for a topological group G, the random walk on a G-space Y associated to a Borel probability measure μ on G. The central concept here is that of μ-stationary probability measure on Y, which plays a similar role to that of invariant measures with respect to a continuous transformation $Y \to Y$ in basic ergodic theory. We show that there are always μ-stationary measures when the space Y is compact, and we present an example to show that this is no longer true for non-compact spaces. Then, we explain how any μ-stationary probability measure on Y can be "desintegrated" into a family of probability measures satisfying a certain equivariance property, the so-called *limit probability measures*. Converseley, an equivariant family of probability measures on Y determines a unique μ-stationary probability measure, so both objects are equivalent. In Section 1.2 we present the concept of *recurrence* of a random walk, and we give a general criterion that allows us to detect this property. After this general introduction, in the last three sections of the first chapter we specialize our discussion to *linear random walks*, that is, when the group G is the general linear group of a finite dimensional real vector space V, and Y is either V or the projective space $\mathbb{P}(V)$. As a preparation for this, in Section 1.3 we discuss some general properties of linear subsemigroups of $GL(V)$, such as irreducibility, strong irreducibility, and proximal dimension, and we explain the relation between these properties for a given linear semigroup and the corresponding ones of its Zariski closure. We also define the limit set of a linear semigroup. Afterwards, in Section 1.4 we prove a classic result of Furstenberg: if the action on V of semigroup generated by the support of μ is strongly irreducible and proximal, there is only one μ-stationary probability measure on $\mathbb{P}(V)$. Finally, in Section 1.5 we give some conditions on the probability measure μ that allow us to control the growth of vectors of the exterior powers of V under the random walk. This is the content of the Law of Large Numbers, which is inspired by the classical result about sums of independent real valued random variables sharing the same distribution. As oposed to the first four sections, in this last one we only quote some theorems that we will need in the second part of this work.

Chapter 2 is divided into three sections: In Section 2.1, we present two statements about the dynamics of a Zariski dense subsemigroup Γ of $SL(3, \mathbb{R})$ on the space X of homothecy classes of 2-lattices of $\mathbb{R}^3$ (Theorems 2.1 and 2.3), and explain how these can be deduced from the following two facts about random walks on X associated to a probability measure μ_Γ whose support generates Γ: there is a unique μ_Γ-stationary probability measure ν_X on X (Theorem 2.4), and the random walk is recurrent (2.5). Finally, in the last two sections we discuss the proofs of these random walk statements: Section 2.2 deals with we deal with Theorem 2.4. Since its proof is quite long and requires a great deal of technical machinery, we will limit ourselves to present the first part of it. First, we define probability measure ν_X, which will be obtained by integrating with respect to the unique μ_Γ-stationary measure on the grassmanian $\mathbb{P}^*(\mathbb{R}^3)$ of planes of $\mathbb{R}^3$ the uniform probability measures on the fibers of the natural projection $X \to \mathbb{P}^*(\mathbb{R}^3)$. Then, we show in Proposition 2.10 that the only μ_Γ-stationary measure whose limit measures are invariant with respect to a certain equivariant family of 1-parameter unipotent subgroups that we will introduce, is ν_X. It is worth remarking that the last part of the proof that we present of this fact is different from the one in the article: the new feature is the use of Lemma 2.11 to conclude. Finally, in Section 2.3 we construct a proper function $X \to \mathbb{R}$, that we show that satisfies the so-called *contraction hypothesis* with respect to the averaging operator of μ. In this way, thanks to the general criterion of Section 1.2 we are able to prove the recurrence of the random walk on X associated to μ_Γ (Theorem 2.5).

Chapter 1

Background material

1.1 Stationary measures

Let X be a topological space and let ν be a Borel measure on X. We will say that ν is **regular** if the measure of any Borel subset A of X can be approximated by the measures of open subsets containing A and closed subsets contained in A, which means that

$$\nu(A) = \inf\{\nu(U) \mid U \text{is open}, A \subseteq U\} = \sup\{\nu(C) \mid C \text{is closed}, C \subseteq A\}.$$

We define a **Radon measure** on X to be a regular Borel measure such that the compact subsets of X have finite measure. Let $\mathcal{M}(X)$ be the set of Radon measures on X.

The main advantage of restricting ourselves to work with Radon measures is that they can be thought as linear functionals in the following way. If ν is a Radon measure on X and $f : X \to \mathbb{R}$ is a continuous function with compact support K, then $\nu(f) < \infty$ because $|f|$ is bounded by a multiple of $\mathbb{1}_K$, which is integrable because $\nu(K)$ is finite. Thus, ν defines a linear functional on the space of continuous functions with compact support $X \to \mathbb{R}$, which we will denote by $\mathscr{C}_c(X)$. Notice that a functional on $\mathscr{C}_c(X)$ defined by a Radon measure assigns non-negative values to non-negative functions. A functional satifying this property is said to be a **positive functional**. Radon measures and positive linear functionals are essentially the same thing for a fairly large class of spaces X, as the next theorem shows.

Theorem 1.1 (Riez representation theorem)**.** *Let Y be a locally compact Hausdorff space, and let I be a positive linear functional on $\mathscr{C}_c(Y)$. There exists a unique Radon measure ν on Y such that*

$$I(f) = \int_Y f \, d\nu$$

for any $f \in \mathscr{C}_c(Y)$.

A proof can be found in [3, Theorem 7.2.8] . From now on we require X to be locally compact, second-countable and metrizable, so in particular it fullfils the hypotheses of the Riesz representation theorem. We will identify $\mathcal{M}(X)$ with the set of positive linear functionals on $\mathscr{C}_c(X)$, and we endow it with the weak-* topology, which by definition is the coarsest topology which makes continuous the evaluations $\ell \to \ell(f)$ for any $f \in \mathscr{C}_c(X)$. The metrizability of X ensures that any finite Borel measure on X is regular (see [6, Lemma 8.4]). We ask X to be second-countable to guarantee that $\mathscr{C}_c(X)$ is separable, as we now prove.

Lemma 1.2. *If X is locally compact, Hausdorff, and second-countable, then the normed vector space $(\mathscr{C}_c(X), || \cdot ||_\infty)$ is separable.*

Proof. Since $\mathscr{C}_c(X)$ is a metric space, it is separable if and only if it is second-countabile. Notice that the topology on $\mathscr{C}_c(X)$ induced by $||\cdot||_\infty$ is precisely the compact-open topology. Let $\mathcal{U}$ be a countable basis of open subsets X with compact closure, and let $\mathcal{V}$ be a countable basis of $\mathbb{R}$. Finite intersections of sets of the form

$$CO(\overline{U}, V) := \{f \in \mathscr{C}_c(X) \mid f(\overline{U}) \subseteq V]\},$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$ form a countable basis for $\mathscr{C}_c(X)$. □

Now we will consider a topological group G and a G-space X, both assumed to be locally compact, second countable and metrizable. Given $\mu \in \mathcal{P}(G)$ and $\nu \in \mathcal{P}(X)$ the **convolution** $\mu * \nu$ is the Borel probability measure on X obtained by pushing-forward the product measure $\mu \otimes \nu$ by the action $G \times X \to X$. Applying Fubini's theorem we see that we may write $\mu * \nu$ as the integral

$$\mu * \nu = \int_G g_* \nu d\mu(g).$$

Since G is a G-space itself (the left action is just the multiplication in G), we can consider convolutions of probability measures on G. The convolution of μ with itself n-times $\mu * \cdots * \mu$ will be denoted by μ^{*n}.

Lemma 1.3. *For any Borel probability measure μ on G, the convolution map $\mu * \cdot : \mathcal{P}(X) \to \mathcal{P}(X)$ is continuous.*

Proof. Consider a sequence (ν_n) in $\mathcal{P}(X)$ converging weakly to a probability measure ν_∞. Fix some $\varphi \in \mathscr{C}_c(X)$, and let $F_n : G \to \mathbb{R}$ be the map

$$F_n(g) = |\nu_\infty(\varphi \circ g) - \nu_n(\varphi \circ g)|$$

for any $n \geq 1$. Since $\varphi \circ g$ in a continuous map with compact support for any $g \in G$, by the weak convergence of (ν_n) we get that $F_n \to 0$ point-wise as $n \to \infty$. Notice also that

$$F_n(g) \leq |\nu_\infty(\varphi \circ g)| + |\nu_n(\varphi \circ g)| \leq \nu_\infty(|\varphi \circ g|) + \nu_n(|\varphi \circ g|) \leq 2||\varphi||_\infty.$$

By Dominated convergence, $\mu(F_n) \to 0$, and since

$$|\mu * \nu_\infty(\varphi) - \mu * \nu_n(\varphi)| = \left| \int_G g_*\nu_\infty(\varphi) - g_*\nu_n(\varphi) \right| \mathrm{d}\mu(g) \leq \int_G F_n \mathrm{d}\mu,$$

we conclude that $\mu * \nu_n(\varphi) \to \mu * \varphi_\infty(\varphi)$ as $n \to \infty$. The function φ is an arbitrary element of $\mathscr{C}_c(X)$, so $(\mu * \nu_n)_n$ converges weakly to $\mu * \nu_\infty$. □

We are interested in studying *random walks* on X arising in the following way: Let $(g_n)_{n \geq 1}$ be an independent sequence of G-valued random variables with law μ, and let x_0 be an X-valued random variable with law ν, independent from the g_n's. By applying succesively $g_1, g_2, \ldots$ to x_0 we obtain a *random orbit*

$$x_0, x_1 = g_1 x_0, \ldots, x_{n+1} = g_{n+1} x_n, \ldots.$$

The study of the random walk consists in describing the dynamical behaviour of a typical random orbit. For example, in the next section we will present a criterion that allows us to conclude that a typical random orbit does not escape to infinity, that is, there are compact subsets of X that the orbit visits infinitely many times.

Notice that the law of x_n is $\mu^{*n} * \nu$. The study of the random walk will be simpler when all the x_n have the same law. We can think of this situation as a sort of equilibrium state of the random walk. To formalize the previous discussion we make some definitions. Let $\mathscr{G}$ be the Borel σ-algebra of G. The

one-sided Bernoulli shift with alphabet $(G, \mathscr{G}, \mu)$ is $(B, \mathscr{B}, \beta, S)$, where $B = G^{\mathbb{N}}$, $\mathscr{B}$ is the product σ-algebra $\mathscr{G}^{\otimes \mathbb{N}}$, β is the product measure $\mu^{\otimes \mathbb{N}}$ and $S : B \to B$ is the continuous (with respect to the product topology) mapping

$$S((b_0, b_1, b_2, \dots)) = (b_1, b_2, b_3, \dots).$$

For any Borel subsets $A_1, \cdots, A_n$ of G, the finite rectangle $R = \{b_0 \in A_0, \cdots, b_n \in A_n\}$ and its preimage $S^{-1}(R)\{b_1 \in A_0, \cdots, b_{n+1} \in A_n\}$ have the same measure $\mu(A_0) \cdots \mu(A_n)$. Since finite rectangles generate $\mathscr{B}$, the shift S preserves β, that is $S_*\beta = \beta$. Moreover, β is an S-ergodic measure, which means that any Borel bounded function $f : B \to \mathbb{R}$ satisfying $f \circ S = F$ β-almost anywhere is constant β-almost surely (see [8, Theorem 1.2.1]).

A Borel probability measure ν on X is said to be **μ-stationary** if $\mu * \nu = \nu$. We start by addressing the question of existence of μ-stationary measures for a fixed $\mu \in \mathcal{P}(G)$. The following results are classical.

Lemma 1.4. *Let μ and ν be Borel probability measures on the topological group G and on the G-space X, respectively. Any Borel measure that is a cluster point of the sequence*

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} \mu^{*j} * \nu \tag{1.1}$$

is μ-stationary.

Proof. Suppose that ν_∞ is the limit of the subsequence (ν_{n_k}). Since $\mu * \underline{\ }$ is a continuous by Lemma 1.3, then, for any $\varphi \in \mathscr{C}_c(X)$ we have

$$|\mu * \nu_\infty(\varphi) - \nu_\infty(\varphi)| = \lim_{k \to \infty} |\mu * \nu_{n_k}(\varphi) - \nu_{n_k}(\varphi)| = \lim_{k \to \infty} \frac{1}{n_k} |\mu^{*n_k} * \nu_\infty(\varphi) - \nu_\infty(\varphi)| \leq \lim_{k \to \infty} \frac{2}{n_k} ||\varphi||_\infty = 0.$$

In other words, ν_∞ is μ-stationary. $\square$

Propositon 1.5. *Let X be a compact metrizable $G-$space, and let μ be a Borel probability measure on G. Then there exists at least one μ-stationary probability measure on X.*

Proof. Since the space X is compact, space of Borel probability measures $\mathcal{P}(X)$ is weakly compact. Consider any probability measure ν on X and the sequence (ν_n) defined in (1.1). Any cluster point of (ν_n) is a probability measure by the weak compactness of $\mathcal{P}(X)$, and it is μ-stationary by Lemma 1.4. $\square$

Now we give an alternate description of μ-stationary measures that will be very useful. Suppose that ν is μ-stationary. Any $b \in B$ gives rise to a sequence of probability measures

$$\nu, (b_0)_*\nu, (b_0 b_1)_*\nu, (b_0 b_1 b_2)_*\nu, \cdots.$$

The key observation is that this sequence converges towards a probability measure ν_b for β-almost any $b \in B$ We will call these ν_b the **limit measures**. Observe that if the limit measures are defined for some b and Sb, then

$$(b_0)_*\nu_{Sb} = \nu_b,$$

which is a kind of equivariance property. To be more precise, let E' be a Borel subset of B of full measure, all of whose elements have well-defined limit measures. The subset

$$E = \bigcap_{n \geq 0} S^{-n}(E')$$

still has full measure because S preserves β, and is forward-invariant under S. Thus, for any $b \in E$ and any positive integer n we have that

$$(b_0 \cdots b_{n-1})_* \nu_{S^n b} = \nu_b.$$

We will say then that the family of limit measures $(\nu_b)_{b \in E}$ is equivariant. We will now show the existence of the limit measures, and that these determine the μ-stationary measure.

Propositon 1.6. *Let ν be a μ-stationary Borel probability measure on X. For β-almost any $b \in B$, the sequence $(b_0 \cdots b_n)_* \nu$ converges towards a probability measure ν_b. Moreover, we can recover ν by integrating the ν_b's:*

$$\nu = \int_B \nu_b \, d\beta(b). \tag{1.2}$$

Proof. We prove the existence of the ν_b's using the Martingale Convergence Theorem. Fix some $\varphi \in \mathscr{C}_c(X)$ and for each $n \in \mathbb{N}$ define $F_n : B \to \mathbb{R}$ by

$$F_n(b) = (b_0 \cdots b_n)_* \nu(\varphi).$$

Notice that F_n is continuous since multiplication in G and $g \mapsto \varphi \circ g$ are continuous. Also, F_n depends only on the coordinates $b_0, \cdots, b_n$, thus it is measurable with respect to the σ-algebra generated by $b_0, \cdots, b_n$, which we will denote by $\mathscr{B}_n$. The fact that ν is μ-stationary implies that $(F_\bullet, \mathscr{B}_\bullet)$ is a martingale. Indeed, let $A_0, \cdots, A_{n-1}$ be measurable subsets of G. We have

$$\begin{aligned}
\int_{\{b_0 \in A_0, \ldots, b_{n-1} \in A_{n-1}\}} F_n(b) \mathrm{d}\beta(b) &= \int_{A_0} \cdots \int_{A_{n-1}} \int_G (b_0 \cdots b_n)_* \nu(\varphi) \mathrm{d}\mu(b_n) \cdots \mathrm{d}\mu(b_0) \\
&= \int_{A_0} \cdots \int_{A_{n-1}} \int_G (b_n)_* \nu(\varphi \circ b_0 \cdots b_{n-1}) \mathrm{d}\mu(b_n) \cdots \mathrm{d}\mu(b_0) \\
&= \int_{A_0} \cdots \int_{A_{n-1}} \nu(\varphi \circ b_0 \cdots b_{n-1}) \mathrm{d}\mu(b_{n-1}) \cdots \mathrm{d}\mu(b_0) \\
&= \int_{\{b_0 \in A_0, \ldots, b_{n-1} \in A_{n-1}\}} F_{n-1}(b) \mathrm{d}\beta(b),
\end{aligned}$$

which means that $\mathbb{E}(F_n \mid \mathscr{B}_{n-1}) = F_{n-1}$. Furthermore, the sequence (F_n) is uniformly bounded since

$$|F_n(b)| = |\nu(\varphi \circ b_0 \cdots b_n)| \leq \nu(|\varphi \circ b_0 \cdots b_n|) \leq \nu(||\varphi||_\infty) = ||\varphi||_\infty, \tag{1.3}$$

so in particular $(F_\bullet, \mathscr{B}_\bullet)$ is an L^1-bounded martingale. By Doob's theorem, there is a full-measure subset $E_\varphi \subseteq B$ such that for each $b \in E_\varphi$, $F_n(b)$ converges towards some number $\nu_b(\varphi)$. Additionally, $b \mapsto \nu_b(\varphi)$ is measurable with respect to $\sigma(\cup_{n \geq 0} \mathscr{B}_n) = \mathscr{B}$ and integrable. By (1.3) we may apply Lebesgue's dominated convergence theorem to deduce that

$$\int_B \nu_b(\varphi) \mathrm{d}\beta(b) = \lim_{n \to \infty} \int_B F_n(b) \mathrm{d}\beta(b) = \mu^{*(n+1)} * \nu(\varphi) = \nu(\varphi). \tag{1.4}$$

Let $D = \{\varphi_0, \varphi_1, \ldots\}$ be a countable dense subset of $\mathscr{C}_c(X)$ and let $E = \cap_{n \geq 0} E_{\varphi_n}$. For any fixed $b \in E$, and for any $n, m \in \mathbb{N}$ we have

$$|\nu_b(\varphi_n) - \nu_b(\varphi_m)| = \lim_{n \to \infty} |(b_0 \cdots b_n)_* \nu(\varphi_n - \varphi_m)| \leq ||\varphi_n - \varphi_m||_\infty,$$

so $\nu_b : D \to \mathbb{R}$ is 1-Lipschitz, in particular continuous. Its unique continuous extension to $\mathscr{C}_c(X)$, is easily seen to be a positive linear functional, so it determines a Radon measure ν_b. The equality (1.2) is established essentially by (1.4).

Notice that at this point we just know that $0 \leq \nu_b(X) \leq 1$, so in principle some of the ν_b's may not be probability measures. If X where compact, then $\mathbb{1}_X \in \mathscr{C}_c(X)$, so it would be immediate that the $\nu_b(X) = 1$. Even if X is not compact, the same conclusion holds for β-almost any $b \in E$ as is seen by applying (1.2) to $\mathbb{1}_X$. This argument needs to be improved

□

Moreover, any equivariant family of probability measures determines a μ-stationary probability of X, hence both objects are essentially the same.

Propositon 1.7. *Let $b \mapsto \widetilde{\nu}_b$ be an equivariant measurable map $B \to \mathcal{P}(X)$ defined $\beta-$almost everywhere. Then, the probability measure*

$$\nu = \int_B \widetilde{\nu}_b \, d\beta(b)$$

is μ-stationary. Moreover, the limit measures ν_b of ν are equal to $\widetilde{\nu}_b$ for $\beta-$almost every b in B.

Proof. The fact that ν is $\mu-$stationary follows from the equality $\beta = \mu \otimes \beta$ and the equivariance of the $\widetilde{\nu}_b$'s as the following calculation shows:

$$\begin{aligned}\mu * \nu &= \int_G g_* \nu \mathrm{d}\mu(g) = \int_G \int_B g_* \widetilde{\nu}_b \mathrm{d}\beta(b) \mathrm{d}\mu(g) = \int_G \int_B \widetilde{\nu}_{(g,b)} \mathrm{d}\beta(b) \mathrm{d}\mu(g) \\ &= \int_{G \times B} \widetilde{\nu}_{(g,b)} \mathrm{d}\beta \otimes \mu(g,b) = \int_B \widetilde{\nu}_b \mathrm{d}\beta(b) = \nu.\end{aligned}$$

To show that $\nu_b = \widetilde{\nu}_b$ for $\beta-$almost every $b \in B$, it suffices to prove that both maps have the same integral in any measurable subset of B. Better still, it is enough to check this on finite rectangles $\{b_0 \in A_0, \ldots, b_n \in A_n\}$ for $A_0, \ldots, A_n$ arbitrary measurable subsets of G, since these generate the $\sigma-$algebra of B. We start by developing the integral for the $\widetilde{\nu}_b$'s:

$$\begin{aligned}\int_{\{b_0 \in A_0, \ldots, b_n \in A_n\}} \widetilde{\nu}_b \mathrm{d}\beta(b) &= \int_{A_0} \cdots \int_{A_n} \int_B \widetilde{\nu}_{(b_0, \ldots, b_n, b')} \mathrm{d}\beta(b') \mathrm{d}\mu(b_n) \ldots \mathrm{d}\mu(b_0) \\ &= \int_{A_0} \cdots \int_{A_n} (b_0 \cdots b_n)_* \int_B \widetilde{\nu}_{b'} \mathrm{d}\beta(b') \mathrm{d}\mu(b_n) \ldots \mathrm{d}\mu(b_0) \\ &= \int_{A_0} \cdots \int_{A_n} (b_0 \cdots b_n)_* \nu \mathrm{d}\mu(b_n) \ldots \mathrm{d}\mu(b_0).\end{aligned}$$

Observe the only thing that we used in the preceding calculation was the equivariance of the $\widetilde{\nu}_b$'s. Since the limit measures are also equivariant, we readily obtain that

$$\int_{\{b_0 \in A_0, \ldots, b_n \in A_n\}} \widetilde{\nu}_b \mathrm{d}\beta(b) = \int_{A_0} \cdots \int_{A_n} (b_0 \cdots b_n)_* \nu \mathrm{d}\mu(b_n) \ldots \mathrm{d}\mu(b_0) = \int_{\{b_0 \in A_0, \ldots, b_n \in A_n\}} \nu_b \mathrm{d}\beta(b),$$

which concludes the proof. □

When the space X is not compact, it can happen that there are no μ-stationary measures on X. We present an example of Furstenberg (see [4, Theorem 1.1])

Propositon 1.8. *Let G be a locally compact, metrizable, and second-countable topological group, and let μ be a Borel probability measure on G. Consider the action of G on itself by left-multiplication. There exists a $\mu-$stationary probability measure ν on G if and only if the support of μ is contained in a compact subgoup of G.*

Proof. Let us denote by A the support of μ. Suppose first that A is contained in a compact subgroup H of G. Denote by $\overline{\mu}$ the restriction of μ to H. By Proposition 1.5, there exists a $\overline{\mu}$ stationary probability measure on H. By extending it to G we obtain a $\mu-$stationary probability measure.

Now suppose that ν is a $\mu-$stationary. The main point of the proof consists in finding a compact subset K of G containing any product of elements of A. Once we have this, it follows that the closed semigroup generated by A is compact, but a compact semigroup contained in a group is a group, so the conclusion follows.

The hypotheses on G imply that we may write it as a countable union of open subsets with compact closure, so we may choose one of these, say U, such that $\mu(U)$ and $\nu(U)$ are greater than $\frac{1}{2}$. From

$$\int_B \nu_b(U)\mathrm{d}\beta(b) = \nu(U) > \frac{1}{2},$$

it follows that the set E_U consisting of those $b \in B$ for which $\nu_b(U)$ is grater that $\frac{1}{2}$ has positive $\beta-$measure. From the characterization of weak convergenge in terms of the measure of open subsets REF we get that, for any $b \in E_U$, there exists a positive integer N_b such that

$$\nu((b_0 \cdots b_n)^{-1}U) > \frac{1}{2}$$

for any $n \geq N_b$, in particular $(b_0 \cdots b_n)^{-1}U$ and U have nonempty intersection. This means that $(b_0 \cdots b_n)^{-1}$ belongs to UU^{-1}, so $b_0 \cdots b_n$ is in the compact set $L = \overline{UU}^{-1}$. Let K be the compact set $L^{-1}L$

To show that $g_1 \cdots g_m$ is in K for $g_1 \dots g_m \in A$, it is enough to prove that we can find an $m-$tuple $(h_1, \dots, h_m)$ arbitrarily close to $(g_1, \cdots, g_m)$, such that $h_1 \cdots h_m$ is in K. Let V_j be an open subset of G to which g_j belongs, and denote by V the product $V_1 \times \cdots \times V_m$. We call E the subset of B consisting of those $b \in B$ such that $(b_{k+1}, \dots, b_{k+m})$ is in V for infinitely many k's. We claim that $\beta(E) = 1$. Write B as $G^m \times G^m \times \cdots$, and let $\pi_j : B \to G^m$ be the projection to the $j-$th factor G^m. If b is not in E, in particular $\pi_j(b)$ is not in V for j large enough, and so

$$b \in \bigcup_{s\geq 0} \left(\bigcap_{j \geq s} \pi_j^{-1}(G^m \setminus V) \right), \tag{1.5}$$

and it is easy to see that each of the $\cap_{j\geq s}\pi_j^{-1}(G^m \setminus V)$ is $\beta-$null. Indeed, for any positive integer ℓ we have the inequality

$$\beta\left(\bigcap_{j\geq s} \pi_j^{-1}(G^m \setminus V) \right) \leq \beta\left(\bigcap_{j=s}^{s+\ell} \pi_j^{-1}(G^m \setminus V) \right) = \mu^{\otimes m}(G^m \setminus V)^{\ell+1},$$

and the conclusion follows as $\mu^{\otimes}(G^m \setminus V)$ is strictly less that 1, since each $\mu(V_j)$ is positive because g_j is in the support of μ. We now know that E and E_U have non-empty intersection.

We are ready to finish the proof. Consider any b in $E_U \cap E$, and let $k \geq N_b$ be such that $(b_{k+1}, \dots b_{k+m})$ belongs to V. Both $b_0 \cdots b_k$ and $b_0 \cdots b_{k+m}$ are in L, so $b_{k+1} \cdots b_{k+m}$ is in $L^{-1}L = K$. □

If μ is a Borel probability measure on the topological group G, we will denote by Γ_μ the closed semigroup generated by the support of μ.

Lemma 1.9. *Let G be a topological group and let X be a topological space, both locally compact, metrizable and second-countable. Consider a Borel probability measure μ on G and a μ-stationary Borel probability measure on X. The support of ν is Γ_μ-invariant.*

Proof. First we observe that

$$\operatorname{supp}(\mu \otimes \nu) = \operatorname{supp}\mu \times \operatorname{supp}\nu, \tag{1.6}$$

because (g, x) is in $\operatorname{supp}(\mu \otimes \nu)$ iff for any open subsets $g \in U \subseteq G$ and $x \in V \subseteq X$, $\mu \otimes \nu(U \times V) = \mu(U)\nu(V)$ is positive, which happens iff g is in $\operatorname{supp}\mu$ and x is in $\operatorname{supp}\nu$.

Let F be the action $G \times X \to X$. By definition of the convolution, $\mu * \nu = F_*(\mu \otimes \nu)$, so it readily follows that

$$\operatorname{supp}(\mu \otimes \nu) \subseteq F^{-1}(\operatorname{supp}\mu * \nu). \tag{1.7}$$

Notice that up to this point we have not used that ν is μ-stationary.

When $\mu * \nu = \nu$, from (1.6) and (1.7) we get that

$$F(\operatorname{supp}\mu \times \operatorname{supp}\nu) \subseteq \operatorname{supp}\nu,$$

or in other words, $\operatorname{supp}\nu$ is $\operatorname{supp}\mu$-invariant. This implies that $\operatorname{supp}\nu$ is Γ_μ-invariant. □

Propositon 1.10. *Let G be a topological group acting continuously on a locally compact, metrizable and second-countable space X. Suppose that μ is a Borel probability measure on G and that ν is a $\mu-$stationary probability measure on X. Then β-almost any $b \in B$ such that the limit measure ν_b exists verifies that*

$$\lim_{n\to\infty} (b_0 \cdots b_n g)_* \nu = \nu_b, \tag{1.8}$$

for any $g \in \Gamma_\mu$. This statement needs to be changed, and the proof completed

Proof. We begin by showing that (1.8) holds for $\beta \otimes \mu-$almost any $(b, g) \in B \times G$. Let us fix a function $\varphi \in \mathscr{C}_c(X)$ and define $F_n : B \times G \to \mathbb{R}$ by

$$F_n(b, g) = \left((b_0 \cdots b_n)_*\nu(\varphi) - (b_0 \cdots b_n g)_*\nu(\varphi)\right)^2.$$

To show that $F_n \to 0$ almost surely it suffices to prove that $\sum_{n\geq 0} \beta \otimes \mu(F_n)$ is finite, because then the series $\sum_{n\geq 0} F_n$ is finite almost surely because it is integrable, which in particular implies that $F_n \to 0$ almost surely.

As we will see know, the F_n's can be expressed in terms of the numbers

$$I_j = (\int_G h_*\nu(\varphi))^2 \mathrm{d}\mu^{*(j)}(h)$$

Developing the square in the formula of F_n we get three terms whose integral we will calculate separately to avoid handling with big formulas. All we need is Fubini's theorem, to remember that ν is $\mu-$stationary, and that μ^{*j} is the push-forward of the product measure $\mu^{\otimes j}$ under the multiplication map $G^j \to G$.

$$\int_{B\times G} \left((b_0 \cdots b_n)_*\nu(\varphi)\right)^2 \mathrm{d}\beta \otimes \mu(b, g) = \int_B \left((b_0 \cdots b_n)_*\nu(\varphi)\right)^2 \mathrm{d}\beta(b) = \int_G (h_*\nu(\varphi))^2 \mathrm{d}\mu^{*(n+1)}(h) = I_{n+1}$$

$$\begin{aligned}\int_{B\times G} ((b_0 \cdots b_n g)_*\nu(\varphi))^2 \mathrm{d}\beta \otimes \mu(b, g) &= \int_G \int_B ((b_0 \cdots b_n g)_*\nu(\varphi))^2 \mathrm{d}\beta(b)\mathrm{d}\mu(g) \\ &= \int_G \int_G ((hg)_*\nu(\varphi))^2 \mathrm{d}\mu^{*(n+1)}(h)\mathrm{d}\mu(g) = I_{n+2}\end{aligned}$$

$$\begin{aligned}\int_{B\times G} (b_0 \cdots b_n)_*\nu(\varphi)(b_0 \cdots b_n g)_*\nu(\varphi)\mathrm{d}\beta \otimes \mu(b, g) &= \int_B (b_0 \cdots b_n)_*\nu(\varphi) \left(\int_G g_*\nu(\varphi \circ b_0 \cdots b_n)\mathrm{d}\mu(g)\right) \mathrm{d}\beta(b) \\ &= \int_B (b_0 \cdots b_n)_*\nu(\varphi)\nu(\varphi \circ b_0 \cdots b_n)\mathrm{d}\beta(b) = I_{n+1}\end{aligned}$$

Combining refe we conclude that

$$\int_{B\times G} F_n(b,g)\mathrm{d}\beta\otimes\mu(b,g) = I_{n+2} - I_{n+1},$$

so

$$\sum_{n_0}^{N}\int_{B\times G} F_n\mathrm{d}\beta\otimes\mu(b,g) = I_{N+2} - I_1 \le |I_{N+2}| + |I_1| \le 2||\varphi||_\infty$$

□

1.2 Recurrence

In this section our standing assumptions are the following: G is a topological group, and X G-space, both assumed to be locally compact, second-countable, and metrizable. Let μ be a Borel probability measure on G. We will say that the action of G on X is μ-recurrent if the following condition is satisfied:

R(μ, X): For any x in X and any $\varepsilon > 0$, there exists a compact subset $K = K(x,\varepsilon)$ of X such that

$$\mu^{*n} * \delta_x(K) \ge 1 - \varepsilon \tag{1.9}$$

for any natural number n.

Recall that a family $\mathcal{A}$ of Borel probability measures on X is **weakly compact** if any sequence with values in $\mathcal{A}$ has a subsequence that converges weakly in $\mathcal{P}(X)$.

Propositon 1.11. *Let μ be a Borel probability measure on X. For any x in X we consider the sequence*

$$\nu_{x,n} = \frac{1}{n}\sum_{j=1}^{n}\mu^{*j} * \delta_x.$$

If $R(\mu^{*n_0}, X)$ holds for some positive integer n_0, then the family $\{\nu_{x,n}\}_{n\ge 1}$ of Borel probability measures on X is weakly compact for any x in X.

Proof. Suppose first that $\mathrm{R}(\mu, X)$ holds. We fix some $x \in X$. We will write simply ν_n instead of $\nu_{x,n}$ to lighten the notation. Let $\widehat{X} = X \cup \{\infty\}$ be the one-point compactification of X. The space $\mathcal{P}(\widehat{X})$ is weakly-compact. Consider a cluster point $\nu_\infty \in \mathcal{P}(\widehat{X})$ of $\{\nu_n\}_{n\ge1}$. For any $\varepsilon > 0$, the compact subset $K = K(x,\varepsilon)$ of X given by $\mathrm{R}(\mu, X)$ verifies

$$\nu_n(K) \ge 1 - \varepsilon,$$

for any $n > 0$. Let $0 \le \varphi_K \le 1$ be a non-negative function in $\mathscr{C}_c(X)$ that is constant equal to 1 on K. Then

$$1 - \varepsilon \le \nu_n(K) \le \nu_n(\varphi_K) \to \nu_\infty(\varphi_K) \le \nu_\infty(\mathbb{1}_X) = \nu_\infty(X).$$

This implies that $\nu_\infty(X) = 1$, so (ν_n) converges to ν_∞ in $\mathcal{P}(X)$.

Suppose now that $\mathrm{R}(\mu^{*n_0})$ holds for some $n_0 > 0$. We define

$$\widetilde{\nu}_n = \sum_{j=1}^{n}\mu^{*jn_0} * \delta_x.$$

By the previous paragraph, the family $\{\widetilde{\nu}_n\}_{n>0}$ is weakly compact. Let m be a positive integer and express it as $kn_0 + r$, with $0 \leq r < n_0$. Then

$$\begin{aligned}\nu_m = \frac{1}{m}\sum_{j=1}^{m}\mu^{*j} * \delta_x &= \frac{1}{m}\left(\sum_{i=1}^{n_0-1}\mu^{*i} * \delta_x - \sum_{l=m+1}^{(k+1)n_0-1}\mu^{*l} * \delta_x + \sum_{s=0}^{n_0-1}\sum_{j=1}^{k}\mu^{*(jn_0+s)} * \delta_x\right)\\ &= \sum_{s=0}^{n_0-1}\frac{k}{m}\mu^{*s} * \widetilde{\nu}_k + \frac{1}{m}\rho_m =, \end{aligned}\tag{1.10}$$

where ρ_m is the sum of last residual terms of ν_m. Since ρ_m is sum of at most $2n_0$ probability measures, then $\rho_m/m \to 0$ as $m \to \infty$. Also notice that

$$\frac{k}{(k+1)n_0} \leq \frac{k}{m} \leq \frac{k}{(k-1)n_0},$$

so $k/m \to 1/n_0$ as $n \to \infty$.

Consider now any subsequence (ν_{n_j}) of (ν_n). By taking a further subsequence if necessary, we may suppose that all the n_j's have the same residue r modulo n_0, so that $n_j = q_j n_0 + r$. We may suppose, again extracting a subsequence if necessary, that the sequence $(\widetilde{\nu}_{q_j})_j$ converges to a probability measure ν_∞. Then, by (1.10),

$$\nu_{n_j} \to \frac{1}{n_0}\sum_{s=0}^{n_0-1}\mu^{*s} * \nu_\infty.$$

This shows that $\{\nu_n\}_{n\geq 0}$ is weakly compact. □

Any Borel probability measure μ on G induces an averaging operator on the space of measurable functions $X \to [0,\infty]$, that we denote by P_μ. It is given by the formula

$$P_\mu f(x) = \int_G f(gx)\mathrm{d}\mu(g),$$

for any $x \in X$ and any Borel function $f : X \to [0,\infty]$. Such an f is said to satisfy the **contraction hypothesis for** μ if there exists constants $a \in (0,1)$ and $b \geq 0$ such that

$$P_\mu f \leq af + b. \tag{1.11}$$

By induction we see that if f satisfies the contraction hypothesis for μ, then

$$(P_\mu^n f)(x) \leq a^n f + (1 + a + \cdots + a^{n-1})b, \tag{1.12}$$

for any positive integer n. As we will see now, the existence of a proper function satisfying the contraction hypothesis for μ implies property $\mathrm{R}(\mu, X)$. Recall that a measurable function $f : X \to [0,\infty]$ is **proper** if $\{f \leq T\}$ is relatively compact in X for any $T < \infty$.

Lemma 1.12. *Suppose that a proper Borel function $f : X \to [0,\infty]$ satisfies the contraction hypothesis for μ. Then, for any $x_0 \in X$ such that $f(x_0)$ is finite, and for any $\varepsilon > 0$, there exists a compact subset $K = K(x_0,\varepsilon)$ of X such that*

$$\mu^{*n} * \delta_{x_0}(K) \geq 1 - \varepsilon, \tag{1.13}$$

for any integer $n \geq 0$.

Proof. Suppose that $f(x_0)$ is finite, and let ε be a positive number. Consider the compact set

$$K = \overline{\left\{ f \leq \frac{2b}{\varepsilon(1-a)} \right\}}.$$

Any point x in X that is not in K in particular satisfies that $f(x) > \frac{2b}{\varepsilon(1-a)}$, so

$$\mathbb{1}_{X \setminus K} \leq \frac{\varepsilon(1-a)}{2b} f.$$

We apply P_μ^n to the previous inequality, evaluate at x_0, and then use (1.12) to deduce that

$$\mu^{*n} * \delta_{x_0}(X \setminus K) \leq \frac{\varepsilon(1-a)a^n f(x_0)}{2b} + \frac{\varepsilon}{2}.$$

Since $a^n \to 0$, the term $\frac{\varepsilon(1-a)a^n f(x_0)}{2b}$ is smaller than $\varepsilon/2$ for n greater than a certain n_0, so (1.13) holds. By enlarging K if necessary we can ensure that (1.13) holds also for $0 \leq n < n_0$. □

It will be convenient to state separately the following direct consequence of Proposition 1.11 and Lemma 1.12

Corollary 1.13. *Let G be a topological group and let X be a G-space, both assumed to be locally compact, second countable, and metrizable. Suppose that μ is a Borel probability measure on G and that there is a Borel proper function $f : X \to [0, \infty)$ satisfying the contraction hypothesis for some power μ^{*n_0} of μ. Then, for any $x \in X$, the sequence of Borel probability measures*

$$\nu_{x,n} = \frac{1}{n} \sum_{j=1}^{n} \mu^{*j} * \delta_x$$

on X is weakly compact.

Proof. Since the proper function f that satisfies the contraction hypothesis for μ^{*n_0} takes only finite values, property $\mathrm{R}(\mu^{*n_0}, X)$ holds by Lemma 1.12. Then, by Proposition 1.11, the family $\{\nu_{x,n}\}_{n \geq 1}$ is weakly compact. □

1.3 Linear semigroups

In this work, the term **linear semigroup** will refer to a subsemigroup of $GL(V)$ for some finite-dimensional vector space V. A linear semigroup $\Gamma \subset GL(V)$ is **irreducible** if the only Γ-stable linear subspaces of V are $\{0\}$ and V itself. An equivalent way to state this is that Γv spans V for any non-zero vector $v \in V$. When Γ verifies the stronger property that no finite union of proper subspaces if Γ-stable, we say that Γ is **strongly irreducible**.

Let us illustrate this concepts with a simple example in the plane $\mathbb{R}^2$, that we identify with $\mathbb{C}$. We think $\mathbb{C}^*$ as subset of $GL(\mathbb{R}^2)$ via the multiplication of complex numbers. Define $\omega_\theta = e^{2\pi\theta i}$, and let Γ_θ be the semigroup generated by ω_θ. We distinguish three cases:

- If θ is irrational and W is any 1-dimensional subspace of $\mathbb{R}^2$, the lines $W, \omega_\theta W, \omega_\theta^2 W, \ldots$ are pairwise distinct. Thus Γ_θ is strongly irreducible in this case.
- Suppose now that θ is rational, say $\frac{p}{q}$ for some relatively prime integers p and q with $q > 2$. On the one hand, Γ_θ is irreducible because v and $\omega_\theta v$ are not collinear for any non-zero $v \in \mathbb{R}^2$. On the other hand, Γ_θ is not strongly irreducible since the union of the lines $\mathbb{R}\omega_{2\pi k/q}$, for $0 \leq k < q$ integer, is Γ_θ-stable.

- Finally, if θ is an integer multiple of $\frac{1}{2}$, any line through the origin is Γ_θ-stable. Hence, Γ_θ is not irreducible.

Suppose now that $(V, ||\cdot||)$ is a finite-dimensional normed real vector space. We endow the ring of endomorphisms $End(V)$ with the operator norm induced by $||\cdot||$, which we denote also by $||\cdot||$. The **proximal dimension** of a linear semigroup $\Gamma \subseteq GL(V)$ is the minimal positive integer r for which there exists a sequence $\gamma_n \in \Gamma$, and real numbers λ_n such that $\lambda_n\gamma_n$ converges towards a rank-r linear endomorphism of V. When $r = 1$ we say that Γ is a **proximal semigroup**. As we shall see later, the linear semigroups that are strongly irreducible and proximal will be important when we discuss *linear random walks*. The **s-limit set** of Γ is the set of the $s-$dimensional subspaces of V that are image of an element of $\overline{\mathbb{R}\Gamma}$, and it is denoted by Λ^s_Γ.

Propositon 1.14. *Let $\Gamma \subseteq GL(V)$ be an irreducible subsemigroup with proximal dimension r.*

(i) The limit set Λ^r_Γ is Γ-invariant and minimal.

(ii) If Γ is proximal, the limit set Λ^1_Γ is the only minimal Γ-invariant closed subset of $\mathbb{P}(V)$.

Proof. Consider any two elements W and W' of Λ^r_Γ that are respectively image of the endomorphisms h and h' belonging to $\overline{\mathbb{R}\Gamma}$. In order to show that Λ^r_Γ is Γ-minimal it suffices to prove that we can move W' arbitrarily close to W by applying elements of Γ. We claim that exists $\gamma \in \Gamma \cup \{e\}$ such that $h\gamma h'$ is non-zero since Γ is irreducible. Variations of this simple fact will appear several times in this section, so we detail the proof this first time: consider any non-null $w' \in W'$, and choose $\gamma = e$ if $h(w') \neq 0$. Otherwise, w' belongs to the proper subspace $\ker h$, and by the irreducibility of Γ there exists some $\gamma \in \Gamma$ such that $\gamma w'$ leaves $\ker h$. Notice that $h\gamma h'$ has rank at most r and belongs to $\overline{\mathbb{R}\Gamma}$, so by the minimality of the proximal dimension, it has rank exactly r. If $h = \lim_{n\to\infty} \lambda_n\gamma_n$, then $W = \lim_{n\to\infty} \gamma_n W'$, which proves (i).

Now lets prove (ii). We must show that Λ^1_Γ is contained in the closure of any $\Gamma-$orbit in $\mathbb{P}(V)$. Suppose that $r = 1$, and let W, h, and γ_n be as in the preceding paragraph. Let v be a non-zero vector in V, and denote by x its corresponding point in $\mathbb{P}(V)$. By the irreducibility of Γ, there is some $\gamma \in \Gamma \cup \{e\}$ verifying $h\gamma v \neq 0$. Then $\lim_{n\to\infty} \gamma_n\gamma x = W$. Since W is an arbitrary element of the limit set, we conclude that Λ^1_Γ is contained in $\overline{\Gamma x}$. □

Lemma 1.15. *Let be a finite dimensional vector space over an infinite field. Any Zariski-dense subsemigroup of $GL(V)$ or $SL(V)$ is strongly irreducible.*

For the proof we need auxiliary observations that we gather in a separate lemma.

Lemma 1.16. *Let V be a finite dimensional vector space over an infinite field F.*

(i) V cannot be written as a finite union of proper subspaces.

(ii) If Z is a finite union of proper subspaces of V and $g(Z)$ is contained in Z for some $g \in GL(V)$, then $g(Z) = Z$.

Proof. We prove (i) by induction on the dimension d of V. It is enough to prove the result when Z is a finite union of codimension 1 subspaces of V. The case $d = 1$ is immediate, since V has no proper subspaces. Suppose that the result holds for a certain $n \geq 1$ and consider some V with $d = n + 1$. There are infinitely many hyperplanes in V because they are in bijective correspondance with $\mathbb{P}(V^*)$, and the map $x \mapsto [x : 1 : \cdots : 1]$ from F to $\mathbb{P}(F^d)$ is injective. Write Z as $\cup_{i=1}^r W_i$, where each W_i is a hyperplane of V. Consider a hyperplane H distinct from all the W_i's. Since $H \cap W_i$ is a proper subspace of H, from the induction hypothesis we know that H is not covered by the $H \cap W_i$'s. In particular Z is strictly contained in V.

We pass to the proof of (ii). Write $Z = \cup_{i=1}^r W_i$ where we suppose that there are no contentions between distinct W_i's. Let d_i be the dimension of W_i and suppose that $d_1 \geq d_2 \geq \cdots \geq d_r$, and that $d_1 = d_s > d_{s+1}$. If $g(Z) \subseteq Z$ for some $g \in GL(V)$, by (i) we get that g permutes the $W_1, \ldots, W_s$, so $g(\cup_{i=1}^s W_i) = \cup_{i=1}^s W_i$. Then, since there are no contentions between different W_i's, g sends $\cup_{i=s+1}^r W_i$ to itself, and thus the result is established by induction on d_1. □

Proof of Lemma 1.15. Suppose for simplicity that $V = F^d$ where F is an infinite field and we consider the coordinates given by the standard basis $e_1, \ldots, e_d$. With this choice we identify $GL(V)$ with $GL(d, F)$. Any finite union $Z = \cup_{i=1}^r W_i$ of proper subspaces of V is the zero-set of a family of polynomials $p_1, \ldots, p_k$ in $F[x_1, \ldots, x_d]$. Let $\{v_1, \cdots v_s\}$ be a finite subset of Z containing a base of each of the W_i's. The subgroup G_Z consisting of those $g \in GL(V)$ for which $g(Z) = Z$ is Zariski closed because by (ii) of Lemma 1.16 it is the zero-set of the polynomials $p_i(Xv_j) \in F[x_{ij}]_{i,j=1}^d$ for $1 \leq i \leq k$ and $1 \leq j \leq s$. Moreover, by (i) of Lemma 1.16 we may take a vector $v \in V \setminus Z$, so any element of $GL(V)$ sending a vector in Z to v is not in G_Z, so $G_Z \neq GL(V)$. A subsemigroup Γ of $GL(V)$ that is not strongly irreducible is contained in some G_Z, so it cannot be Zariski-dense in $GL(V)$. □

Let μ be a probability measure in G. Studying the random walk on X associated to μ amounts to understand the dynamics in X when we apply succesively homeomorphisms $b_0, b_1, b_2, \ldots$ chosen randomly according to the law μ. It is then natural to believe that the semigroup generated the support of μ will be relevant for this purpose. Having this in mind, it might seem a little bit odd that the authors of the article [7] present their main results by dividing in cases that depend on the Zariski closure of the **group** generated by the support of μ. Wouldn't it be more natural to consider the Zariski closure of the semigroup generated by the support of μ? Even more importantly, what does the Zariski has to do at all with the subject at hand? The aim of this section is to provide answers these questions, which rely on two facts whose precise statement and proof we present after the informal discussion that follows.

By *linear group* we refer to a subgroup of $GL(V)$, for some finite-dimensional vector space V. A similiar remark applies to the term *linear semigroup*. The first fact is that the proximal dimensions of an irreducible linear semigroup over $\mathbb{R}$ and of its Zariski closure coincide. Secondly, the Zariski closure of a linear semigroup is always a linear group. This makes our life easier because it tells us that we can find the proximal dimension of any linear irreducible semigroups if we know how to compute it for irreducible, Zariski-closed linear groups (and conceivably there are much less groups of the latter type than of the former).

What allows us to detect a natural candidate to be the unique stationary measure in the space X of rank-2 discrete subgroups of $\mathbb{R}^3$ was the map

Propositon 1.17. *If $\mathbb{K}$ is a field and $\Gamma \subset GL(d, \mathbb{K})$ is a semigroup, the Zariski closure H of Γ in $GL(d, \mathbb{K})$ is a group.*

Proof. Let $p(X) \in \mathbb{K}[X_{ij}]_{i,j=1}^d$. Any $A \in GL(d, \mathbb{K})$ defines new polynomials by the formulas

$$(L_A p)(X) = p(AX), \qquad (R_A p)(X) = p(XA).$$

The maps $L_A, R_A : \mathbb{K}[X_{ij}]_{i,j=1}^d \to \mathbb{K}[X_{ij}]_{i,j=1}^d$ are linear isomorphisms since their inverses are $L_{A^{-1}}$ and $R_{A^{-1}}$. They also preserve the degree of the polynomials. Indeed, it is clear that the degree of $L_A p$ is less or equal that the degree of p, and since $p = L_{A^{-1}}(L_A p)$, the equality holds.

Denote by $I \subseteq \mathbb{K}[X_{ij}]_{i,j=1}^d$ the ideal of polynomials vanishing at Γ. Then $\overline{\Gamma}$ consists of all the $A \in GL(d, \mathbb{K})$ such that $p(A) = 0$ for every $p \in I$.

We begin by proving that H is stable under composition. Consider $A, B \in \Gamma$ and $p \in I$. Notice that $p(AB) = 0$ since AB also belongs to Γ. Since B is an arbitrary element of Γ, this implies that $L_A(I) \subseteq I$. Moreover, the equality holds: if I_m is the subset of polynomials in I of degree $\leq m$, then $L_A : I_m \to I_m$

is bijective since it is an injective linear map and I_m is finite-dimensional. Suppose now that $A \in \Gamma$ and $B \in H$. Observe that $p(AB) = (L_A p)(B) = 0$ for any $p \in I$ since $L_A p \in I$. Again, $A \in \Gamma$ is arbitrary, so $R_B(I) = I$. Finally take $A, B \in H$ and $p \in I$. From $R_B p \in I$ we get that

$$p(AB) = (R_B p)(A) = 0,$$

which tells us that $AB \in H$, just as we wanted.

To finish we show that $A \in H$ implies that $A^{-1} \in H$. We know that $R_{A^2}(I) = I$ because $A^2 \in H$, so $I = R_{A^{-2}}(I)$. For any $p \in I$,

$$p(A^{-1}) = p(AA^{-2}) = (R_{A^{-2}} p)(A) = 0,$$

so $A^{-1} \in H$. □

Theorem 1.18. *Let Γ be an irreducible subsemigroup of $GL(d, \mathbb{R})$, and let H be its Zariski closure in $GL(d, \mathbb{R})$. Then Γ and H have the same proximal dimension.*

Proof. Since Γ is contained in H, the inequality $r_H \leq r_\Gamma$ follows from the definition of proximal dimension. We need some preparation to prove the reverse inequality.

Consider $\pi \in \overline{\mathbb{R}\Gamma}$ with rank r_Γ and such that $\pi^2 \neq 0$ (if $\pi^2 = 0$, by the irreducibility of Γ there is some $\gamma \in \Gamma$ such that $(\gamma\pi)^2 \neq 0$). Denote $\mathrm{ker}\pi$ by W_0, and $\mathrm{Im}\pi$ by W_1. The rank of π^2 is also r_Γ because it belongs to $\overline{\mathbb{R}\Gamma}$, and its rank is not greater than the rank of π. Thus we have a decomposition

$$\mathbb{R}^d = W_0 \oplus W_1,$$

which allows us to identify $End(W_1)$ with the subset of $End(\mathbb{R}^d)$ consisting of the maps f that verify $W_0 \subseteq \mathrm{ker} f$, and $\mathrm{Im} f \subseteq W_1$. Notice that $End(W_1) = \pi End(V)\pi$. The map $P : g \mapsto \pi g \pi$ is polynomial, so it is continuous with respect to the Zariski topology and the topology of $End(V)$ coming from the operator norm, which we will call metric topology. Denote $\pi\overline{\mathbb{R}\Gamma}\pi$ by G'. It follows from the metric continuity of P that G' is contained in $\overline{\mathbb{R}\Gamma}$, which in turn implies that any non-zero element of $\pi\overline{\mathbb{R}\Gamma}\pi$ is in $GL(W_1)$. Notice that G' is closed with respect to the metric topology because it is the image under the linear map P of the closed cone $\overline{\mathbb{R}\Gamma}$. Let

$$G = \{g \in G' | \det_{W_1} g = \pm 1\},$$

so that $G' = \mathbb{R}G$. Observe that G is bounded, otherwise there would be a sequence (g_n) in G such that $||g_n|| \to \infty$, but then any cluster point of $(g_n/||g_n||)$ would be non-nul and would have determinant 0, contrary to the fact that G is contained in $GL(W_1)$. Thus G is a compact semigroup contained in $GL(W_1)$, so it is a group by Lemma REF. Let Q_0 be a scalar product in W_1 such that $G \subseteq O(Q_0)$. Such a Q_0 can be obtained from an arbitrary scalar product Q in W_1, by setting

$$Q_0 = \int_G g^* Q \mathrm{d}g,$$

where $\mathrm{d}g$ is the Haar probability measure of G. Hence G' is contained in $\mathbb{R}O(Q_0)$, which is Zariski closed in $End(V)$. Since $\pi\Gamma\pi \subseteq \mathbb{R}O(Q_0)$, by the Zariski-continuity of P we deduce that $\pi H \pi \subseteq \mathbb{R}O(Q_0)$, and by the metric continuity of P we deduce that $\pi\overline{\mathbb{R}H}\pi$ is contained in $\mathbb{R}O(Q_0)$. In particular any nonzero element of $\pi\overline{\mathbb{R}H}\pi$ has rank $\dim W_1 = r_\Gamma$.

We are ready to conclude. Suppose that $\tau \in \overline{\mathbb{R}H}$ has rank r_H. Again, by the irreducibility of Γ, there are $\gamma_1, \gamma_2 \in \Gamma \cup \{e\}$ such that $\pi\gamma_1\tau\gamma_2\pi$ is non-zero. Then

$$r_H = \mathrm{rank}\, \tau \geq \mathrm{rank}\, \pi\gamma_1\tau\gamma_2\pi = r_\Gamma,$$

which is what we seeked. □

1.4 Linear random walks on projective spaces

Lemma 1.19. *Let μ be a Borel probability measure on $GL(V)$ such that Γ_μ is strongly irreducible, and let ν be μ-stationary measure on $\mathbb{P}(V)$. Then*

$$\nu(\mathbb{P}(W)) = 0,$$

for any proper subspace W of V.

Proof. Let d be the dimension of V and let d_0 be the minimal dimension of a subspace W of V such that $\nu(\mathbb{P}(W))$ is positive. We must show that $d_0 = d$.

Observe that for any two distinct $W_1, W_2 \in \mathbb{G}_{d_0}(V)$ we have

$$\nu(\mathbb{P}(W_1) \cup \mathbb{P}(W_2)) = \nu(\mathbb{P}(W_1)) + \nu(\mathbb{P}(W_2)),$$

due to the fact that $\nu(\mathbb{P}(W_1) \cap \mathbb{P}(W_2)) = \nu(\mathbb{P}(W_1 \cap W_2)) = 0$ because dimension of $W_1 \cap W_2$ is less that d_0. By induction we can prove that

$$\nu\left(\bigcup_{j=1}^{n} \mathbb{P}(W_j)\right) = \sum_{j=1}^{n} \nu(\mathbb{P}(W_j)), \tag{1.14}$$

for any finite collection $W_1, \dots W_n$ of parwise distinct d_0-dimensional subspaces of V. Thus, the supremum α of the values $\nu(\mathbb{P}(W))$ for $W \in \mathbb{G}_{d_0}(V)$ is attained (otherwise there would be infinitely many W's such that $\nu(\mathbb{P}(W)) > \frac{1}{2}\alpha$, which is impossible by (1.14)). Let $\mathcal{M}$ be the collection of all the W's for which α is reached. From (1.14) we deduce that $\mathcal{M}$ is finite. We claim that $\cup\mathcal{M}$ is Γ_μ stable. Let W be in $\mathcal{M}$. Since ν is μ-stationary,

$$\alpha = \nu(W) = \int_{GL(V)} g_*\nu(\mathbb{P}(W))\mathrm{d}\mu(g) = \int_{GL(V)} \nu(g^{-1}\mathbb{P}(W))\mathrm{d}\mu(g) \leq \int_{GL(V)} \alpha\mathrm{d}\mu(g) = \alpha.$$

Hence $g^{-1}W \in \mathcal{M}$ for μ-almost any $g \in \gamma$. As $\mathcal{M}$ is finite, we conclude that there exists some $E \subset GL(V)$ of full μ-measure such that

$$g^{-1}(\cup\mathcal{M}) \subseteq \cup\mathcal{M}$$

for any $g \in E$. The stabilizer H of $\cup\mathcal{M}$ in $GL(V)$ is a closed subgroup of $GL(V)$ that contains E^{-1}, so it also contains E and

$$\mathrm{supp}\mu \subseteq \overline{E} \subseteq H,$$

which in turn implies that $\Gamma_\mu \subseteq H$. By the strong irreducibility of Γ_μ we find that $\mathcal{M} = \{V\}$, and $d_0 = d$. □

Theorem 1.20. *Let μ be a Borel probability measure of $GL(V)$ such that Γ_μ is strongly irreducible and has proximal dimension r. Consider a μ-stationary probability measure ν on X. The following holds:*

1. *There exists a measurable map $\xi : B \to \mathbb{G}_r(V)$ such that, for β-almost any b in B, any non-zero cluster point of a sequence of the form $\lambda_n b_0 \cdots b_n$ has image $\xi(b)$.*
2. *For β-almost any $b \in B$, $\xi(b)$ is the smallest subspace W of V such that $\nu_b(\mathbb{P}(W)) = 1$.*

Proof. We begin by observing that $f_*\nu$ is a well-defined probability measure for any non-zero linear map $f : V \to V$, and that this measures depend continuously on f. The map f defines a continuous application, that by abuse of notation we will still denote by f, from $\mathbb{P}(V) \setminus \mathbb{P}(\mathrm{ker}f)$ to $\mathbb{P}(V)$. We may extend it to a measurable function on defined on $\mathbb{P}(V)$ by declaring it to have any constant value on the ν-null set $\mathbb{P}(\ker f)$. Now we prove that $(f_n)_*\nu$ converges weakly to $f_*\nu$ if $f_n \to f$ in $End(V)$. Let φ be any continuous real-valued function on $\mathbb{P}(V)$. The set of lines contained in the union of the kernels of f and the f_n's is still ν-null, so its completement is an open subset U of full measure of $\mathbb{P}(V)$ where all the corresponding proyective maps f, f_n are continuous. Hence $f_n \to f$ $\nu-$almost surely. By Egoroff's theorem, for any $\varepsilon > 0$ there is some measurable subset E_ε of $\mathbb{P}(V)$ where the convergence is uniform and such that $\nu(E_\varepsilon) > 1-\varepsilon$. We have the following bound:

$$|f_*\nu(\varphi) - (f_n)_*\nu(\varphi)| \leq \nu(|\varphi\circ f - \varphi\circ f_n|) \leq \int_{K_\varepsilon} |\varphi\circ f(x) - \varphi\circ f_n(x)| \mathrm{d}\nu(x) + \int_{\mathbb{P}(V)\setminus K_\varepsilon} |\varphi\circ f(x) - \varphi\circ f_n(x)| \mathrm{d}\nu(x)$$
$$\leq ||(\varphi\circ f - \varphi\circ f_n)\mathbb{1}_{K_\varepsilon}|\mathrm{d}\nu(x) + 2\varepsilon||\varphi||_\infty.$$

Notice that the last term can be made arbitrarily small by choosing ε small enough and n large enough because φ is uniformly continuous and the convergence $f_n \to f$ is uniform on K_ε.

Now define $\xi(b)$ to be the smallest subspace W of V such that $\nu_b(\mathbb{P}(W)) = 1$ for any b such that ν_b is defined. As we will see, the definition of ξ does not depend on the stationary measure ν and $\xi(b)$ has dimension r almost-surely.

Suppose the ν_b is well-defined and that $f = \lim_{k\to\infty} \lambda_{n_k} b_0 \cdots b_{n_k}$. Then

$$f_*\nu = \lim_{k\to\infty} (\lambda_{n_k} b_0 \cdots b_{n_k})_*\nu = \lim_{k\to\infty} (b_0 \cdots b_{n_k})_*\nu = \nu_b.$$

From $\nu(\mathbb{P}(f^{-1}\xi(b))) = \nu_b(\mathbb{P}(\xi(b))) = 1$ and the strong irreducibility of Γ_μ it follows that $\mathbb{P}(f^{-1}\xi(b))$ is equal to V, and so $\mathrm{Im} f \subseteq \xi(b)$. Additionally, from

$$\nu_b(\mathbb{P}(\mathrm{Im} f)) = f_*\nu(\mathbb{P}(\mathrm{Im} f)) = \nu(\mathbb{P}(V)) = 1$$

and the definition of $\xi(b)$ we conclude that $\xi(b)$ is contained in $\mathrm{Im} f$, establishing thus the equality. We remark that this proves that $\xi(b)$ does not depend on ν.

To conclude, let us show that the dimension of $\xi(b)$ is r. Let h be a rank-r linear endomorphism of V that is limit of some sequence $(\lambda'_m g_m)$ with each g_m belonging to Γ_μ. By Lemma REF for β-almost any b, the measures $(b_0 \cdots b_n g)_*\nu$ converge weakly towards ν_b. We keep the same hypotheses for f as in the previous paragraph. For each fixed m we get

$$(fg_m)_*\nu = \lim_{k\to\infty} (b_0 \cdots b_{n_k} g_m))_*\nu = \nu_b,$$

which combined with $fg_m \to fh$ yields that $(fh)_*\nu = \nu_b$. By the same argument as before we conclude that $\xi(b) = \mathrm{Im} fh$, so the dimension of $\xi(b)$ is r. If $fh = 0$, by the irreducibility of Γ_μ there is some $g \in \Gamma_\mu$ such that fgh is non-zero and the same argument proves what we wanted. □

The map $\xi : B \to \mathbb{G}_r(V)$ in the previous theorem is known as the **bounday map** of μ. A direct and important consequence of the existence and the definingand property of the boundary map is the following.

Corollary 1.21. *Let μ be a Borel probability measure on $GL(V)$ such that Γ_μ is strongly irreducible and proximal. Then there is a unique μ-stationary measure on $\mathbb{P}(V)$, and it is μ-proximal.*

Proof. The boundary map ξ of μ is $\mathbb{P}$-valued because the proximal dimension of Γ_μ is 1. Let ν be any μ-stationary probability measure on $\mathbb{P}(V)$. By *(ii)* of Theorem 1.20, for almost any $b \in B$, the support of ν_b is $\mathbb{P}(\xi(b)) = \xi(b)$, hence $\nu_b = \delta_{\xi(b)}$. We recover ν by integrating the limit probability measures

$$\nu = \int_B \nu_b \mathrm{d}\beta(b) = \int_B \delta_{\xi(b)} \mathrm{d}\beta(b) = \xi_*\beta,$$

so ν is uniquely determined by μ. □

If we are interested more generally in studying linear random walks on grassmanians of any dimension, it is often useful to keep in mind the *linear duality*: Let V be a d-dimensional real vector space, and let V^* be the dual of V (i.e. the vector space of linear maps $V \to \mathbb{R}$). For any $g \in GL(V)$ we will denote by g^* the adjoint map defined for any $\ell \in V^*$ by

$$g^*(\ell) = \ell \circ g.$$

The *duality* between the grassmanians of V and V^* is given by the maps $\perp\colon \mathbb{G}_s(V) \to \mathbb{G}_{d-s}(V^*)$ defined by

$$W^\perp = \{\ell \in V^* \mid W \subseteq \ker \ell\},$$

for $W \in \mathbb{G}_s(V)$. From the definitions it follows that

$$g^* \circ \perp \circ g = \perp,$$

for any $g \in GL(V)$. It is then immediate that if $D : g \mapsto (g^*)^{-1}$, then

$$\perp \circ g = D(g) \circ \perp \tag{1.15}$$

In the following lemma we gather some statements that tell us how probability measures on $GL(V)$ and on the grassmanians of V interact with the maps $\perp$. If ν is a Borel measure on $\mathbb{G}_s(V)$, we denote $\perp_* \nu$ as $\nu^\perp$.

Lemma 1.22. *Let μ and ν be Borel probability measures on $GL(V)$ and $\mathbb{G}_s(V)$, respectively.*

- (i) *The convolution satisfies: $(\mu * \nu)^\perp = (D_*\mu) * \nu^\perp$.*
- (ii) *The duality $\mathbb{G}_s(V) \to \mathbb{G}_{d-s}(V^*)$ induces a bijection between μ-stationary and $D_*\mu$-stationary probability measures.*
- (iii) *$\Gamma_{D^*\mu}$ is equal to $D(\Gamma_\mu)$.*
- (iv) *Γ_μ is irreducible if and only if $\Gamma_{D_*\mu}$ is irreducible.*
- (v) *Γ_μ is strongly irreducible if and only if $\Gamma_{D_*\mu}$ is strongly irreducible.*

Proof. For any g in $GL(V)$, by (1.15) we have

$$(g_*\nu)^\perp = (\perp \circ g)_*\nu = (D(g) \circ \perp)_*\nu = D(g)_*\nu^\perp.$$

Using this equality we get the identity in (i):

$$(\mu * \nu)^\perp = \int_{GL(V)} (g_*\nu)^\perp \mathrm{d}\mu(g) = \int_{GL(V)} D(g)_*\nu^\perp \mathrm{d}\mu(g) = \int_{GL(V^*)} h_*\nu^\perp \mathrm{d}D_*\mu(h) = (D_*\mu) * \nu^\perp.$$

The assertion in (ii) is a direct consequence of (i) applied to the dualities

$$\mathbb{G}_s(V) \to \mathbb{G}_{d-s}(V^*) \to \mathbb{G}_s(V^{**}) \cong \mathbb{G}_s(V).$$

Another possibility is to notice that D is a group morphism since

$$D(g_1g_2) = ((g_1g_2)^*)^{-1} = (g_2^*g_1^*)^{-1} = D(g_1)D(g_2),$$

so $GL(V)$ acts in V^* via D. The equality (1.15) tells us that the duality map is a $GL(V)$-equivariant homeomorphism, thus (i) and (ii) follow.

As D is a continuous isomorphism $GL(V) \to GL(V^*)$, it sends the support of μ to the support of $D_*\mu$. This implies (iii).

Let Z be a finite union of subspaces of V, and denote by $Z^\perp$ the union of the duals of the subspaces forming Z. If $gZ = Z$, from (1.15) we deduce that

$$D(g)Z^\perp = (gZ)^\perp = Z^\perp.$$

The duality sends proper subspaces of V to proper subspaces of V^* and viceversa, and since $\Gamma_{D_*\mu} = D(\Gamma_\mu)$, the assertions (iv) and (v) follow.

□

We end this section remarking that proximal dimensions do not behave well in general with respect to the map D. For example, if $V = \mathbb{R}^3$ and g_1, g_2 are the diagonal matrices

$$g_1 = \mathrm{diag}\,(3,2,1), \quad g_2 = \mathrm{diag}\,(2,1,1),$$

the proximal dimension of the semigroups generated by g_1, g_2 and $D(g_1)$ is 1, since they have proximal elements, but the proximal dimension of the semigroup generated by

$$D(g_2) = \mathrm{diag}\,(1/2,1,1)$$

is 2.

1.5 The Law of Large Numbers

Let V be a normed finite dimensional real vector space with a norm $||\cdot||$ induced from a scalar product. This norm induces an operator norm on $End(V)$ that we will also denote by $||\cdot||$. The general linear group $GL(V)$ will be denoted by G. The *norm cocycle* $\sigma : G \times \mathbb{P}(V) \to \mathbb{R}$ is defined as

$$\sigma(g,x) = \log \frac{||gv||}{||v||},$$

where v is any vector in x. For $g \in G$ we define

$$N(g) = \max\{||g||, ||g^{-1}||\}.$$

A Borel probability measure μ on G has **finite first moment** if

$$\int_G \log N(g) \mathrm{d}\mu(g) < \infty.$$

This notion does not depend on the norm we choose for V. From the inequality $\log N(g_1g_2) \le \log N(g_1) + \log N(g_2)$ we deduce that the family of Borel probability measures on G with finite first moment is stable under convolutions.

Let us fix a probability measure μ on G with finite first moment, and let (B,β,S) be the one-sided shift with alphabet $(G,\mathscr{G},\mu)$. The function $g \mapsto \log ||g||$ is integrable because $|\log ||g||| \le \log N(g)$. The sequence

$$a_n := \int_G \log ||g|| \mathrm{d}\mu^{*n}(g)$$

is subadditive as the following calculation shows:

$$a_{n+m} = \int_G \int_G \log ||g_1g_2|| \mathrm{d}\mu^{*n}(g_1)\mathrm{d}\mu^{*m}(g_2) \le \int_G \int_G \log ||g_1|| + \log ||g_2|| \mathrm{d}\mu^{*n}(g_1)\mathrm{d}\mu^{*m}(g_2) = a_n + a_m.$$

Thus a_n/n converges. Its limit is known as the **first Lyapunov exponent** of μ, and is denoted by $\lambda_{1,\mu}$. As we will now see, the first Lyapunov exponent controls the asymptotic growth of the norm of the vectors in V when we apply succesively independent random elements of G with law μ.

Theorem 1.23 (Law of Large Numbers V1)**.** *Let μ be a Borel probability measure on $GL(V)$ with finite first moment, and such that Γ_μ is irreducible. Let ν be a μ-stationary Borel probability measure on $\mathbb{P}(V)$.*

(i) The norm cocycle σ is $\mu \otimes \nu$-integrable, i.e.

$$\int_{GL(V)} \int_{\mathbb{P}(V)} |\sigma| d\nu d\mu < \infty,$$

and its mean is the first Lyapunov exponent of μ:

$$\lambda_{1,\mu} = \int_{GL(V)} \int_{\mathbb{P}(V)} \sigma \, d\nu \, d\mu.$$

In particular, it does not depend on the μ-stationary measure ν.

(ii) For any x in $\mathbb{P}(V)$, the equality

$$\lim_{n\to\infty} \frac{1}{n} \sigma(b_{n-1} \cdots b_0, x) = \lambda_{1,\mu}$$

holds for β-almost any b in B. The sequence also converges in $L^1(B, \beta)$ uniformly for $x \in \mathbb{P}(V)$.

(iii) For any x in $\mathbb{P}(V)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \int_{GL(V)} \sigma(g, x) d\mu^{*n}(g) = \lambda_{1,\mu},$$

and the convergence is uniform for $x \in \mathbb{P}(V)$.

The proof can be found in [2, Theorem 4.28] . The uniform convergence in (iii) will play an important role in Chapter 2.

Our next objective is to define all the Lyapunov exponents of a Borel probability measure μ on $GL(V)$. This will control the growth of vectors in the random walk associated to μ on exterior powers $\wedge^s V$. The technical conditions we will require μ to fullfill will be those that allow us to apply Theorem 1.23 to the probability measures $(\wedge^s)_*\mu$ on $GL(\wedge^s V)$.

Let d be the dimension of V, and consider an integer $1 \leq s \leq d$. Occasionally we will use the multi-index notation: A *multi-index* of size s is a finite sequence

$$I = (n_1, \cdots, n_s) \tag{1.16}$$

such that the terms are strictly increasing integers between 1 and d. We denote by $\mathcal{I}_s$ the set of multi-indices of size s.

Consider some $I \in \mathcal{I}_s$ like in (1.16). For any fixed choice of real numbers $\lambda_1, \ldots, \lambda_d$ and of vectors $v_1, \ldots, v_d$ in V, we denote

$$\lambda_I = \lambda_{n_1} \cdots \lambda_{n_s}, \quad \text{and} \quad v_I = v_{n_1} \wedge \cdots \wedge v_{n_s}. \tag{1.17}$$

If q is the scalar product on V, we endow the s-th exterior power $\wedge^s V$ with the scalar product:

$$q_s(v_1 \wedge \cdots \wedge v_s, v'_1 \wedge \cdots \wedge v'_s) = |\det\left(q(v_i, v'_j)\right)_{1\leq i,j\leq s}| \tag{1.18}$$

In other words, if $\{\overline{v}_1, \dots, \overline{v}_d\}$ is an orthonormal basis of V, then

$$\{\overline{v}_I \mid I \in \mathcal{I}_s\} \tag{1.19}$$

is a orthonormal basis of $\wedge^s V$.

For any $g \in GL(V)$ we will denote by $\wedge^s g$ the linear endomorphism of $\wedge^s V$ such that

$$\wedge^s g(v_1 \wedge \cdots \wedge v_s) = (gv_1) \wedge \cdots \wedge (gv_s), \tag{1.20}$$

for any $v_1, \dots, v_s \in V$. This defines a continuous homomorphism

$$\wedge^s : GL(V) \to GL(\wedge^s V).$$

We will define the higher order Lyapunov exponents for Borel probability measures on $GL(V)$ by using the first Lyapunov exponents of the measures $\wedge^s_* \mu$. We will need that this exponents are finite, which we can guarantee using (i) of Theorem 1.23, so the probability measures we consider are those who satisfy the hypothesis of this theorem. The next lemma takes care of the finite moment condition.

Lemma 1.24. *Let μ be a Borel probability measure on $GL(V)$. If μ has finite first moment, then $\wedge^s_* \mu \in \mathcal{P}(\wedge^s V)$ also has finite first moment for any $1 \leq s \leq d$.*

Proof. To simplify the notation we will suppose that V is $\mathbb{R}^d$ with the standard scalar product. We identify any $g \in GL(\mathbb{R}^d)$ with its matrix with respect to the canonical basis $\{e_1, \dots, e_d\}$ of $\mathbb{R}^d$. According to the Cartan Decomposition REF, for any $g \in GL(d, \mathbb{R})$, there exists a unique diagonal matrix

$$a(g) = \operatorname{diag}(\kappa_1(g), \dots, \kappa_d(g)),$$

where

$$\kappa_1(g) \geq \cdots \geq \kappa_d(g), \tag{1.21}$$

and such that $g \in O(d)a(g)O(g)$. Here $O(d)$ is the group of orthogonal matrices of size $d \times d$. Lets write $g = h_1 a(g) h_2$ for some $h_1, h_2 \in O(d)$. From the definition of the scalar product in the exterior powers it easily follows that $\wedge^s h_1$ and $\wedge^s h_2$ are isometries, so

$$|| \wedge^s g|| = || \wedge^s a(g)|| = \kappa_1(g) \cdots \kappa_s(g).$$

The last step of the inequality comes from the observation that for vectors $w \in \wedge^s \mathbb{R}^d$ of unit length, $|| \wedge^s a(g) w||$ attains its maximum value for $w = e_1 \wedge \cdots \wedge e_s$ because of (1.21). Notice that for $s = 1$ we get that

$$\kappa_1(g) = ||g||,$$

hence from (1.21) it follows that

$$|| \wedge^s g|| = \kappa_1(g) \cdots \kappa_s(g) \leq \kappa_1(g)^s$$

for any $g \in GL(V)$. This tells us that

$$\log N(\wedge^s g) \leq s \log N(g),$$

from where the desired result follows. $\square$

Now we take care of the irreducibility condition. Let μ be a Borel probability measure on $GL(V)$. Consider the following property:

Irr(μ)*: For any* $1 \leq s \leq d$*, the subsemigroup* $\wedge^s(\Gamma_\mu)$ *of* $GL(\wedge^s V)$ *is irreducible.*

Notice that property Irr(μ) implies that $\Gamma_{\wedge^s_*\mu}$ is irreducible because it contains $\wedge^s(\Gamma_\mu)$.

Suppose that μ has finite first moment and that it satisfies property Irr(μ). Part (i) of Theorem 1.23 guarantees that the first Lyapunov exponent of $\wedge^s\mu$ is finite, so we can define the **Lyapunov exponents** $\lambda_{1,\mu}, \ldots, \lambda_{d,\mu}$ of μ inductively by the formula

$$\lambda_{1,\mu} + \cdots + \lambda_{s,\mu} = \lambda_{1,\wedge^s_*\mu}. \tag{1.22}$$

In other words,

$$\lambda_{1,\mu} + \cdots + \lambda_{s,\mu} = \lim_{n\to\infty} \frac{1}{n} \int_{GL(V)} \log || \wedge^s g|| \mathrm{d}\mu^{*n}(g). \tag{1.23}$$

For convenience for later reference, we will restate part of Theorem 1.23 for this kind of probability measures. We denote by $\sigma_s : GL(V) \times \mathbb{P}(\wedge^s V) \to \mathbb{R}$ the norm cocycle in the natural representation of $GL(V)$ in $\wedge^s V$, in other words

$$\sigma_s(g, x_s) = \log \frac{||(\wedge^s g)w}{||w||},$$

for any non-zero vector w in the line x_s.

Theorem 1.25. *Let* μ *be a Borel probability measure on* $GL(V)$ *with finite first moment satisfying property Irr*(μ)*. For any* $1 \leq s \leq d$*, let* ν_s *be a* μ*-stationary Borel probability measure on* $\mathbb{P}(\wedge^s V)$*.*

(i) *The norm cocycle* σ_s *is* $\mu \otimes \nu_s$*-integrable and its mean is the sum of the first s Lyapunov exponents of* μ*:*

$$\lambda_{1,\mu} + \cdots + \lambda_{s,\mu} = \int_{GL(V)} \int_{\mathbb{P}(\wedge^s V)} \sigma_s \, d\nu_s \, d\mu.$$

(ii) *For any non-zero vector* w *in* $\wedge^s V$*, the equality*

$$\lim_{n\to\infty} \log \left(\frac{1}{n} \frac{|| \wedge^s (b_{n-1} \cdots b_0)w||}{||w||} \right) = \lambda_{1,\mu} + \cdots + \lambda s, \mu$$

holds for β*-almost any* b *in* B*. The sequence also converges in* $L^1(B, \beta)$ *uniformly in* w*.*

(iii) *For any non-zero vector* w *in* $\wedge^s V$*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \int_{GL(V)} \log \left(\frac{|| \wedge^s gw||}{||w||} \right) d\mu^{*n}(g) = \lambda_{1,\mu} + \cdots + \lambda s, \mu,$$

and moreover, the convergence is uniform in w*.*

We will now prove that a Zariski dense Borel probability measure μ on $SL(V)$ satisfies property Irr(μ).

Lemma 1.26. *Any Zariski dense Borel probability measure of* $SL(V)$ *satisfies property Irr*(μ)*.*

This is a consequence of the fact that the action of $SL(V)$ on the exterior powers of V is irreducible.

Lemma 1.27. *Let* V *be a d-dimensional vector space. The linear action of* $\wedge^s SL(V)$ *on* $\wedge^s V$ *is irreducible.*

Proof. Let us fix an integer $1 \leq s \leq d$. Suppose that W is a non-zero $\wedge^s SL(V)$-invariant linear subspace of $\wedge^s V$. Consider a sequence of real numbers $a_1, \dots, a_d$ such that $a_1 \cdots a_d = 1$ and $a_I \neq a_J$ for any two distinct multi-indices $I, J \in \mathcal{I}_s$. For any $I \in \mathcal{I}_s$, let $p_I(x)$ be a polynomial with real coefficients such that $p_I(\lambda_I) = 1$ and $p_I(\lambda_J) = 0$ for any $J \in \mathcal{I}_s \setminus \{I\}$.

Let $\{v_1, \dots v_d\}$ be a basis of V and consider the linear endomorphism g of V sending v_i to $a_i v_i$. The eigenvalues of $\wedge^s g$ are the numbers $a_I, I \in \mathcal{I}_s$, all distinct by construction. Any non-zero vector $w \in W$ can be written as

$$w = w_{I_1} + \cdots + w_{I_l},$$

where the w_{I_j} are non-zero and $\wedge^s g w_{I_j} = a_{I_j} w_{I_j}$. Since $w_{I_j} = p_{I_j}(\wedge^s g)(w)$ and W is $p_{I_j}(\wedge^s g)$-stable, we conclude that W is a direct sum of some of the lines $\mathbb{R}v_I$. We can always interchange any two lines $\mathbb{R}v_I$ and $\mathbb{R}v_J$ with a map in $SL(V)$ that permutes the base $\{v_1, \dots v_d\}$, so necessarily $W = \wedge^s V$. □

Proof of Lemma 1.26. Let μ be a Zariski dense Borel probability measure on $SL(V)$. Suppose that W is a $\wedge^s(\Gamma_\mu)$-stable linear subspace of $\wedge^s V$. The stabilizer $S(W)$ of W in $GL(\wedge^s V)$ is Zariski closed in $GL(\wedge^s V)$, and the map $\wedge^s : GL(V) \to GL(\wedge^s V)$ is continuous (it is polynomial), so W is $\wedge^s SL(V)$-stable:

$$\wedge^s SL(V) = \wedge^s \mathrm{cl}_Z \Gamma_\mu \subseteq \mathrm{cl}_Z(\wedge^s \Gamma_\mu) \subseteq S(W)$$

By Lemma 1.26, $W = \wedge^s V$, as we wanted. □

The Lyapunov exponents of a Borel probability measure μ are always decreasing

$$\lambda_{1,\mu} \geq \cdots \geq \lambda_{d,\mu}.$$

To show this we use the notation of the proof of Lemma **??**. Observe that for any $g \in GL(V)$ and any $1 < s < d$ we have

$$\frac{|| \wedge^{s+1} g||}{|| \wedge^s g||} = \kappa_{s+1}(g) \leq \kappa_s(g) \leq \frac{|| \wedge^s g||}{|| \wedge^{s-1} g||}$$

thus

$$\log || \wedge^{s-1} g|| + \log || \wedge^{s+1} g|| \leq 2 \log || \wedge^s g|| \tag{1.24}$$

If we integrate (1.24) with respect to $\frac{1}{n}\mu^{*n}$ and then take the limit as $n \to \infty$, from (1.23) we deduce that

$$(\lambda_{1,\mu} + \cdots \lambda_{s-1,\mu}) + (\lambda_{1,\mu} + \cdots \lambda_{s+1,\mu}) \leq 2(\lambda_{1,\mu} + \cdots \lambda_{s,\mu}),$$

so $\lambda_{s,\mu} \leq \lambda_{s+1,\mu}$.

We say that the Lyapunov exponents of μ are simple if they are all distinct, that is

$$\lambda_{1,\mu} > \cdots > \lambda_{d,\mu}.$$

We cite following theorem that gives sufficient conditions to guarantee the simplicity of the Lyapunov exponents of μ. It is part of [2, Corollary 10.15].

Theorem 1.28. *Let $V = \mathbb{R}^d$ and let μ be a Borel probability measure on $GL(V)$ with finite first moment, that is Zariski dense in $GL(V)$ or in $SL(V)$. Then the Lyapunov exponents of μ satisfy*

$$\lambda_{1,\mu} > \cdots > \lambda_{d,\mu}.$$

Chapter 2

The space of 2-lattices in $\mathbb{R}^3$

2.1 Presentation of the main results

In the discussion that follows we equip $\mathbb{R}^2$ and $\mathbb{R}^3$ with their standard scalar product. The space $\mathbb{G}_2(\mathbb{R}^3)$ of 2-dimensional subspaces of $\mathbb{R}^3$ will be denoted by $\mathbb{P}^*(\mathbb{R}^3)$, because by duality it is in correspondance with $\mathbb{P}((\mathbb{R}^3)^*)$. We say that two lattices Δ_1 and Δ_2 of $\mathbb{R}^2$ have the same shape if there is a linear isometry $T : \mathbb{R}^2 \to \mathbb{R}^2$ and a positive number λ such that $\lambda T \Delta_1 = \Delta_2$, which is an equivalence relation. The **shape of a lattice** Δ is its equivalence class with respect to this relation. Let us denote by $\mathcal{S}$ the set shapes of lattices of $\mathbb{R}^2$, which we endow with its topology of $SL^{\pm}(2,\mathbb{R})$-homogeneous space.

A **2-lattice of** $\mathbb{R}^3$ is by definition a rank 2 discrete subgroup of $\mathbb{R}^3$. The covolume of a 2-lattice Λ is defined as $||v_1 \wedge v_2||$, where v_1, v_2 is an arbitrary basis of Λ. From this point onwards X will denote the space of 2-lattices of $\mathbb{R}^3$ modulo homothecies. If Λ is a 2-lattice of $\mathbb{R}^3$, we will denote by $[\Lambda]$ its respective class in X. From now on G will denote the group $SL(3,\mathbb{R})$. The natural action of G on $\mathbb{R}^3$ induces a transitive action on X, for which the stabilizer of $x_0 = [\mathbb{Z}^2 \times \{0\}]$ is

$$S(x_0) = \left\{ \begin{pmatrix} a & b & * \\ c & d & * \\ 0 & 0 & e^{-1} \end{pmatrix} : e \in \mathbb{R}^\times, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in eSL(2,\mathbb{Z}) \right\}. \tag{2.1}$$

so we may identify X with the homogeneous space $G/S(x_0)$. Let $\pi : X \to \mathbb{P}^*(\mathbb{R}^3)$ be the map that sends any point in X to the plane it spans. Consider some $x = [\Lambda] \in X$. We equip the plane spanned by Λ with the restriction of the scalar product of $\mathbb{R}^3$, and consider some linear isometry $I : \pi(x) \to \mathbb{R}^2$. The shape of $I\Lambda$ does not depend on I, nor on the representative Λ of x, so we can denote it by $\mathbf{s}(x)$. This defines the **shape map** $\mathbf{s} : X \to \mathcal{S}$.

The goal of the article [7] is to study the dynamics of G and its subgroups on the space X, having as motivation a conjecture related to the open problem of determining if any cubic irrational number α is well-approximable (which means that the sequence of integers appearing in the continued fraction development of α is unbounded). This leads the authors to study the dynamics of subgroups of G in X whose Zariski closure is either G or $SO(2,1)$. Here we will treat only the first case, studying the dynamics of Zariski dense subsemigroups of G. The reason for choosing to work with semigroups, and not just groups, is that the technical tools used to prove the theorems come from the study of random walks in X associated to Borel probability measures μ on G, and the natural object that appears here is the closed semigroup generated by the support of μ.

After having introduced the objects we are interested in studying, we now present the main results of this work. We begin with *dynamical statements* (Theorems 2.1 and 2.3), which will be established by *random walk* results (Theorems 2.4 and 2.5). Our goal is to make the link between the two kinds of results, and showing various implications between them.

The first theorem is inspired by the conjecture that motivates [7].

Theorem 2.1. *Let Γ be a Zariski-dense subsemigroup of $G = SL(3, \mathbb{R})$. Then, for any x in X, the set $\boldsymbol{s}(\Gamma x)$ is dense in $\mathcal{S}$.*

The following Lemma implies that it enough to prove Theorem 2.1 for finitely-generated Zariski dense semigroups of G.

Lemma 2.2. *Any Zariski dense semigroup of G contains a finitely generated Zariski dense semigroup.*

Proof. Let Γ be a Zariski dense semigroup of G and consider a countable dense subset of Γ with respect to the topology inherited from the manifold topology of G,

$$D = \{\gamma_n \mid n \in \mathbb{N}\}.$$

Notice that D is Zariski dense in G because $\mathrm{cl}_Z\, D$ contains $\overline{D}$, and $\overline{D}$ contains Γ, so

$$G = \mathrm{cl}_Z\, \Gamma \subseteq \mathrm{cl}_Z\, D.$$

Let Γ_n be the semigroup generated by $\gamma_0, \ldots, \gamma_n$, and let $G_n = \mathrm{cl}_Z\, \Gamma_n$, which is a group by Proposition 1.17. We will show that $G_n = G$ for $n >> 0$.

Let us denote by G_n^o the Zariski connected component of e in G_n, which is a finite index normal subgroup, and in fact is equal to the irreducible component of e in G_n (see [2], Lemma 6.21). The increasing sequence of Zariski irreducible, Zariski closed subsets $(G_n^o)_n$ must become eventually constant, say equal to a Zariski closed subgroup H of G. The normalizer of H in G is a Zariski closed subgroup containing D (because it contains G_n for $n >> 0$), so necessarily H is normal in G. We can conclude then that $H = G$ or H is discrete, because the Lie algebra of H is an ideal of the simple Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$. If $H = G$, then $G_n = G$ for $n >> 0$, so Γ_n is Zariski dense in G for $n >> 0$. If this were not the case, the groups G_n would be finite because they are discrete and Zariski closed. A Theorem of Jordan (see [**?**, Theorem 36.14]) tells us that there is a positive integer $r = r(m)$ such that any finite subgroup of $GL(m, \mathbb{C})$ has an abelian normal subgroup of index at most r. Any two elements g, h of D are in some G_n, so their N-th powers are in the abelian normal subgroup of G_n given by Jordan's Theorem, where $N = r(3)!$. They satisfy then the polynomial equation

$$g^N h^N = h^N g^N. \tag{2.2}$$

Since (2.2) is verified for any two elements of the Zariski dense subset D of G, it must be satisfied for any two elements in G. This is impossible because any element of G that is in the image of the exponential map $\mathfrak{g} \to G$ is an N-th power, in particular a neighborhood of the $e \in G$ would be abelian, which is absurd. □

The second dynamical result gives us information about the orbit closures $\overline{\Gamma x}$, but before we present it we need a definition. Recall that $D : GL(\mathbb{R}^3) \to GL((\mathbb{R}^3)^*)$ is the map

$$D(g) = (g^*)^{-1}$$

Let Γ be a Zariski dense semigroup of G. The semigroup $D(\Gamma)$ of $GL((\mathbb{R}^3)*)$ is irreducible because Γ is. Also, $D(\Gamma)$ is Zariski dense in $SL((\mathbb{R}^3)^*)$ because $D : SL(\mathbb{R}^3) \to SL((\mathbb{R}^3)*)$ is a homeomorphism with respect to the Zariski topology (the transposition and inversion of matrices with determinant 1 are polynomial maps). Since $D(\Gamma)$ is irreducible and $SL((\mathbb{R}^3)*)$ is proximal, then $D(\Gamma)$ is also proximal by

Theorem 1.18. Then, by (ii) of Proposition 1.14, the limit set $\Lambda^1_{D(\Gamma)}$ is the only $D(\Gamma)$-invariant minimal closed subset of $\mathbb{P}((\mathbb{R}^3)^*)$. Let

$$\Lambda_\Gamma^\perp = (\Lambda^1_{D(\Gamma)})^\perp$$

be the corresponding dual set in $\mathbb{P}^*(\mathbb{R}^3)$. Since the dual map $\perp\colon \mathbb{P}((\mathbb{R}^3)^*) \to \mathbb{P}^*(\mathbb{R}^3)$ is $GL(\mathbb{R}^3)$-equivariant homeomorphism ($GL(\mathbb{R}^3)$ acting on $(\mathbb{R}^3)^*$ via D), then $\Lambda_\Gamma^\perp$ is the only Γ-invariant minimal closed subset of $\mathbb{P}^*(\mathbb{R}^3)$.

Theorem 2.3. *If Γ is a Zariski dense subsemigroup of G, the only Γ-invariant minimal closed subset of X is $\pi^{-1}(\Lambda_\Gamma^\perp)$.*

Theorem 2.3 $\Rightarrow$ Theorem 2.1. We fix some $x \in X$ and consider any shape $s_0 \in \mathcal{S}$. We take any plane $W \in \Lambda_\Gamma^\perp$. There is some 2-lattice $\Lambda \subseteq W$ such that $s([\Lambda]) = s_0$. Let $(\gamma_n)_{n\geq 1}$ be a sequence in Γ such that $\gamma_n(x) \to [\Lambda]$ as $n \to \infty$. By the continuity of the shape map we get that $\mathbf{s}(\gamma_n(x)) \to \mathbf{s}([\Lambda]) = s_0$. Since s_0 is an arbitrary element of $\mathcal{S}$, then $\mathbf{s}(\Gamma x)$ is dense in $\mathcal{S}$. □

Now we present the random walk statements that we will use to prove Theorem 2.3.

Theorem 2.4. *Let μ be a Zariski dense Borel probability measure on $SL(3,\mathbb{R})$ with compact support. There exist a unique μ-stationary probability measure ν_X on X.*

Theorem 2.4 is implied by points (a) and (b) of the Theorem 2.1 of [7]. The proof is quite long, and requires a lot of technical machinery. Here we will admit it, contenting ourselves with sketching the first steps of the proof.

The next theorem is related to the recurrence of the random walk on X associated to a Zariski dense Borel probability measure on G.

Theorem 2.5. *Let μ be a Zariski dense Borel probability measure on G with compact support. For any x in X, the sequence*

$$\nu_{x,n} = \frac{1}{n}\sum_{j=1}^{n} \mu^{*j} * \delta_x$$

is weakly compact, and any cluster point is a μ-stationary Borel probability measure on X.

To be more concrete, will define explicitely the natural candidate ν_X to be the unique μ-stationary measure in the situation of Theorem 2.4. We begin by recalling that the set $\mathcal{L}_2^1$ of covolume 1 lattices of $\mathbb{R}^2$ is identified with $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ as follows: the natural action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ induces a transitive action on $\mathcal{L}_2^1$ for which the stabilizer of $\mathbb{Z}^2 \in \mathcal{L}_2^1$ is $SL(2,\mathbb{Z})$. The identification $SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \to \mathcal{L}_2^1$ is given by the orbital map of $\mathbb{Z}^2$.

$$gSL(2,\mathbb{Z}) \mapsto g\mathbb{Z}^2.$$

Consider now the space Y of lattices of $\mathbb{R}^2$ modulo homothecies. Any lattice of $\mathbb{R}^2$ is homothetic to a unique lattice of covolume 1, so we can identify Y with $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$. Let α be the unique $SL(2,\mathbb{R})$-invariant Borel probability measure on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.

Let W be a plane of $\mathbb{R}^3$. Any linear isomorphism $L : \mathbb{R}^2 \to W$ determines a bijection $\vartheta_L : SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \to \pi^{-1}(W)$ given by

$$\vartheta_L : gSL(2,\mathbb{Z}) \mapsto [Lg\mathbb{Z}^2]. \tag{2.3}$$

We claim that the probability measure $(\vartheta_L)_*\alpha$ does not depend on the choice of L. Indeed, if $L' : \mathbb{R}^2 \to W$ is another linear isomorphism, the map $(\vartheta_{L'})^{-1} \circ \vartheta_L$ is given by

$$gSL(2,\mathbb{Z}) \mapsto g_0 gSL(2,\mathbb{Z}),$$

where g_0 is the matrix of $|\det((L')^{-1}L)|^{-1/2}(L')^{-1}L$ with respect to the basis e_1, e_2, so

$$((\vartheta_{L'})^{-1} \circ \vartheta_L)_*\alpha = (g_0)_*\alpha = \alpha,$$

which tells us that

$$(\vartheta_L)_*\alpha = (\vartheta_{L'})_*\alpha.$$

Now we have the right to denote by m_W the probability measure $(\vartheta_L)_*\alpha$. We remark that the same considerations prove that

$$g_* m_{W_1} = m_{W_2} \tag{2.4}$$

if $g \in GL(3,\mathbb{R})$ sends the plane W_1 to W_2.

We are ready to define ν_X. Let $(B, \mathscr{B}, S)$ be the one-sided shift with alphabet $(G, \mathscr{G}, \mu)$. We will prove in Proposition 2.7 that if μ is a Zariski dense Borel probability measure on $G = SL(3,\mathbb{R})$, there is only one μ-stationary Borel probability measure $\nu_{\mathbb{P}^*}$ on $\mathbb{P}^*(\mathbb{R}^3)$, and that it is μ-proximal. Let $\xi^* : B \to \mathbb{P}*(\mathbb{R}^3)$ be the unique measurable map (up to β-null sets) that satysfies

$$(\nu_{\mathbb{P}^*})_b = \delta_{\xi^*(b)},$$

and let ν_b be $m_{\xi^*(b)}$ considered as a probability measure on X. Since $b_0\xi^*(Sb) = \xi^*(b)$ holds β-almost surely, then by **??** we have

$$(b_0)_*\nu_{Sb} = (b_0)_* m_{\xi^*(Sb)} = m_{b_0\xi^*(Sb)} = m_{\xi^*(b)} = \nu_b,$$

and hence (ν_b) is a equivariant family of probability measures on X. We define ν_X as the integral

$$\nu_X = \int_B \nu_b \mathrm{d}\beta(b).$$

By Proposition 1.7, ν_X is μ-stationary and its limit measures are the ν_b that we used to define it. We can describe also ν_X as an integral with respect to $\nu_{\mathbb{P}^*}$. Recall that $\nu_{\mathbb{P}^*} = \int_B \delta_{\xi^*(b)}\mathrm{d}\beta(b) = (\xi^*)_*\beta$, so by the change of variables formula we get that

$$\nu_X = \int_B m_{\xi(b)}\mathrm{d}\beta(b) = \int_{\mathbb{P}^*(\mathbb{R}^3)} m_W \mathrm{d}\xi_*\beta(W) = \int_{\mathbb{P}^*(\mathbb{R}^3)} m_W \mathrm{d}\nu_{\mathbb{P}^*}(W).$$

We can think then ν_X as a sort of natural lift to X of $\nu_{\mathbb{P}^*}$ because we are giving to each $\pi^{-1}(W)$ the natural uniform probability measure m_W.

Let us show how the random walk statements imply Theorem 2.3 for finitely generated Zariski dense semigroups of G (which is enough to prove Theorem 2.1 by Lemma 2.2). First, we combine Theorem 2.4 and Theorem 2.5 to get the following corollary, which in turn will imply Theorem 2.1.

Corollary 2.6. *Let μ be a Zariski dense Borel probability measure on $SL(3,\mathbb{R})$ with compact support. The only Γ_μ-invariant minimal closed subset of X is $\operatorname{supp}\nu_X$.*

Theorem 2.4 and Theorem 2.5 $\Rightarrow$ Corollary 2.6. We need to show that for any $x \in X$, $\operatorname{supp}\mu$ is contained in $\overline{\Gamma_\mu x}$. Let y be any point in the support of ν_X, and condider an open neighborhood U of y. There is a non-negative function $\varphi_U \in \mathscr{C}_c(X)$ taking a positive value on y (so $\nu_X(\varphi_U) > 0$) and vanishing outside of U. Notice that the sequence $(\nu_{x,n})_n$ converges weakly to ν_X since there are no other μ-stationary measures on X. Since $\nu_{x,n}(\varphi_U) \to \nu_X(\varphi_U)$, in particular

$$\nu_{x,m}(\varphi_U) = \frac{1}{m}\sum_{j=1}^{n}\int_G \varphi_U(gx)\mathrm{d}\mu^{*j}(g) > 0$$

for some n, so $\Gamma_\mu x$ intersects U. This is valid for any open neighborhood U of any y in $\operatorname{supp}\nu_X$, hence $\operatorname{supp}\nu_X$ is contained in $\overline{\Gamma_\mu x}$. $\square$

Corollary 2.6 $\Rightarrow$ (Theorem 2.3 when Γ is finitely generated). Suppose that Γ is a Zariski dense subsemigroup of G generated by $\gamma_1, \cdots, \gamma_n$. Consider the Borel probability measure

$$\mu = \frac{1}{n}(\delta_{\gamma_1} + \cdots + \delta_{\gamma_n}).$$

The closed subsemigroup Γ_μ of G generated by the support of μ is precisely $\overline{\Gamma}$. Then Γ_μ and Γ have the same invariant closed subsets of X, and $\overline{\Gamma x} = \overline{\overline{\Gamma} x}$, so $\operatorname{supp} \nu_X$ is the only Γ-invariant minimal closed subset of X. But we know by Lemma 2.9 that

$$\operatorname{supp} \nu_X = \pi^{-1}(\operatorname{supp} \nu_{\mathbb{P}^*}) = \pi^{-1}(\Lambda^{\perp}_{\Gamma_\mu}),$$

and from the definition of limit set we easily get that $\Lambda^1_{D(\Gamma_\mu)} = \Lambda^1_{D(\overline{\Gamma})} = \Lambda^1_{D(\Gamma)}$, and by duality $\Lambda^{\perp}_{\Gamma_\mu} = \Lambda^{\perp}_{\Gamma}$. $\square$

To end this section we discuss how the rest of the work is organized. In Section 2.2 we present the start of the proof of Theorem 2.4, which consists in showing that the μ-stationary Borel probability measure ν_X is characterized by the invariance of its limit measures with respect to an equivariant family 1-parameter unipotent groups $(U_b)_{b\in B}$. In section 2.3 we give a complete proof of Theorem 2.3 using the results of recurrence of random walks that we introduced in the last section of Chapter 1. More precisely, we will show that the random walk on X associated to a Zariski dense Borel probability measure μ is recurrent by constructing a proper continuous function $u : X \to [0, \infty)$ satisfying the contraction hypothesis for a power μ^{*n_0} of μ.

2.2 First steps of the proof Theorem 2.4

We will now introduce notation that we will use in this chapter. Let P be the subgroup of upper-triangular matrices of G,

$$P = \left\{ \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} : abc = 1 \right\}. \tag{2.5}$$

The homogeneous space G/P is called the **flag variety** of G, and it is denoted by $\mathscr{P}$. Let us explain the terminology. A *flag* $\mathcal{F}$ on a vector space V is nested sequence of subspaces of V:

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l. \tag{2.6}$$

We say that $\mathcal{F}$ is a maximal flag if $l = \dim V$. The action of G on $\mathbb{R}^3$ gives rise to a transitive action on the set $\mathscr{F}_m$ of maximal flags of $\mathbb{R}^3$, and the stabilizer of the flag

$$\mathcal{F}_0 : \{0\} \subsetneq W_1 \subsetneq W_2 \subsetneq \mathbb{R}^3, \tag{2.7}$$

where $W_1 = \langle e_1 \rangle$ and $W_2 = \langle e_1, e_2 \rangle$, is precisely P. Thus the orbital map of $\mathcal{F}$ induces a bijection between $\mathscr{P}$ and $\mathscr{F}_m$. In the sequel we identify both spaces. Since a maximal flag of $\mathbb{R}^3$ like in (2.6) is determined by V_1 and V_2, tha flag variety $\mathscr{P}$ can also be thought as a subset of the product $\mathbb{P}(\mathbb{R}^3) \times \mathbb{P}^*(\mathbb{R}^3)$.

Propositon 2.7. *Let μ be a Zariski dense Borel probability measure on G. There is only one μ-stationary probability measure on each of the spaces $\mathbb{P}(\mathbb{R}^3), \mathbb{P}^*(\mathbb{R}^3)$, and $\mathscr{P} = G/P$. Moreover, each of these measures is μ-proximal.*

Proof. The semigroup Γ_μ is srongly irreducible by Lemma 1.15, and it is proximal by Theorem 1.18 because $SL(3,\mathbb{R})$ is proximal. Corollary 1.21 then implies that there is only one μ-stationary probability measure $\nu_{\mathbb{P}}$ on $\mathbb{P}(\mathbb{R}^3)$, and that it is μ-proximal. We will denote by $\xi : B \to \mathbb{P}(\mathbb{R}^3)$ its boundary map.

Recall that $D : GL(\mathbb{R}^3) \to GL((\mathbb{R}^3)^*)$ is the map $g \mapsto (g^*)^{-1}$. By Lemma 1.22 (iv), $D(\Gamma_\mu) = \Gamma_{D_*\mu}$ is strongly irreducible. Also, $\Gamma_{D_*\mu}$ is Zariski dense in $SL((\mathbb{R}^3)^*)$ because the restriction $D : SL(\mathbb{R}^3) \to SL((\mathbb{R}^3)^*)$ is an homeomorphism with respect to the Zariski topology (matrix inversion is a polynomial map for matrices with determinant 1). So, $\Gamma_{D_*\mu}$ is proximal by Theorem 1.18. Applying once more Corollary 1.21 and using the duality to come back to $\mathbb{R}^3$, we conclude that there is a unique μ-stationary probability measure $\nu_{\mathbb{P}^*}$ on $\mathbb{P}^*(\mathbb{R}^3)$, and additionaly it is μ-proximal. Let $\xi^* : B \to \mathbb{P}(\mathbb{R}^3)$ be the boundary map of $\nu_{\mathbb{P}^*}$.

Finally we will prove the statement on the flag variety $\mathscr{P}$ with the help of ξ and ξ^*. Since $\mathscr{P}$ is compact, by Proposition 1.5 we can consider a μ-stationary probability measure on $\mathscr{P}$. The projections

$$\mathbb{P}(\mathbb{R}^3) \xleftarrow{p_1} \mathscr{P} \xrightarrow{p_2} \mathbb{P}^*(\mathbb{R}^3)$$

are G-equivariant, hence the image of ν under these is μ-stationary, and the only possibility is

$$(p_1)_*\nu = \nu_{\mathbb{P}}, \quad \text{and} \quad (p_2)_*\nu = \nu_{\mathbb{P}^*}.$$

Looking at the limit measures we deduce that β-almost surely we have the equalities

$$(p_1)_*\nu_b = (\nu_{\mathbb{P}})_b = \delta_{\xi(b)}, \quad \text{and} \quad (p_2)_*\nu_b = (\nu_{\mathbb{P}^*})_b = \delta_{\xi^*(b)}.$$

This means that $\nu_b = \delta_{(\xi(b),\xi^*(b))}$ because the sets $p_1^{-1}(\xi(b))$ and $p_2^{-1}(\xi(b))$ have full ν_b-measure and the only possible point in its intersection is $(\xi(b), \xi^*(b))$ (in particular $\xi(b)$ is contained in $\xi^*(b)$ β-almost surely). Let us denote by ζ the map $(\xi, \xi^*) : B \to \mathscr{P}$. We have shown that the only μ-stationary measure on $\mathscr{P}$ is $\zeta_*\beta$ □

Corollary 2.8. *Let μ be a Zariski dense Borel probability measure on G, and let ν be a μ-stationary probability measure on X. For β-almost any $b \in B$, the limit measures ν_b are supported in $\pi^{-1}(\xi^*(b))$.*

Proof. The map $\pi : X \to \mathbb{P}^*(\mathbb{R}^3)$ is G-equivariant, hence for β-almost any b we have

$$\pi_*(\nu_b) = (\nu_{\mathbb{P}^*})_b = \delta_{\xi^*(b)},$$

which implies that

$$\nu_b(\pi^{-1}(\xi^*(b))) = \delta_{\xi^*(b))}(\{\xi^*(b)\}) = 1.$$

□

We prove here the auxiliary lemma that we used to see that Corollary 2.6⇒ Theorem 2.3.

Lemma 2.9. *Let μ be a Zariski dense Borel probability measure of G. We have the following equalities:*

- (i) *supp* $\nu_{\mathbb{P}^*} = \Lambda^{\perp}_{\Gamma_\mu}$,
- (ii) *supp* $\nu_X = \pi^{-1}($*supp* $\nu_{\mathbb{P}^*})$.

Proof. First we prove (i). Since $D(\Gamma_\mu)$ is proximal and irreducible, the limit set $\Lambda^1_{D(\Gamma_\mu)}$ is the unique $D(\Gamma_\mu)$-invariant minimal closed subset of $\mathbb{P}((\mathbb{R}^3)^*)$, so $\Lambda^{\perp}_{\Gamma_\mu} = (\Lambda^1_{D(\Gamma_\mu)})^{\perp}$ is the unique Γ_μ invariant minimal closed subset of $\mathbb{P}^*(\mathbb{R}^3)$. The support of $\nu_{\mathbb{P}^*}$ is Γ_μ invariant by Lemma 1.9, hence it contains

$\Lambda^{\perp}_{\Gamma_\mu}$. To prove that $\operatorname{supp}\nu_{\mathbb{P}^*}$ is contained in $\Lambda^{\perp}_{\Gamma_\mu}$, we consider a point $x \in \Lambda^{\perp}_{\Gamma_\mu}$. The sequence of probability measures

$$\nu_n = \frac{1}{n}\sum_{j=1}^{n} \mu^{*j} * \delta_x$$

is weakly compact because $\mathbb{P}^*(\mathbb{R}^3)$ is compact, and any cluster point is μ-stationary by Lemma 1.4. Then (ν_n) converges weakly to $\nu_{\mathbb{P}^*}$, the unique μ-stationary Borel probability measure. This implies that $\operatorname{supp}\nu_X$ is contained in $\overline{\Gamma_\mu x} = \Lambda^{\perp}_{\Gamma_\mu}$.

Now we prove (ii). Since $\pi_*(\nu_X) = \nu_{\mathbb{P}^*}$, we have

$$\nu_X(\pi^{-1}(\operatorname{supp}\nu_{\mathbb{P}^*})) = \nu_{\mathbb{P}^*}(\operatorname{supp}\nu_{\mathbb{P}^*}) = 1,$$

which implies that $\operatorname{supp}\nu_X \subseteq \pi^{-1}(\operatorname{supp}\nu_{\mathbb{P}^*})$. Now we take $x \in \pi^{-1}(\operatorname{supp}\nu_{\mathbb{P}^*})$ and denote $W_0 = \pi(x)$. To prove that x is in $\operatorname{supp}\nu_X$ is the same as showing that $\nu_X(\varphi) > 0$ for any non-negative function $\varphi \in \mathscr{C}_c(X)$ taking a positive value on x. For any such φ, $m_{W_0}(\varphi|_{\pi^{-1}(W_0)}) > 0$. The map $\overline{\varphi} : \mathbb{P}^*(\mathbb{R}^3) \to \mathbb{R}$ given by

$$\overline{\varphi}(W) = m_W(\varphi|_{\pi^{-1}(W)})$$

is non-negative and continuous, and takes a positive value at $W_0 \in \operatorname{supp}\nu_{\mathbb{P}^*}$, hence $\nu_{\mathbb{P}^*}(\overline{\varphi}) > 0$. By definition of ν_X we have that $\nu_X(\varphi) = \nu_{\mathbb{P}^*}(\overline{\varphi})$, which concludes the proof. $\square$

We will now define the equivariant family of 1-parameter unipotent subgroups $(U_b)_{b\in B}$.

Let G_0 be the stabilizer in G of the plane $W_0 = \langle e_1, e_2\rangle$.

$$G_0 = \left\{ \begin{pmatrix} a & b & * \\ c & d & * \\ 0 & 0 & e \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}), e \in \mathbb{R}^\times \right\} \tag{2.8}$$

Let $L : \mathbb{R}^2 \to W_0$ the inclusion of $\mathbb{R}^2$ in the first two coordinates of $\mathbb{R}^3$ and consider the homeomorphism $\vartheta_L : SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \to \pi^{-1}(W_0)$ defined in (2.3). The action of G_0 on $\pi^{-1}(W_0)$ read through ϑ_L is the following: Let $p : G_0 \to GL(2,\mathbb{R})$ be the projection

$$\begin{pmatrix} a & b & * \\ c & d & * \\ 0 & 0 & e \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.9}$$

The diagram

$$\begin{tikzcd} \pi^{-1}(W_0) \arrow[r, "\phi"] & \pi^{-1}(W_0) \\ SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \arrow[u, "\vartheta_L"] \arrow[r, "q(g)"] & SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \arrow[u, "\vartheta_L"'] \end{tikzcd} \tag{2.10}$$

is commutative, where $q(g) = \frac{p(g)}{|\det p(g)|^{1/2}}$. For example, the action of the group

$$L_0 = \left\{ \begin{pmatrix} a & * & * \\ 0 & a & * \\ 0 & 0 & a^{-2} \end{pmatrix} : a \in \mathbb{R}^\times \right\} \tag{2.11}$$

on $\pi^{-1}(W_0)$ corresponds to the action on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of the group of unipotent matrices

$$U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}. \tag{2.12}$$

Notice that the group L_0 is 4-dimensional, way bigger than U. This difference is due to the fact that L_0 contains a lot of elements that act trivially on $\pi^{-1}(W_0)$, they form the normal subgroup of L_0:

$$R_0 = \left\{ \begin{pmatrix} a & 0 & * \\ 0 & a & * \\ 0 & 0 & a^{-2} \end{pmatrix} : a \in \mathbb{R}^\times \right\}. \tag{2.13}$$

Then, the action of the 1-dimensional quotient group $U_0 := L_0/R_0$ on $\pi^{-1}(W_0)$ corresponds to the action of U on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$. Observe that both L_0 and R_0 are normalized by P, so any flag $\mathcal{F} = gP$ defines a pair of subgroups of G

$$L_{gP} = gL_0g^{-1}, \quad \text{and} \quad R_{gP} = gR_0g^{-1}$$

We define also $U_{gP} = L_{gP}/R_{gP}$. Since Ł$_0$ fixes $\mathcal{F}_0$ (the canonical flag (2.7)), then $L_{\mathcal{F}}$ fixes the flag $\mathcal{F}$, in particular $L_{\mathcal{F}}$ and $U_{\mathcal{F}}$ act on $\pi^{-1}(p_2(\mathcal{F}))$. We will write U_b instead of $U_{\zeta(b)}$, where ζ is the boundary map $B \to \mathscr{P}$. The groups U_b are defined for β-almost surely and they satisfy

$$U_b = b_0 U_{Sb} b_0^{-1}$$

for β-almost any $b \in B$. We remark (once more) that the action of U_b on the fiber $\pi^{-1}(\xi^*(b))$ corresponds to the action of U on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.

Propositon 2.10. *Let μ be a Zariski dense Borel probability measure of $SL(3,\mathbb{R})$ with finite first moment and let ν be a μ-stationary, μ-ergodic Borel probability measure on X. If ν_b is U_b-invariant for β-almost any $b \in B$, then $\nu = \nu_X$.*

For the proof we need a lemma which is very interesting in its own right. Recall that $\xi : B \to \mathbb{P}(\mathbb{R}^3)$ and $\xi^* : B \to \mathbb{P}^*(\mathbb{R}^3)$ are defined for β-almost any $b \in B$ as:

$$\xi(b) = (\nu_{\mathbb{P}})_b \quad \text{and} \quad \xi^*(b) = (\nu_{\mathbb{P}^*})_b,$$

where $\nu_{\mathbb{P}}$ and $\nu_{\mathbb{P}^*}$ are the unique μ-stationary Borel probability measures on $\mathbb{P}(\mathbb{R}^3)$ and $\mathbb{P}^*(\mathbb{R}^3)$, respectively (which are also μ-proximal according to Proposition 2.7).

Lemma 2.11. *Let μ be a Zariski dense Borel probability measure of G with finite first moment.*

(i) For β-almost any $b \in B$ we have that, for any non-zero $v \in \xi(b)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{||b_{n-1}^{-1} \cdots b_0^{-1} v||}{||v||} = -\lambda_{1,\mu}.$$

(ii) For β-almost any $b \in B$ we have that, for any non-zero $w \in \wedge^2 \xi^*(b)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{|| \wedge^2 (b_{n-1}^{-1} \cdots b_0^{-1}) w||}{||w||} = -\lambda_{1,\mu} - \lambda_{2,\mu}.$$

Proof. We begin by proving (i). The idea of the proof is to interpret the quantities

$$\frac{1}{n} \log \frac{||b_{n-1}^{-1} \cdots b_0^{-1} v||}{||v||}$$

as the Birkhoff sums of a suitable map $B \to \mathbb{R}$.

Consider a measurable subset E of B such that $\beta(E) = 1$, and such that, for any $b \in E$ and any $n \geq 1$, we have

$$\xi(b) = b_0 \cdots b_{n-1}\xi(S^n b).$$

Let $\sigma : G \times \mathbb{P}(\mathbb{R}^3) \to \mathbb{R}$ be the norm cocycle, that is

$$\sigma(g, x) = \log \frac{||gv||}{||v||},$$

for any non-zero $v \in x$. If $b \in E$ and we take a non-zero $v \in \xi(b)$, then

$$\log \frac{||b_{n-1}^{-1} \cdots b_0^{-1} v||}{||v||} = -\log \frac{||b_0 \cdots b_{n-1}(b_{n-1}^{-1} \cdots b_0^{-1} v)||}{||b_{n-1}^{-1} \cdots b_0^{-1} v||} = -\sigma(b_0 \cdots b_{n-1}, \xi(S^n b)). \tag{2.14}$$

We define the map $\Theta : B \to \mathbb{R}$ by

$$\Theta(b) = \sigma(b_0, \xi(Sb)).$$

If we take $b \in E$, then

$$\Theta(b) + \Theta(Sb) = \sigma(b_0, \xi(Sb)) + \sigma(b_1, \xi(S^2 b)) = \sigma(b_0, b_1\xi(S^2 b)) + \sigma(b_1, \xi(S^2 b)) = \sigma(b_0 b_1, \xi(S^2 b)),$$

and by an inductive argument we get that

$$\Theta(b) + \cdots + \Theta(S^{n-1} b) = \sigma(b_0 \cdots b_{n-1}, \xi(S^n b)). \tag{2.15}$$

Now we calculate the integral of Θ. Observe that Γ_μ is irreducible because it is Zariski dense in G and G is irreducible, and by assumption μ has finite first moment, so Theorem 1.23 applies. We use it to deduce:

$$\begin{aligned} \int_B \Theta \mathrm{d}\beta &= \int_G \int_B \sigma(g, \xi(b'))\mathrm{d}\beta(b')\mathrm{d}\mu(g) = \int_G \int_{\mathbb{P}(\mathbb{R}^3)} \sigma(g, x)\mathrm{d}\xi_*\beta(x)\mathrm{d}\mu(g) \\ &= \int_G \int_{\mathbb{P}(\mathbb{R}^3)} \sigma(g, x)\mathrm{d}\nu_{\mathbb{P}}(x)\mathrm{d}\mu(g) = \lambda_{1,\mu}. \end{aligned} \tag{2.16}$$

Since the shift S is β-ergodic, by Birkhoff's Theorem we know that for β-almost any $b \in E$ we have

$$\lim_{n\to\infty} \frac{1}{n}(\Theta(b) + \cdots + \Theta(S^{n-1} b)) = \int_B \Theta \mathrm{d}\beta = \lambda_{1,\mu}. \tag{2.17}$$

Combining 2.14, 2.15 and 2.17 we get that for β-almost any $b \in E$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{||b_{n-1}^{-1} \cdots b_0^{-1} v||}{||v||} = \lim_{n\to\infty} -\frac{1}{n}\sigma(b_0 \cdots b_{n-1}, \xi(S^n b)) = \lim_{n\to\infty} -\frac{1}{n}(\Theta(b) + \cdots + \Theta(S^{n-1} b)) = -\lambda_{1,\mu},$$

for any $v \in \xi(b)$, which concludes the proof of (i).

The proof of (ii) follows the same lines. As usual, we let G act on $\wedge^2\mathbb{R}^3$ via the map $\wedge^2$. Consider the norm cocycle $\sigma' : G \times \mathbb{P}(\wedge^2\mathbb{R}^3) \to \mathbb{R}$. The natural map $\alpha : \mathbb{P}^*(\mathbb{R}^3) \to \mathbb{P}(\wedge^2\mathbb{R}^3)$ given by

$$W = \langle v_1, v_2 \rangle \mapsto \mathbb{R}(v_1 \wedge v_2)$$

is a G-equivariant homeomorphism, so $\zeta' = \alpha \circ \xi^* : B \to \mathbb{P}(\wedge^2\mathbb{R}^3)$ is an almost sure G-equivariant map, and hence the probability measure $\nu' = (\zeta')_*\beta$ is μ-stationary. This time we consider the (β-almost surely defined) map $\Theta' : B \to \mathbb{R}$ given by

$$\Theta'(b) = \sigma'(b_0, \zeta'(Sb)).$$

Let E' be a measurable subset of B such that $\beta(E') = 1$, and such that, for any $b \in E'$ and any $n \geq 1$, we have

$$\xi^*(b) = b_0 \cdots b_{n-1}\xi^*(S^n b).$$

If $b \in E'$ and w is a non-zero element of $\wedge^2\xi^*(b)$, then

$$\frac{1}{n}\log\frac{|| \wedge^2 (b_{n-1}^{-1} \cdots b_0^{-1})w||}{||w||} = -\frac{1}{n}(\Theta'(b) + \cdots + \Theta'(S^{n-1}b)). \tag{2.18}$$

Now we calculate $\int_B \Theta' \mathrm{d}\beta$ using Theorem 1.25, which applies because μ satisfies property $\mathrm{Irr}(\mu)$ by Lemma 1.26 and it has finite first moment.

$$\int_B \Theta' \mathrm{d}\beta = \int_G \int_B \sigma'(g, \zeta'(b'))\mathrm{d}\beta(b')\mathrm{d}\mu(g) = \int_G \int_{\mathbb{P}(\wedge^2\mathbb{R}^3)} \sigma'(g, x')\mathrm{d}\nu'(x')\mathrm{d}\mu(g') = \lambda_{1,\mu} + \lambda_{2,\mu}.$$

Using 2.18 and Birkhoff's Theorem we conclude that almost any $b \in E'$ satisfies that, for any non-zero $w \in \wedge^2\xi^*(b)$,

$$\lim_{n\to\infty} \frac{1}{n}\log\frac{|| \wedge^2 (b_{n-1}^{-1} \cdots b_0^{-1})w||}{||w||} = -\int_B \Theta' \mathrm{d}\beta = -\lambda_{1,\mu} - \lambda_{2,\mu}.$$

□

Proof of Proposition 2.10. By Corollary 2.8 we know that β-almost surely, ν_b is supported on $\pi^{-1}(\xi^*(b))$. By Ratner's Theorem REF, the U_b-invariant and U_b-ergodic measures on $\pi^{-1}(\xi^*(b))$ are either $m_{\xi^*(b)}$, or supported in a closed U_b-orbit. Hence, if ν_b is U_b-invariant, we can express it as a convex combination

$$\nu_b = t_b m_{\xi^*(b)} + (1 - t_b)\widetilde{\nu_b},$$

where $(1 - t_b)\widetilde{\nu_b}$ is supported in a (possibly infinite) union of closed U_b-orbits. By the definition of the U_b's, we have that β-almost surely,

$$U_b = b_0 U_{Sb} b_0^{-1},$$

so $b_0 : \pi^{-1}(\xi^*(Sb)) \to \pi^{-1}(\xi^*(b))$ sends closed U_{Sb}-orbits to closed U_b-orbits, and then $(b_0)_*\widetilde{\nu_{Sb}}$ is also supported in a union of closed U_b-orbits. When $(b_0)_*\nu_{Sb} = \nu_b$, we obtain

$$\begin{aligned} t_b m_{\xi^*(b)} + (1 - t_b)\widetilde{\nu_b} = \nu_b = (b_0)_*\nu_{Sb} &= t_{Sb}(b_0)_* m_{\xi^*(Sb)} + (1 - t_{Sb})(b_0)_*\widetilde{\nu_{Sb}} \\ &= t_{Sb} m_{\xi^*(b)} + (1 - t_{Sb})(b_0)_*\widetilde{\nu_{Sb}}, \end{aligned}$$

and hence $t_b = t_{Sb}$ for β-almost any $b \in B$. Since the shift S is β-ergodic, then t_b is equal to a constant a β-almost surely, thus

$$\nu = \int_B \nu_b \mathrm{d}\beta(b) = a\int_B m_{\xi^*(b)}\mathrm{d}\beta(b) + (1 - a)\int_B \widetilde{\nu_b}\mathrm{d}\beta(b) = a\nu_X + (1 - a)\widetilde{\nu}. \tag{2.19}$$

We need to show that $a = 1$. If this is not the case, then

$$\widetilde{\nu_b} = \frac{1}{1 - a}(\nu_b - a m_{\xi^*(b)})$$

for β-almost any $b \in B$, and since the measures ν_b and $m_{\xi^*(b)}$ are β-almost surely equivariant, the $\widetilde{\nu_b}$'s are as well. In turn, this implies that $\widetilde{\nu}$ is μ-stationary. Since ν is μ-stationary and μ-ergodic, from 2.19 we conclude that $a = 0$, so for β-almost any $b \in B$, the support of ν_b is contained the union of U_b-periodic orbits of $\pi^{-1}(\xi^*(b))$, which can also be written as

$$D_b = \{[\Lambda] \in X \mid \Lambda \subseteq \xi^*(b), \Lambda \cap \xi(b) \neq \{0\}\}.$$

Suppose that E is a measurable subset of B such that $\beta(E) = 1$, ξ and ξ^* are defined in E, for any $b \in E$ the equalities

$$b_0 \cdots b_{n-1}\xi(S^n b) = \xi(b) \quad \text{and} \quad b_0 \cdots b_{n-1}\xi^*(S^n b) = \xi^*(b)$$

hold for any $n \geq 1$, and such that both conclutions of Lemma 2.11 hold for any $b \in E$. Let us consider the *backwards dynamical system* $(B \times X, \beta^X, T)$, where β^X is the Borel probability measure

$$\int_B \delta_b \otimes \nu_b \mathrm{d}\beta(b),$$

and $T(b, x) = (Sb, b_0^{-1}x)$. The map T preserves the probability measure β^X (see BQ Proposition 2.23). The subset of $B \times X$ defined by

$$C = \{(b, x) \mid b \in E, x \in D_b\}$$

has full β^X-measure. For any $z = (b, [\Lambda]) \in C$, let $v_z(\Lambda)$ be a generator of $\Lambda \cap \xi(b)$ and let $w_z(\Lambda)$ be the wedge product of a basis of Λ. The map $f : C \to \mathbb{R}$ given by

$$f(z) = \log ||v_z(\Lambda)|| - \frac{1}{2}\log|\Lambda| = \log ||v_z(\Lambda)|| - \frac{1}{2}\log||w_z(\Lambda)||$$

is well defined (it does not depend on the 2-lattice Λ, just in its homothecy class). If $z = (b, [\Lambda]) \in C$, the linear map b_0^{-1} sends ξ^* to $\xi^*(Sb)$, and $\xi^(b)$ to $\xi^(Sb)$, so $b_0^{-1}v_z(\Lambda)$ generates $b_0^{-1}\Lambda \cap \xi(Sb)$. This implies that

$$f(Tz) = f((Sb, [b_0^{-1}\Lambda])) = \log ||b_0^{-1}v_z(\Lambda)|| - \frac{1}{2}\log|| \wedge^2 (b_0^{-1})w_z(\Lambda)||,$$

and by induction

$$f(T^n z) = \log ||b_{n-1}^{-1} \cdots b_0^{-1}v_z(\Lambda)|| - \frac{1}{2}\log|| \wedge^2 (b_{n-1}^{-1} \cdots b_0^{-1})w_z(\Lambda)||,$$

for any $z \in C$ and for any $n \geq 1$. Lemma 2.11 tells us that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} f(T^n z) &= \lim_{n\to\infty}\left(\frac{1}{n}\log ||b_{n-1}^{-1} \cdots b_0^{-1}v_z(\Lambda)|| - \frac{1}{2}\cdot\frac{1}{n}\log|| \wedge^2 (b_{n-1}^{-1} \cdots b_0^{-1})w_z(\Lambda)||\right) \\ &= -\lambda_{1,\mu} + \frac{1}{2}(\lambda_{1,\mu} + \lambda_{2,\mu}) = \frac{1}{2}(\lambda_{2,\mu} - \lambda_{1,\mu}),\end{aligned}$$

but $\lambda_{2,\mu} - \lambda_{1,\mu} < 0$ by Theorem 1.28, thus

$$\lim_{n\to\infty} f(T^n z) = -\infty \tag{2.20}$$

for any $z \in C$. Let us show that this is not possible: we can pick a constant $R > -\infty$ such that $\beta^X(F_R) > 0$, where

$$F_R = \{z \in C \mid f(z) \geq R\}.$$

Since the Borel map $T : B \times X \to B \times X$ preserves β^X, by Poincare's Recurrence Theorem, the T-orbit of almost any $z \in F_R$ returns to F_R infinitely many often, which contradicts (2.20). $\square$

2.3 Proof of Theorem 2.5

If Λ is a 2-lattice of $\mathbb{R}^3$, its systole $\alpha_1(\Lambda)$ is the norm of the shortest non-zero vector of Λ. Observe that the quantity $\frac{|\Lambda|^{1/2}}{\alpha_1(\Lambda)}$ does not change if we multiply Λ by a non-zero real number. Then the map $u : X \to \mathbb{R}$ given by

$$u([\Lambda]) = \frac{|\Lambda|^{1/2}}{\alpha_1(\Lambda)}$$

is well-defined. It is also continuous because the covolume and the systole depend continuoulsy on the 2-lattice. Our objective in this section is to show that, for δ small enough, u^δ is a proper function that satisfies the contraction hypothesis for a power μ^{n_0} of μ. We begin by proving that u is proper.

Lemma 2.12. *The map $u : X \to [0, \infty)$ is proper.*

Proof. Let $\mathcal{L}_2^1$ be the space of covolume 1 lattices of $\mathbb{R}^2$. By Mahler's Compactness Criterion, the map $\alpha_1^{-1} : \mathcal{L}_2^1 \to [0, \infty)$ is proper. Consider a plane W in $\mathbb{R}^3$, a linear isomorphism $L : \mathbb{R}^2 \to W$ and the homeomorphism $\vartheta_L : \mathcal{L}_2^1 \to \pi^{-1}(W)$ given by

$$\Delta \mapsto [L\Delta].$$

The diagram

$$\begin{array}{ccc} \mathcal{L}_2^1 & \xrightarrow{\alpha_1^{-1}} & \mathcal{L}_2^1 \\ \downarrow{\scriptstyle \vartheta_L} & & \downarrow{\scriptstyle \vartheta_L} \\ \pi^{-1}(W) & \xrightarrow{u} & \pi^{-1}(W) \end{array} \tag{2.21}$$

commutes, so

$$\{u \le M\} \cap \pi^{-1}(W)$$

is compact for any $M > 0$.

The space X is a fiber bundle with fiber $\mathcal{L}_2^1$ and projection $\pi : X \to \mathbb{P}^*(\mathbb{R}^3)$. Then, for any $W \in \mathbb{P}^*(\mathbb{R}^3)$, there exists a compact neighborhood K of W in $\mathbb{P}^*(\mathbb{R}^3)$ and a local trivialization of X in which we see $\pi^{-1}(K) \cap \{u \le M\}$ as

$$K \times \{\Delta \in \mathcal{L}_2^1 \mid \alpha_1^{-1}(\Delta) \le M\}, \tag{2.22}$$

which is a compact set. Since $\mathbb{P}^*(\mathbb{R}^3)$ is also compact, we may express $\{u \le M\}$ as a finite union of subsets of the form (2.22). This implies that $\{u \le M\}$ is compact. □

Lemma 2.13. *Let μ be a Zariski dense probability measure on G with finite first moment. For any $M > 0$, there exists a natural number $n_0 = n_0(M)$ such that any non-zero vectors $v \in \mathbb{R}^3$ and $w \in \wedge^2\mathbb{R}^3$ verify the inequality*

$$\int_G \log\left(\frac{||gv||}{||v||} \cdot \frac{||w||^{1/2}}{||\wedge^2 gw||^{1/2}}\right) d\mu^{*n_0} > M. \tag{2.23}$$

Proof. The probability measure μ has finite first moment and it satisfies property $\mathrm{Irr}(\mu)$ by Lemma 1.26. The seconf version of the Law of Large Numbers (Theorem 1.25) guarantees that, uniformly for v and w,

$$\frac{1}{n}\int_G \log\frac{||gv||}{||v||}\mathrm{d}\mu^{*n}(g) \to \lambda_{1,\mu} \quad \text{and} \quad \frac{1}{n}\int_G \log\frac{||\wedge^2 gw||}{||w||}\mathrm{d}\mu^{*n}(g) \to \lambda_{1,\mu} + \lambda_{2,\mu} \tag{2.24}$$

Combining (2.24) and (**??**) we obtain that, uniformly for v and w,

$$\frac{1}{n}\int_G \log\left(\frac{||gv||}{||v||} \cdot \frac{||w||^{1/2}}{||\wedge^2 gw||^{1/2}}\right) \mathrm{d}\mu^{*n} \to \lambda_{1,\mu} - \frac{1}{2}(\lambda_{1,\mu} - \lambda_{2,\mu}).$$

Since $\lambda_{1,\mu} - \lambda_{2,\mu}$ is positive by Theorem 1.28, the conclusion follows. □

Propositon 2.14. *Let μ be a Zariski dense probability measure on G with compact support. There exists a positive integer n_0 and a real number $\delta_0 > 0$ verifying the following property: For any $\delta \in (0, \delta_0)$, there is a constant $a = a(\delta, n_0)$ in the interval $(0, 1)$ such that, for any non-zero vectors $v \in \mathbb{R}^3$ and $w \in \wedge^2\mathbb{R}^3$, one has*

$$\int_G \left(\frac{||\wedge^2 gw||^{1/2}}{||gv||}\right)^\delta d\mu^{*n_0}(g) \le a \left(\frac{||w||^{1/2}}{||v||}\right)^\delta \tag{2.25}$$

Proof. The result will be obtained by looking at the expansion of order 2 of the map

$$\delta \mapsto \left(\frac{||\wedge^2 gw||^{1/2}}{||w||^{1/2}} \cdot \frac{||v||}{||gv||} \right)^{\delta}.$$

The main ingredient of the proof is Lemma 2.13. We begin by introducing some notation and establishing some auxiliary inequalities.

Denote by $\sigma_1 : G \times (\mathbb{R}^3 \setminus \{0\}) \to \mathbb{R}$ and $\sigma_2 : G \times (\wedge^2\mathbb{R}^3 \setminus \{0\}) \to \mathbb{R}$ the cocycles

$$\sigma_1(g, v) = \log \frac{||gv||}{||v||}, \quad \sigma_2(g, w) = \log \frac{||\wedge^2 gw||}{||w||},$$

and consider the map $\tau : G \times (\mathbb{R}^3 \setminus \{0\}) \times (\wedge^2\mathbb{R}^3 \setminus \{0\}) \to \mathbb{R}$ given by

$$\tau = \sigma_1 - \frac{1}{2}\sigma_2.$$

Let n_0 be a positive integer such that

$$\int_G \tau(g, v, w) \mathrm{d}\mu^{*n_0}(g) \geq 1 \tag{2.26}$$

for any non-zero $v \in \mathbb{R}^3$ and $w \in \wedge^2\mathbb{R}^3$, whose existence is guaranteed by Lemma 2.13. Recall that we use the notation

$$N(h) = \max\{||h||, ||h^{-1}||\}$$

for any $h \in GL(V)$, where V is a normed vector space. Observe that

$$\begin{aligned} |\sigma_1(g, v)| &= \max \left\{ \log \frac{||gv||}{||v||}, \log \frac{||v||}{||gv||} = \log \frac{||g^{-1}gv||}{||gv||} \right\} \\ &\leq \max\{||g||, ||g^{-1}||\} = \log N(g), \end{aligned} \tag{2.27}$$

and similarly

$$|\sigma_2(g, w)| \leq \log N(\wedge^2 g). \tag{2.28}$$

We obtain an upper bound for τ^2 using (2.27) and (2.28):

$$\begin{aligned} \tau(g, v, w)^2 &\leq |\sigma_1(g, v)|^2 + |\sigma_1(g, v)| \cdot |\sigma_2(g, w)| + \frac{1}{4}|\sigma_2(g, w)|^2 \\ &\leq (\log N(g))^2 + \log N(g) \log N(\wedge^2 g) + \frac{1}{4} \log N(\wedge^2 g))^2 =: \rho(g) \end{aligned} \tag{2.29}$$

This inequality will be useful because the upper bound $\rho(g)$ of $\tau(g, v, w)^2$ depends only on g. Since μ has compact support and ρ is continuous, the integral

$$C_{n_0} := \int_G e^{\rho(g)} \mathrm{d}\mu^{*n_0}(g)$$

is finite.

Now consider

$$\delta_0 = \min \left\{ 1, \left(\frac{2}{C_{n_0}} \right)^{1/2} \right\}, \tag{2.30}$$

and take any $\delta \in (0, \delta_0)$. We use the inequalities

$$e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}, \quad \text{and} \quad x^2 \leq e^{|x|},$$

that are valid for any real number x, to obtain

$$\begin{aligned}\left(\frac{||\wedge^2 gw||^{1/2}}{||w||^{1/2}} \cdot \frac{||v||}{||gv||}\right)^\delta &= \exp(-\delta\tau(g,v,w)) \leq 1 - \delta\tau(g,v,w) + \frac{1}{2}\delta^2\tau(g,v,w)^2\exp(\delta|\tau(g,v,w)|) \\ &\leq 1 - \delta\tau(g,v,w) + \frac{1}{2}\delta^2\exp(\delta_0\tau(g,v,w)^2) \\ &\leq 1 - \delta\tau(g,v,w) + \frac{1}{2}\delta^2\exp(\rho(g)), \end{aligned} \tag{2.31}$$

where we applied (2.29) in the last step. Integrating (2.31) with respect to μ^{*n_0} and applying (2.26) we obtain

$$\int_G \left(\frac{||\wedge^2 gw||^{1/2}}{||w||^{1/2}} \cdot \frac{||v||}{||gv||}\right)^\delta \mathrm{d}\mu^{*n_0}(g) \leq 1 - \delta + \frac{1}{2}\delta^2 C_{n_0} = 1 - \delta\left(1 - \frac{1}{2}\delta^2 C_{n_0}\right),$$

but

$$\frac{1}{2}\delta^2 C_{n_0} < \frac{1}{2}\delta_0^2 C_{n_0} \leq \frac{1}{2} \cdot \frac{2}{C_{n_0}} \cdot C_{n_0} = 1,$$

so $a(\delta, n_0) = 1 - \delta(1 - \frac{1}{2}\delta^2 C_{n_0})$ is strictly smaller that 1. □

Propositon 2.15. *Let μ be a Zariski dense probability measure on G with compact support. There exists a positive integer n_0 such that, for any sufficiently small $\delta > 0$, the proper map u^δ verifies the contraction hypothesis for μ^{*n_0}.*

Proof. Let n_0 and δ_0 be as in the conclusion of Proposition 2.14 We rename μ^{*n_0} as $\widetilde{\mu}$. Let M be a positive number, and consider the following partition of X in two pieces: the compact set $X^{\leq M} = \{u \leq M\}$ and $X^{>M} = \{u > M\}$. The map u is bounded in $X^{\leq M}$, so the additive constant b in the contraction hypothesis will take care of it. We have only to worry abour $X^{>M}$. The main idea of the proof is to realize that we can choose M large enough so that u^δ is contracted on $X^{>M}$ by $P_{\overline{\mu}}$ for any $\delta < \delta_0$ (i.e. $P_{\overline{\mu}}(u^\delta) \leq au^\delta$ for some $a \in (0,1)$).

Denote by A the support of $\widetilde{\mu}$ (which is compact because the support of μ is compact). Let us fix

$$M = \max\{(||g|| \cdot ||g^{-1}||)^{1/2} \mid g \in A\}. \tag{2.32}$$

For any 2-lattice Λ of $\mathbb{R}^3$, let $\{v_\Lambda, w_\Lambda\}$ be a basis of Λ such that $||v_\Lambda|| = \alpha_1(\Lambda)$. Any $w \in \Lambda$ that is not a multiple of v_Λ verifies the inequality

$$|\Lambda| \leq ||v_\Lambda \wedge w|| \leq ||v_\Lambda|| \cdot ||w||,$$

which we rewrite as

$$||w|| \geq \frac{|\Lambda|}{\alpha_1(\Lambda)}. \tag{2.33}$$

We claim that if $u([\Lambda]) > M$, then gv_Λ realizes $\alpha_1(g\Lambda)$ for any g in A. If $w \in \Lambda$ is not multiple of v_Λ, using (2.33) and then (2.32) we deduce that

$$\begin{aligned}||gw|| &= \frac{||g^{-1}|| \cdot ||gw||}{||g^{-1}||} \geq \frac{||w||}{||g^{-1}||} \geq \frac{|\Lambda|}{\alpha_1(\Lambda)} \cdot \frac{1}{||g^{-1}||} = \frac{|\Lambda|}{\alpha_1(\Lambda)^2} \cdot \frac{||v_\Lambda||}{||g^{-1}||} \\ &\geq \frac{M^2}{||g|| \cdot ||g^{-1}||}||g|| \cdot ||v_\Lambda|| \geq ||g|| \cdot ||v_\Lambda|| \geq ||gv_\Lambda||.\end{aligned}$$

In addition, by the choice of v_Λ, we know that $g\Lambda \cap \mathbb{R}gv_\Lambda = \mathbb{Z}gv_\Lambda$, hence there are no non-zero vectors in $g\Lambda$ shorter than gv_Λ. Consider any $\delta \in (0, \delta_0)$, and let $a = a(\delta, n_0)$ be as in Proposition 2.14. It $u([\Lambda]) > M$, then

$$(P_{\widetilde{\mu}} u^\delta)([\Lambda]) = \int_A u(g[\Lambda])^\delta \mathrm{d}\widetilde{\mu}(g) = \int_A \frac{|| \wedge^2 g(v_\Lambda \wedge w_\Lambda)||^{\delta/2}}{||gv_\Lambda||^\delta} \mathrm{d}\mu^{*n_0}(g) \le a \frac{||v_\Lambda \wedge w_\Lambda||^{\delta/2}}{||v_\Lambda||^\delta} = au^\delta([\Lambda]).$$

We are ready to conclude. Let b be the maximum value attained by the continuous function $P_{\widetilde{\mu}}(u^\delta)$ on the compact set $X^{\le M}$. By the choice of b, and since any $[\Lambda] \in X^{>M}$ verifies (2.34), we get that

$$P_{\widetilde{\mu}}(u^\delta) \le au^\delta + b.$$

□

Proof of Theorem 2.5. Proposition 2.15 guarantees the existence of a (continuous) proper Borel map $X \to (0, \infty)$ verifying the contraction hypothesis for a power μ^{*n_0} of μ, so Theorem 2.5 follows from Corollary 1.13. □

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