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# More than two fifths of the zeros of the Riemann zeta function are on the critical line

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## 1. Introduction

In this paper we show that at least 2/5 of the zeros of the Riemann zeta-function are simple and on the critical line. Our method is a refinement of the method Levinson [11] used when he showed that at least 1/3 of the zeros are on the critical line (and are simple, as observed by Heath-Brown [10] and, independently, by Selberg). The main new element here is the use of a mollifier of length $y=T^{\theta}$ with $\theta=4/7-\varepsilon$ whereas in Levinson's theorem the mollifier has $\theta=1/2-\varepsilon$. The work [6] of Deshouillers and Iwaniec on averages of Kloosterman sums is what allows us to use a longer mollifier. In fact, in their paper [7], they obtain an upper bound of essentially the right magnitude for an integral of the modulus squared of the zeta function multiplied by a mollifier of length $T^{4/7}$.

In order to obtain our result, we need asymptotic formulas; obtaining these involves technical but familiar details. In fact, this paper is essentially a synthesis of three papers: Balasubramanian, Conrey, and Heath-Brown [2], Conrey [3], and Deshouillers and Iwaniec [7].

The first paper has the analytic machinery which reduces the integral in question to a main term involving a sum of coefficients of the mollifier and an error term involving sums of incomplete Kloosterman sums; the second paper has the arithmetic machinery for giving an asymptotic formula for the main term; the third paper has the key lemma for bounding the error term.

We mention also that in [3], it is shown that at least 0.365 of the zeros are on the critical line. This paper uses a mollifier with $\theta=1/2-\varepsilon$ as in Levinson, but the coefficients are more elaborate. Levinson uses

$$b(n)=\mu(n)\left(\frac{\log y/n}{\log y}\right)$$

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whereas in [3] and here we will use

$$b(n, P) = \mu(n)\, P\left(\frac{\log y/n}{\log y}\right) \tag{1}$$

where $P$ is a polynomial with $P(0)=0$, $P(1)=1$ which can be chosen optimally by the calculus of variations at the end of the argument. Also in [3], we start from a somewhat more general situation than in Levinson [11]. Levinson's first step is to observe that if the proportion of zeros of

$$\zeta(s) + a(s)\,\zeta'(s)$$

to the right of the critical line is $\leqq p$, then the proportion of zeros of $\zeta(s)$ on the critical line is $\geqq 1-2p$. Here $a(s)$ is a simple function which is essentially $\left(\log \frac{s}{2\pi}\right)^{-1}$. In [3] we make the same observation about a more general combination

$$\sum a_n(s)\,\zeta^{(n)}(s)$$

with simple functions $a_n$ which can be chosen with a certain amount of freedom. Here we use this more general treatment, but we approach it in an easier way as outlined in [5].

## 2. Background material and statement of theorem

We recall some basic information about the Riemann zeta-function $\zeta(s)$, where $s=\sigma+it$ (see Titchmarsh [15]). It is defined for $\sigma>1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and has a meromorphic continuation to the whole plane with its only pole, a simple pole at $s=1$ with residue 1. It satisfies the functional equation

$$\xi(s) = \xi(1-s) \tag{2}$$

where the entire function $\xi(s)$ is defined by

$$\xi(s) = H(s)\,\zeta(s) \tag{3}$$

with

$$H(s) = (1/2)\, s(s-1)\,\pi^{-s/2}\,\Gamma(s/2). \tag{4}$$

In asymmetrical form, the functional equation is

(5) $$\zeta(s)=\chi(s)\,\zeta(1-s)$$

where, because of familiar properties of the $\Gamma$-function,

(6) $$\chi(s)^{-1}=\chi(1-s)=2(2\pi)^{-s}\,\Gamma(s)\cos\frac{\pi s}{2}.$$

Because of the Euler-product representation (for $\sigma>1$),

(7) $$\zeta(s)=\prod_p(1-p^{-s})^{-1},$$

$\zeta(s)$ has no zeros in $\sigma>1$. By (5) and (6) it is seen that $\zeta(s)$ has simple zeros at

$$s=-2,\ -4,\ -6,\dots$$

and nowhere else in $\sigma<0$. Hadamard and de la Vallée-Poussin showed, independently in 1885, that $\zeta(s)$ has no zeros on $\sigma=1$; hence all the non-real zeros of $\zeta(s)$ are in the critical strip $0<\sigma<1$. The zeros of $\zeta(s)$ in the critical strip are denoted by $\varrho=\beta+i\gamma$. Von Mangoldt proved that

(8) $$\begin{aligned}N(T)&=\#\{\varrho:0<\gamma<T\}\\&=\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T).\end{aligned}$$

If we let

(9) $$N_0(T)=\#\{\varrho:0<\gamma<T\quad\text{and}\quad\beta=1/2\},$$

then Riemann [13] conjectured that $N_0(T)=N(T)$ for all $T$; i.e., that all the zeros are on $\sigma=1/2$. Hardy was the first to show that $N_0(T)$ goes to infinity with $T$; later he and Littlewood showed [9] that $N_0(T)\gg T$. Selberg [14] was the first to prove that $N_0(T)\gg T\log T$ i.e., that

(10) $$\kappa=\liminf_{T\to\infty}\frac{N_0(T)}{N(T)}>0.$$

We call $\kappa$ the proportion of zeros on the critical line. Let now

(11) $$N_0^*(T)=\#\{\varrho:0<\gamma<T,\ \beta=1/2,\ \zeta'(\varrho)\neq 0\}$$

be the number of simple zeros on the critical line up to a height $T$, and let

(12) $$\kappa^*=\liminf_{T\to\infty}\frac{N_0^*(T)}{N(T)}.$$

Then the work of Levinson [11] implies that

$$\kappa^* \geqq 0.3474. \tag{13}$$

In Conrey [3] it was shown that

$$\kappa \geqq 0.3658 \tag{14}$$

and in [4] that

$$\kappa^* \geqq 0.3485. \tag{15}$$

Anderson showed in [1] that

$$\kappa^* \geqq 0.3532. \tag{16}$$

Here we prove

**Theorem 1.** *With the above notation,* $\kappa \geqq 0.4077$ *and* $\kappa^* \geqq 0.401$. *In particular, at least* $2/5$ *of the zeros of* $\zeta(s)$ *are simple and on the critical line.*

## 3. Beginning of the proof

It follows from (2) that $\xi^{(n)}(s)$ is real for $s = 1/2 + it$ when $n$ is even and is purely imaginary when $n$ is odd. Let $g_n$, $n \geqq 0$, be complex numbers with $g_n$ real if $n$ is odd and $g_n$ purely imaginary if $n$ is even. Let $g \neq 0$ be real. Let $T$ be a large parameter (we will be working with the strip $0 < t \leqq T$) and let

$$L = \log T. \tag{17}$$

Now define

$$\eta(s) = g\,\xi(s) + \sum_{n=0}^{N} g_n\,\xi^{(n)}(s)\,L^{-n} \tag{18}$$

for some fixed $N$. Then, for $s = 1/2 + it$,

$$g\,\xi(s) = \operatorname{Re} \eta(s) \tag{19}$$

so that $\xi(s) = 0$ on $\sigma = 1/2$ if and only if $\operatorname{Re} \eta(s) = 0$. The idea, then, is to show that $\operatorname{Re} \eta(1/2 + it)$ vanishes at least

$$(\kappa + o(1))\,\frac{T}{2\pi}\log T \tag{20}$$

times for $t$ in $[0, T]$. We do this by showing that the change in argument of $\eta(1/2 + it)$ as $t$ varies from 0 to $T$ is at least

$$(\kappa + o(1))\frac{T}{2}\log T \tag{21}$$

since for every change of $\pi$ in the argument of some function $f(z)$ it must be the case that $\operatorname{Re} f(z)$ has at least one zero.

To estimate the change in argument of $\eta(s)$ on the 1/2-line, we let $\eta(s) = H(s)\, V_1(S)$ where $H$ is defined in (4) and

$$V_1(s) = g\zeta(s) + \sum_{n=0}^{N} \frac{g_n}{L^n} \sum_{k=0}^{n} \binom{n}{k} \frac{H^{(n-k)}(s)}{H(s)} \zeta^{(k)}(s). \tag{22}$$

By Lemma 1 of [3], for $|t| \geqq 2$,

$$\frac{H^{(m)}}{H}(s) = (1/2 \log s/(2\pi))^m\,(1 + O(1/|t|)) \tag{23}$$

so that

$$\begin{aligned} V_1(s) &= g\zeta(s) + \sum_{n=0}^{N} g_n \left(\frac{\log s/(2\pi)}{2L} + \frac{1}{L}\frac{d}{ds}\right)^n \zeta(s)\,(1 + O(1/|t|)) \\ &= \left(Q_1\left(\frac{\log s/(2\pi)}{2L} + \frac{1}{L}\frac{d}{ds}\right)\zeta(s)\right)(1 + O(1/|t|)) \end{aligned} \tag{24}$$

where

$$Q_1(x) = g + \sum_{n=0}^{N} g_n x^n. \tag{25}$$

A useful approximation to $V_1(s)$ in $0 < t < T$, $|\sigma| \ll 1$, is given by

$$V(s) = Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s) \tag{26}$$

where

$$Q(x) = Q_1(1/2 - x). \tag{27}$$

Note that for $\sigma > 1$,

$$V(s) = \sum_{n=1}^{\infty} \frac{Q(\log n/L)}{n^s}. \tag{28}$$

Also, the condition that $g_{2n}$ is imaginary and $g_{2n+1}$ is real is equivalent to $\operatorname{Re} Q_1(ix) = g$ for real $x$ or $\operatorname{Re} Q(1/2 + ix) = g$ for real $x$. Equivalently,

$$Q(z) + \bar{Q}(1 - z) \equiv g. \tag{29}$$

By Lemma 1 of [3],

$$\arg H(1/2+it)=\frac{t}{2}\log\frac{|t|}{2\pi e}+O(1) \tag{30}$$

so that

$$\Delta \arg\eta(1/2+it)|_2^T=\frac{T}{2}\log T+\Delta \arg V_1(1/2+it)|_2^T+O(T). \tag{31}$$

Now if $Q(0)=1$ (i.e. $Q_1(1/2)=1$), then it is not hard to show (see Conrey [3], Section 4) using the argument principle and standard estimates that

$$\Delta \arg V_1(1/2+it)|_2^T=-2\pi N_{V_1}^*(T)+O(T) \tag{32}$$

where $N_{V_1}^*(T)$ is the number of zeros of $V_1(s)$ with $\sigma\geqq 1/2$ and $0<t<T$ counted with multiplicity, except that zeros on $\sigma=1/2$ only count with weight 1/2 (times their multiplicity). We need an upper bound for $N_{V_1}^*(T)$.

We introduce the mollifier

$$B(s,P)=\sum_{n\leqq y}\frac{b(n,P)\,n^{\sigma_0-1/2}}{n^s} \tag{33}$$

where $\sigma_0=1/2-R/L$ with $R>0$ a free parameter; also

$$y=T^\theta \tag{34}$$

(and eventually $\theta=4/7-\varepsilon$) and $b(n,P)$ is defined in (1). (Recall that $P$ denotes a polynomial with $P(0)=0$ and $P(1)=1$ so that $b(1,P)=1$ and $b(n,P)\to 0$ as $n\to y$.)

Then

$$N_{V_1}^*(T)\leqq N_{V_1B}^*(T) \tag{35}$$

where $N_{V_1B}^*(T)$ is the number of zeros of $V_1B$ in $0<t\leqq T$, $\sigma\geqq\frac{1}{2}$ counted with multiplicity, except that zeros on the 1/2-line have only one-half their usual weight.

Now, as above

$$\sigma_0=1/2-R/L \tag{36}$$

where $R$ is a positive real number, $R\ll 1$. Then by Littlewood's lemma and the arithmetic-mean, geometric-mean inequality exactly as in Levinson [11] or Conrey [3],

$$2\pi N_{V_1B}(T)\leqq\frac{TL}{2R}\log\left(\frac{1}{T}\int_1^T|V_1B(\sigma_0+it)|^2\,dt\right)+O(T) \tag{37}$$

where $N_{V_1B}(T)$ is the number of zeros of $BV_1(s)$ in $0<t\leqq T$, $\sigma\geqq 1/2$ counted with multiplicity (and no convention about zeros on the 1/2-line). This leads to

$$\kappa\geqq 1-\frac{1}{R}\log\left(\frac{1}{T}\int_1^T|V_1B(\sigma_0+it)|^2\,dt\right)+o(1). \tag{38}$$

We will prove an asymptotic formula for the integral here. In fact, such a formula will follow by an integration by parts from an asymptotic formula for

$$\int_1^T |VB(\sigma_0+it)|^2\,dt.$$

Thus,

$$\kappa \geqq 1-\frac{1}{R}\log\left(\frac{1}{T}\int_1^T |BV(\sigma_0+it)|^2\,dt\right)+o(1) \tag{39}$$

where $R>0$, $\sigma_0=1/2-R/L$, $V(s)=Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s)$ where $Q$ is a polynomial satisfying $Q(0)=1$ and $Q(z)+\bar{Q}(1-z)\equiv g$ for some real number $g$; and $B(s)=\sum_{n\leqq y} b(n,P)n^{-s+\sigma_0-1/2}$ with $b(n,P)=\mu(n)\,P\left(\frac{\log y/n}{\log y}\right)$ where $P$ is a polynomial with $P(0)=0$ and $P(1)=1$.

Regarding simple zeros, we note that our argument thus far has shown that

$$\begin{aligned}\frac{1}{\pi}\Delta\arg\eta(1/2+it)|_2^T &= \frac{TL}{2\pi}-2N^*_{V_1}(T)+O(T)\\ &\geqq \frac{TL}{2\pi}-2N^*_{V_1B}(T)+O(T)\\ &\geqq \frac{TL}{2\pi}-2N_{V_1B}(T)+N_{0,V_1}(T)+O(T)\end{aligned} \tag{40}$$

where $N_{0,V_1}(T)$ is the number of zeros of $V_1(s)$ (or equivalently of $\eta(s)$) on the half-line. Thus, we actually have

$$\begin{aligned}\frac{1}{\pi}\Delta\arg\eta(1/2+it)|_2^T-N_{0,\eta}(T) &\geqq \frac{TL}{2\pi}-2N_{V_1B}(T)+O(T)\\ &\gtrsim \frac{TL}{2\pi}\left(1-\frac{1}{R}\log\frac{1}{T}\int_1^T |B\,V(\sigma_0+it)|^2\,dt\right).\end{aligned} \tag{41}$$

Now the left hand side here is a lower bound for the number of $t$ in $(2, T)$ for which $\operatorname{Re}\eta(1/2+it)=0$ but $\eta(1/2+it)\neq 0$. In the event that

$$\eta(s)=g\,\xi(s)+g_0\,\xi(s)+g_1\,\xi'(s)L^{-1} \tag{42}$$

we have $\operatorname{Re}\eta(1/2+it)=g\,\xi(1/2+it)$ so that $\operatorname{Re}\eta(1/2+it)=0$ but $\eta(1/2+it)\neq 0$ implies that $\xi(1/2+it)=0$ but $\xi'(1/2+\mathrm{i}t)\neq 0$, i.e., that $1/2+it$ is a simple zero of $\xi$.

Hence we have

$$\kappa^* \geqq 1 - \frac{1}{R} \log\left(\frac{1}{T} \int_2^T |VB(\sigma_0 + it)^2\, dt\right) + o(1) \tag{43}$$

subject to all the conditions on $R$, $V$, $B$, and $Q$ mentioned above and with the additional condition that $Q$ be a polynomial of degree 1.

## 4. The mean value theorem

Apart from some numerical calculations, Theorem 1 will be a consequence of the following mean value theorem.

**Theorem 2.** *Let $B(s, P)$ be as in* (1), (33), *and* (34). *Suppose that* $R \ll 1$ *and* $\sigma_0 = 1/2 - R/L$. *Let* $V(s) = Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s)$ *for some polynomial $Q$. If $\theta < 4/7$, then*

$$\int_2^T |VB(\sigma_0 + it)|^2\, dt \sim c(P, Q, R)\, T$$

*as $T \to \infty$ where*

$$c(P, Q, P) = |P(1)\, Q(0)|^2 + \frac{1}{\theta} \int_0^1 \int_0^1 \left| \frac{d}{du} \left(e^{R(y+\theta u)} Q(y + \theta u)\, P(x+u)\right)\Big|_{u=0} \right|^2 dx\, dy$$

$$= |P(1)\, Q(0)|^2 + \frac{1}{\theta} \int_0^1 \int_0^1 e^{2Ry} |Q(y)\, P'(x) + \theta Q'(y)\, P(x) + \theta R Q(y)\, P(x)|^2\, dx\, dy.$$

To deduce Theorem 1 from Theorem 2, we make our choices for $P$, $Q$, and $R$. For the moment, let

$$w(y) = e^{Ry} Q(y). \tag{44}$$

Then

$$c(P, Q, R) = |w(0)|^2 + \frac{1}{\theta} \int_0^1 \int_0^1 \left| \frac{d}{du} \left(w(y+\theta u)\, P(x+u)\right)\Big|_{u=0} \right|^2 dx\, dy \tag{45}$$

$$= |w(0)|^2 + \frac{1}{\theta} \int_0^1 \int_0^1 |w(y)\, P'(x) + \theta w'(y)\, P(x)|^2\, dx\, dy.$$

The double integral here is

$$I(P) = A \int_0^1 |P'(x)|^2 \, dx + 2 \operatorname{Re} \left(B \int_0^1 P'(x) \, \bar{P}(x) \, dx\right) + C \int_0^1 |P(x)|^2 \, dx$$

where

$$A = \int_0^1 |w(y)|^2 \, dy, \quad B = \theta \int_0^1 w(y) \, \bar{w}'(y) \, dy, \quad C = \theta^2 \int_0^1 |w'(y)|^2 \, dy. \tag{46}$$

By the Euler-Lagrange equations, $I(P)$ will be minimized by a function $P$ satisfying

$$AP'' - (B - \bar{B}) P' - CP = 0, \quad P(0) = 0, \quad P(1) = 1. \tag{47}$$

By an integration by parts,

$$\int_0^1 |P'(x)|^2 \, dx = \bar{P} P'(x)|_0^1 - \int_0^1 \bar{P}(x) \, P''(x) \, dx.$$

We use (47) here to substitute for $P''$; then by (46) and an easy calculation,

$$I(P) = AP'(1) + \bar{B} \, 2 \operatorname{Re} \int_0^1 P(x) \, \bar{P}'(x) \, dx.$$

But

$$\int_0^1 P(x) \, \bar{P}'(x) \, dx + \int_0^1 \bar{P}(x) \, P'(x) = P(x) \, \bar{P}(x)|_0^1 = 1$$

whence

$$I(P) = AP'(1) + \bar{B}.$$

The solution of (47) is easily seen to be

$$P(x) = \frac{e^{rx} - e^{sx}}{e^r - e^s}$$

where $r$ and $s$ are roots of $Az^2 - (B - \bar{B}) z - C = 0$. (Although $P$ is not a polynomial it can be uniformly approximated by polynomials of the right sort.) Solving for $r$ and $s$ we may write

$$P(x) = \frac{e^{i\beta x}}{e^{i\beta}} \, \frac{e^{\alpha x} - e^{-\alpha x}}{e^{\alpha} - e^{-\alpha}}$$

where

$$\alpha = \frac{((B - \bar{B})^2 + 4AC)^{1/2}}{2A}, \quad i\beta = \frac{B - \bar{B}}{2A}. \tag{48}$$

Then

$$P'(1) = \alpha \frac{e^{\alpha} + e^{-\alpha}}{e^{\alpha} - e^{-\alpha}} + i\beta$$

from which we see from (48) that

$$I(P) = AP'(1) + \bar{B} = A\alpha \coth\alpha + \frac{B+\bar{B}}{2}$$
$$= A\alpha \coth\alpha + \frac{\theta}{2}(|w(1)|^2 - |w(0)|^2)$$

since

$$2\,\mathrm{Re}\,B = \theta\left(\int_0^1 w\bar{w}' + \int_0^1 \bar{w}w'\right) = \theta|w|^2\Big|_0^1.$$

By (45) we now have

$$c = c(Q, R) = 1/2(|w(0)|^2 + |w(1)|^2) + \frac{A\alpha\coth\alpha}{\theta}. \tag{49}$$

We illustrate the computation in the case of simple zeros. We are forced here to take

$$Q(z) = 1 + \lambda z$$

for some real number $\lambda$. Then

$$A = \int_0^1 |w(y)|^2\,dy = \int_0^1 e^{2Ry}(1+\lambda y)^2\,dy = \lambda^2 I_2 + 2\lambda I_1 + I_0$$

where

$$I_n = \int_0^1 e^{2Ry} y^n\,dy.$$

Also, $B = 0$,

$$C = \theta^2 \int_0^1 e^{2Ry}(R(1+\lambda y)+\lambda)^2\,dy = \theta^2((R+\lambda)^2 I_0 + 2R\lambda(R+\lambda)I_1 + R^2\lambda^2 I_2)$$

and

$$\alpha = (C/A)^{1/2}.$$

Then with $\theta = 4/7$, $R = 1.2$, $\lambda = -1.02$ we get (using $I_0 = (e^{2R}-1)/(2R)$, $I_1 = (e^{2R} - I_0)/(2R)$, and $I_2 = (e^{2R} - 2I_1)/(2R)$), $I_0 = 4.17\ldots$, $I_1 = 2.85\ldots$, $I_2 = 2.21\ldots$, $A = 0.66\ldots$, $B = 0$, $C = 0.71\ldots$, $\alpha = 1.04\ldots$, $\coth\alpha = 1.28\ldots$,

$$c = (1/2)\,(1 + e^{2R}(1+\lambda)^2) + \frac{A}{\theta}\alpha\coth\alpha = 2.05\ldots,$$

and $1 - (\log c)/R = 0.4013\ldots$. The 0.4077 result arises from choosing $Q(y)$ as the $\varphi(y)$ in Conrey [3], Section 7, with $m = 0$.

## 5. The proposition

In this section, we deduce our mean value theorem, Theorem 2, from the following

**Proposition.** *Let $a, b \in \mathbb{C}$ with $a, b \ll 1$, and put $\alpha = a/L$, $\beta = b/L$ where $L = \log T$. Let $s_0 = 1/2 + iw$ with $T \leqq w \leqq 2T$. Suppose that $\delta > 0$, $\Delta = T^{1-\delta}$ and that $y = T^{\theta}$ with $0 < \theta < 4/7$. Let*

$$g(a, b, w, P_1, P_2)$$
$$= \frac{1}{i\Delta\pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta(s+\alpha)\,\zeta(1-s+\beta)\,\mathscr{B}(s, P_1)\,\mathscr{B}(1-s, P_2)\,ds$$

*where $(c)$ denotes the straight line path from $c - i\infty$ to $c + i\infty$ and where*

$$\mathscr{B}(s, P_i) = \sum_{n \leqq y} \frac{b(n, P_i)}{n^s} = B(s + \sigma_0 - 1/2, P_i)$$

*with $P_i(0) = 0$ for $i = 1, 2$. Then*

$$g(a, b, w, P_1, P_2)$$
$$= \frac{1}{\theta} \int_0^1 e^{-(b+a)y}\,dy\,\frac{\partial}{\partial u}\frac{\partial}{\partial v}\,e^{-a\theta u - b\theta v} \int_0^1 P_1(x+u)\,P_2(x+v)\,dx\Big|_{u=v=0}$$
$$+ P_1(1)\,P_2(1) + o_\delta(1)$$

*uniformly in $a$, $b$, and $w$.*

*Proof of Theorem* 2. To prove Theorem 2 it suffices to show, in the notation of Theorem 2, that

$$\frac{1}{\Delta\pi^{1/2}} \int_{-\infty}^{\infty} e^{-(t-w)^2\Delta^{-2}} |VB(\sigma_0 + it)|^2\,dt = c(P, Q, R) + o_\delta(1) \tag{50}$$

uniformly for $T \leqq w \leqq 2T$, with $\Delta = T^{1-\delta}$. For then Theorem 2 follows exactly as in Section 3 of Balasubramanian, Conrey, and Heath-Brown [2]. To prove (50) we write the left side as a complex integral

$$\frac{1}{i\Delta\pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}}\,VB(s+\sigma_0-1/2)\,\overline{VB}(1-s+\sigma_0-1/2)\,ds$$

where $s_0 = 1/2 + iw$; by (26) this is

$$= Q\left(\frac{-d}{da}\right)\bar{Q}\left(\frac{-d}{db}\right)$$
$$\left(\frac{1}{i\Delta\pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta(s+\alpha)\,\zeta(1-s+\beta)\,\mathscr{B}(s, P)\,\mathscr{B}(1-s, \bar{P})\,ds\Big|_{a=b=-R}\right)$$

where $\alpha = a/L$ and $\beta = b/L$. In the notation of the proposition, this is

$$= Q\left(\frac{-d}{da}\right) \bar{Q}\left(\frac{-d}{db}\right) g(a, b, w, P, \bar{P})|_{a=b=-R}.$$

Thus, by the proposition, the left side of (50) is

$$= Q\left(\frac{-d}{da}\right) \bar{Q}\left(\frac{-d}{db}\right)$$

$$\left(|P(1)|^2 + \frac{1}{\theta} \frac{\partial}{\partial u} \frac{\partial}{\partial v} \int_0^1 e^{-a(y+\theta u)-b(y+\theta v)} dy \int_0^1 P(x+u) \bar{P}(x+v) dx|_{u=v=0} + o_\delta(1)\right)|_{a=b=-R}.$$

Clearly, $g$ is analytic in the complex variables $a$ and $b$ if $a, b \ll 1$. Thus, we may use Cauchy's integral formula and the fact that

$$Q\left(\frac{-d}{da}\right) e^{-ay}|_{a=-R} = Q(y) e^{Ry} \tag{51}$$

to conclude that the left side of (50) is

$$= |Q(0) P(1)|^2$$
$$+ \frac{1}{\theta} \frac{\partial}{\partial u} \frac{\partial}{\partial v} \int_0^1 \int_0^1 P(x+u) \bar{P}(x+v) e^{2Ry} Q(y+\theta u) \bar{Q}(y+\theta v) dx dy|_{u=v=0} + o_\delta(1).$$

Equation (50) easily follows from this.

## 6. Initial Lemmas

In this section we prove that the proposition is a consequence of the following two lemmas.

**Lemma 1.** *Suppose that* $y = T^\theta$, $0 < \theta < 1$, $P_1$ *and* $P_2$ *are polynomials with* $P_1(0) = P_2(0) = 0$,

$$b(n, P) = \mu(n) P\left(\frac{\log y/n}{\log y}\right),$$

$L = \log T$, $\alpha = a/L$, $\beta = b/L$ *with* $a, b \ll 1$ *and*

$$\Sigma(\alpha, \beta, P_1, P_2) = \sum_{h,k \leqq y} \frac{b(h, P_1)\, b(k, P_2)}{h^{1+\alpha} k^{1+\beta}} (h, k)^{1+\alpha+\beta}.$$

*Then, as* $T \to \infty$

$$\Sigma(\alpha, \beta, P_1, P_2) \sim \frac{1}{\theta L} \frac{\partial}{\partial u} \frac{\partial}{\partial v} e^{a\theta u + b\theta v} \int_0^1 P_1(x+u) P_2(x+v)\, dx|_{u=v=0}$$

$$= \frac{1}{\theta L} \int_0^1 (P_1'(x) + a\theta P_1(x))\,(P_2'(x) + b\theta P_2(x))\, dx.$$

*Proof.* We give a sketch, as this sort of mean is worked out in Conrey [3], Section 6, using Lemmas 10 and 11 of that paper. We write

$$(h, k)^{1+\alpha+\beta} = \sum_{\substack{d|h \\ d|k}} \sum_{e|d} \mu(e) \left(\frac{d}{e}\right)^{1+\alpha+\beta}$$

$$= \sum_{\substack{d|h \\ d|k}} d^{1+\alpha+\beta} F(d, 1+\alpha+\beta)$$

where

$$F(d, s) = \prod_{p|d} (1 - p^{-s}).$$

Next we change the order of summation, so that the sum over $d$ is on the outside and on the inside we have a product of a sum over $h' = h/d$ and a sum over $k' = k/d$. The sums over $h'$ and $k'$ are evaluated using Lemma 10 and the result is evaluated using Lemma 11. After some simplification, we have our result. (More details may be found in Section 6 of Conrey [3].)

**Lemma 2.** *Let $g$ be as in the proposition and let $\Sigma$ be as in Lemma* 1. *Assume the hypotheses of the proposition. Then*

$$g(a, b, w, P_1, P_2) = \frac{\Sigma(b, a, P_1, P_2) - e^{-a-b}\Sigma(-a, -b, P_1, P_2)}{\alpha+\beta} + o_\delta(1)$$

*uniformly in $a$, $b$, and $w$.*

*Proof of the proposition.* Let

$$\sigma(a, b, P_1, P_2) = \frac{1}{\theta} \int_0^1 (P_1'(x) + a\theta P_1(x))\,(P_2'(x) + b\theta P_2(x))\, dx$$

so that by Lemma 1,

$$\Sigma(a, b, P_1, P_2) \sim \frac{1}{L} \sigma(a, b, P_1, P_2). \tag{52}$$

Then

$$\sigma(a, b, P_1, P_2) - \sigma(-b, -a, P_1, P_2) = (a+b) \int_0^1 (P_1'(x) P_2(x) + P_1(x) P_2'(x))\, dx \tag{53}$$

$$= (a+b) P_1(x) P_2(x)|_0^1 = (a+b) P_1(1) P_2(1)$$

for any $a$ and $b$. Thus, by Lemma 2 and (52) and (53),

$$g \sim (a+b)^{-1}[\sigma(-a,-b)+(a+b)P_1(1)P_2(1)-e^{-a-b}\sigma(-a,-b)]$$

$$=\frac{1-e^{-a-b}}{a+b}\sigma(-a,-b,P_1,P_2)+P_1(1)P_2(1)$$

$$=\int_0^1 e^{-(a+b)y}dy\,\frac{1}{\theta}\frac{\partial}{\partial u}\frac{\partial}{\partial v}e^{-a\theta u-b\theta v}\int_0^1 P_1(x+u)P_2(x+v)\,dx\big|_{u=v=0}+P_1(1)P_2(1)$$

as stated in the proposition.

## 7. The main term

In this section, we produce the main term of $g$ in Lemma 2 after some preparatory lemmas.

**Lemma 3.** *Suppose that $1<c<2$ and as usual,*

$$\chi(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}.$$

*Let*

$$J(y,s_0,\delta,\Delta)=\frac{1}{i\Delta\pi^{1/2}}\int_{(c)} e^{(s-s_0)^2\Delta^{-2}}\chi(1-s+\beta)y^{-s}ds.$$

*Then*

$$J=y^{-\beta}\int_0^\infty v^{s_0-\beta}\exp\left(-\frac{\Delta^2\log^2 v}{4}\right)(e(-yv)+e(yv))\frac{dv}{v}$$

*for any $y\neq 0$, $\Delta>0$, $s_0$ and $\beta$ with* $\operatorname{Re}\beta<c$.

*Proof.* By a change of variables,

$$J=\frac{y^{-\beta}}{i\Delta\pi^{1/2}}\int_{(c-\operatorname{Re}\beta)} e^{(s+\beta-s_0)^2\Delta^{-2}}\chi(1-s)y^{-s}ds.$$

The lemma now follows from Lemma 2 of Balasubramanian, Conrey, and Heath-Brown [2].

**Lemma 4.** *Let*

$$D(s,\alpha,\beta,H/K)=\sum_{m,n} m^{-s-\alpha}n^{-s-\beta}e(mnH/K)$$

*where $H$, $K$ are integers ($K\geqq 1$) with $(H,K)=1$ and $\alpha,\beta,s\in\mathbb{C}$. Then*

$$D(s, \alpha, \beta, H/K) - K^{1-\alpha-\beta-2s} \zeta(s+\alpha)\, \zeta(s+\beta)$$

*is an entire function of s. Also, D satisfies the functional equation*

$$D(s, \alpha, \beta, H/K) = -2\left(\frac{K}{2\pi}\right)^{1-2s-\alpha-\beta} \Gamma(1-s-\alpha)\, \Gamma(1-s-\beta)$$

$$\left[\cos\frac{\pi}{2}(2s+\alpha+\beta)\, D\left(1-s, -\alpha, -\beta, \frac{-\bar{H}}{K}\right) - \cos\frac{\pi}{2}(\alpha-\beta)\, D\left(1-s, -\alpha, -\beta, \frac{\bar{H}}{K}\right)\right].$$

*Moreover, if* $\alpha, \beta \ll (\log K)^{-1}$, *then* $D(0, \alpha, \beta, H/K) \ll_\varepsilon K^{1+\varepsilon}$ *for any* $\varepsilon > 0$.

All of these assertions are easily proven using the techniques of Estermann's original paper [8]. Basically, one uses the fact that

$$D(s, \alpha, \beta, H/K) = \sum_{a=1}^{K} \zeta(s+\alpha, a, K)\, \zeta(s+\beta, aH/K)$$

where

$$\zeta(s, a, K) = \sum_{n \equiv a \bmod K} n^{-s}$$

and

$$\zeta(s, a/K) = \sum_{n=1}^{\infty} e(an/K) n^{-s}.$$

These functions satisfy the functional equations

$$\zeta(s, a, K) = G(s)\, K^{-s}[e^{\pi i s/2} \zeta(1-s, a/K) - e^{-\pi i s/2} \zeta(1-s, -a/K)]$$

and

$$\zeta(s, a/K) = G(s)\, K^{1-s}[e^{\pi i s/2} \zeta(1-s, -a, K) - e^{-\pi i s/2} \zeta(1-s, a, K)]$$

where

$$G(s) = -i(2\pi)^{s-1}\, \Gamma(1-s).$$

The details are in Estermann's paper [8].

**Lemma 5.** *Let H and K be relatively prime integers with* $K > 0$. *Suppose that* $\alpha, \beta, x \in \mathbb{C}$ *with* $\operatorname{Im} x > 0$ *and let*

$$S(x, \alpha, \beta, H/K) = \sum_{m,n} m^{-\alpha} n^{-\beta} e(mnH/K)\, e(mnx).$$

*If* $c > 1$, *then*

$$\begin{aligned}&S(x,\alpha,\beta,H/K)\\&\quad=\zeta(1-\alpha+\beta)K^{-1+\alpha-\beta}z^{-1+\alpha}\Gamma(1-\alpha)+\zeta(1-\beta+\alpha)K^{-1+\beta-\alpha}z^{-1+\beta}\Gamma(1-\beta)\\&\qquad+D(0,\alpha,\beta,H/K)+\frac{1}{\pi i}\int_{(c)} z^{s-1}\Gamma(1-s)\,\Gamma(s-\alpha)\,\Gamma(s-\beta)\,(K/2\pi)^{2s-1-\alpha-\beta}\\&\qquad\times[\cos\pi/2(2s+\alpha+\beta)\,D(s,-\alpha,-\beta,-\bar{H}/K)\\&\qquad+\cos\pi/2(\alpha-\beta)\,D(s,-\alpha,-\beta,\bar{H}/K)]\,ds.\end{aligned}$$

*Proof.* By Mellin's formula,

$$S=\sum_{m,n} m^{-\alpha}n^{-\beta}e(mnH/K)\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\,(-2\pi imnx)^{-s}\,ds$$

where we take $c>1$. Thus,

$$S=\frac{1}{2\pi i}\int_{(c)} D(s,\alpha,\beta,H/K)\,\Gamma(s)z^{-s}\,ds$$

where $z=-2\pi ix$. We move the path of integration to $(1-c)$ and then make the change of variable $s\longrightarrow 1-s$. Thus, by Cauchy's theorem and Lemma 4,

$$\begin{aligned}S&=\zeta(1-\alpha+\beta)K^{-1+\alpha-\beta}z^{-1+\alpha}\Gamma(1-\alpha)\\&\quad+\zeta(1-\beta+\alpha)K^{-1+\beta-\alpha}z^{-1+\beta}\Gamma(1-\beta)\\&\quad+D(0,\alpha,\beta,H/K)+\frac{1}{\pi i}\int_{(c)} z^{s-1}\Gamma(1-s)\,\Gamma(s-\alpha)\,\Gamma(s-\beta)\,(K/2\pi)^{2s-1-\alpha-\beta}\\&\quad\times[\cos\pi/2\,(2s+\alpha+\beta)\,D(s,-\alpha,-\beta,-\bar{H}/K)\\&\quad+\cos\pi/2(\alpha-\beta)\,D(s,-\alpha,-\beta,\bar{H}/K)]\,ds\end{aligned}$$

which completes the proof of the lemma.

**Lemma 6.** *Let $w$ be real with $T\leqq w\leqq 2T$ and let $s_1=1/2+\beta+iw$ where $\beta=b/L$ with $b\in\mathbb{C}$, $b\ll 1$. Let $\delta>0$, $\pi/2>\lambda>0$, $\Delta=T^{1-\delta}$, and $\alpha=a/L$ with $a\in\mathbb{C}$, $a\ll 1$. Define*

$$r(s_1,a)=\int_{L_\lambda} v^{s_1}\exp(-\Delta^2(\log^2 v)/4)\,(v-1)^{-1+\alpha}\,dv/v,$$

*where $L_\lambda$ is the half-line $L_\lambda=\{re^{i\lambda}:r\geqq 0\}$. Then*

$$r(s_1,a)=-\pi ie^{-a}+o_\delta(1)$$

*as $T\longrightarrow\infty$, uniformly in $a$ and $s_1$.*

*Proof.* We change the path of integration to the positive real axis except for a small semicircular indentation into the upper half-plane centered at $v=1$. Now let $v=e^x$. Writing $w=\omega T$, $\alpha=a/L$, $\beta=b/L$, and $\Delta=T^{1-\delta}$, we have

$$r(s_1, a) = \int_{\mathscr{C}} e^{bx/L} e^{i\omega T x} \exp\left(-\frac{T^{2-2\delta} x^2}{4}\right) (e^x - 1)^{a/L} \frac{dx}{2\sinh x/2}$$

where $\mathscr{C}$ is the path consisting of the entire real axis from $-\infty$ to $\infty$ apart from a small semicircular indentation into the upper half plane centered at $x=0$. Now let

$$R(s_1, a) = \int_{\mathscr{C}} e^{bx/L} e^{i\omega T x} \exp\left(-\frac{T^{2-2\delta} x^2}{4}\right) x^{a/L}\, dx/x,$$

and consider $r(s_1, a) - R(s_1, a)$. Since

$$q(a, x, T) = \frac{(e^x - 1)^{a/L}}{2\sinh x/2} - \frac{x^{a/L}}{x} \ll |x|^{a/L}$$

as $|x| \to 0$, it follows that

$$R(s_1, a) - r(s_1, a) = \lim_{\eta \to 0^+} \int_{-\infty}^{-\eta} + \int_{-\eta}^{\infty} e^{bx/L} e^{i\omega T x} \exp\left(-\frac{T^{2-2\delta} x^2}{4}\right) q(a, x, T)\, dx.$$

The convergence is uniform in $T$, $\omega$, $b$, and $a$ for $T \geqq 1$, and $a, b, \omega \ll 1$. Taking the limit as $T \to \infty$, we get 0 for the right side whence

$$r(s_1, a) = R(s_1, a) + o_\delta(1)$$

uniformly in $\omega$, $b$, and $a$. In the integral defining $R$ let $y = Tx$. Then

$$R(s_1, a) = e^{-a} \int_{\mathscr{C}} e^{by/TL} e^{i\omega y} \exp\left(-\frac{T^{2-2\delta} x^2}{4}\right) y^{a/L}\, dy/y.$$

Again, for fixed $\delta > 0$ the convergence is uniform in $a$, $b$, and $\omega \ll 1$. Letting $T \to \infty$ we have, by the residue theorem,

$$\lim_{T \to \infty} R(s_1, a) = e^{-a} \int_{\mathscr{C}} e^{i\omega y}\, dy/y = e^{-a} \int_{\mathscr{C}} e^{iy}\, dy/y = -\pi i e^{-a}.$$

Thus, $R(s_1, a) = -\pi i e^{-a} + o_\delta(1)$ whence the lemma follows.

Now we begin the proof of Lemma 2. First of all we move the path of integration in the definition of $g$ to $(c)$ where $c = 1 + \eta$ with $\eta > 0$ small and fixed. Since $\alpha, \beta \ll 1/L$ it will be the case that $|\alpha|, |\beta| < \eta$ if $T$ is sufficiently large. Thus, in moving the path of integration we cross a pole at $s = 1 - \alpha$. The contribution from the residue is negligible since for $s \ll 1$,

$$\exp((s - s_0)^2 \Delta^{-2}) \ll \exp(T^{-2\delta}) \ll T^{-20} \tag{54}$$

because of the definition of $\Delta$ and $s_0$. We use the functional equation (5) on $\zeta(1 - s + \beta)$; then we interchange summation and integration and have

$$g=\sum_{h,k\leqq y}\frac{b(h,P_1)\,b(k,P_2)}{k}\sum_{m,n}m^{-\alpha}n^{\beta}J(mnh/k,s_0,\beta,\Delta)+o_\delta(1)$$

$$=\sum_{h,k\leqq y}\frac{b(h,P_1)\,b(k,P_2)}{k^{1-\beta}h^{\beta}}\sum_{m,n}m^{-\alpha-\beta}\int_0^\infty v^{s_0-\beta}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)$$

$$\times\left(e\left(\frac{mnhv}{k}\right)+e\left(-\frac{mnhv}{k}\right)\right)\frac{dv}{v}+o_\delta(1).$$

We express the integral as a sum of two integrals and use Cauchy's theorem to move one path to $L_\lambda$ and the other to $L_{-\lambda}$ where $\lambda>0$ is small and $L_\lambda$ is the half-line $\{re^{i\lambda}:r\geqq 0\}$. We interchange summation over $m$ and $n$ with the integration and have

$$g=\sum_{h,k\leqq y}\frac{b(h,P_1)\,b(k,P_2)}{k^{1-\beta}h^{\beta}}(I_1+I_2)+o_\delta(1)\tag{55}$$

where, in the notation of Lemma 5,

$$I_1=\int_{L_\lambda}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)S(h/k(v-1),\alpha+\beta,0,h/k)\frac{dv}{v}\tag{56}$$

and

$$I_2=\int_{L_{-\lambda}}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)S(-h/k(v-1),\alpha+\beta,0,h/k)\frac{dv}{v}.\tag{57}$$

Then, by Lemma 5, with $H=h/(h,k)$ and $K=k/(h,k)$,

$$I_1=M_1+R_1+E_1\tag{58}$$

where

$$\begin{aligned}M_1=\int_{L_\lambda}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)&\\ \times\Big[\zeta(1-\alpha-\beta)\,\Gamma(1-\alpha-\beta)\,K^{-1+\alpha+\beta}&\left(-2\pi i\frac{h}{k}(v-1)\right)^{-1+\alpha+\beta}\\ +\zeta(1-\alpha-\beta)\,K^{-1-\alpha-\beta}&(-2\pi ih/k(v-1))^{-1}\Big]\frac{dv}{v},\end{aligned}\tag{59}$$

$$R_1=D(0,\alpha+\beta,H/K)\int_{L_\lambda}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)\frac{dv}{v},\tag{60}$$

and

$$E_1=\int_{L_\lambda}v^{s_1}\exp\left(\frac{-\Delta^2\log^2 v}{4}\right)F_1(v)\frac{dv}{v}\tag{61}$$

with

$$F_1(v)=\frac{1}{\pi i}\int_{(c)}(-2\pi i h/k(v-1))^{s-1}\,\Gamma(1-s)\,\Gamma(s-\alpha-\beta)\,\Gamma(s)\left(\frac{K}{2\pi}\right)^{2s-1-\alpha-\beta}$$
$$\times[\cos(\pi/2(2s+\alpha+\beta))\,D(s,\,-\alpha-\beta,0,\,-\bar{H}/K)$$
$$+\cos(\pi/2(\alpha+\beta))\,D(s,\,-\alpha-\beta,0,\,\bar{H}/K)]\,ds. \tag{62}$$

There are similar expressions for $I_2=M_2+R_2+E_2$.

Now in the notation of Lemma 6,

$$M_1=\zeta(1-\alpha-\beta)\,\Gamma(1-\alpha-\beta)\,(-2\pi i)^{-1+\alpha+\beta}H^{-1+\alpha+\beta}r(s_1,\alpha+\beta)$$
$$+\zeta(1+\alpha+\beta)\,(-2\pi i)^{-1}H^{-1}K^{-\alpha-\beta}r(s_1,1). \tag{63}$$

Now $\zeta(s)\sim 1/(s-1)$ for $s$ near 1 and $\Gamma(1)=1$. Thus, by Lemma 6,

$$\sum_{h,k\leqq y}\frac{b(h,P_1)\,b(k,P_2)}{k^{1-\beta}h^{\beta}}M_1$$
$$=\frac{-1}{2(\alpha+\beta)}(e^{-a-b}\Sigma(-a,\,-b,P_1,P_2)-\Sigma(b,a,P_1,P_2))\,(1+o(1)). \tag{64}$$

We get exactly the same expression for the sum of the $M_2$. Thus, from the terms with $M_1$ and $M_2$ we get the main term of $g$ in Lemma 2.

## 8. The error terms

In this section we complete the proof of Lemma 2 which completes the proofs of the theorems. This section is where the important work of Deshouiller and Iwaniec on averages of Kloosterman sums enters.

It remains to bound

$$\sum_{h,k\leqq y}\frac{b(h,P_1)\,b(k,P_2)}{h^{\beta}k^{1-\beta}}(R_i+E_i) \tag{65}$$

for $i=1$ and 2. As the situation is identical for $i=2$ we deal with $i=1$ only. As far as $R_i$ is concerned, since, by Lemma 4 of [2]

$$\int_{L_\lambda}v^{s_1}\exp\left(-\Delta^2\frac{\log^2 v}{4}\right)\frac{dv}{v}=\frac{2\pi^{1/2}}{\Delta}\exp\left(\frac{s_1^2}{\Delta^2}\right)\ll\exp(T^{-2\delta}) \tag{66}$$

it follows from Lemma 4 that

$$R_1\ll T^{-20}$$

whence the contribution to (65) from $R_1$ is $\ll T^{-10}$.

Now by (61) and (62), the part of (65) which involves $E_i$ may be written as a sum of two terms, one of which is

$$Z = \int_{L_\lambda} \int_{(c)} G(\alpha+\beta, v, s_1, \Delta, s)\, \mathcal{M}(\alpha, \beta, s)\, ds\, dv \tag{67}$$

where

$$G(\alpha, v, s_1, \Delta, s) = -i v^{s_1} \exp\left(-\Delta^2 \frac{\log^2 v}{4}\right) \Gamma(s)\, \Gamma(1-s)\, \Gamma(s-\alpha)\, (2\pi)^{\alpha-s} \times \cos(\pi/2(2s+\alpha))\, e^{-\pi i s/2} (v-1)^{s-1} v^{-1} \tag{68}$$

and

$$\mathcal{M}(\alpha, \beta, s) = \sum_{m,n} m^{\alpha+\beta-s} n^{-s} \sum_{h,k \leqq y} \frac{b(h, P_1)\, b(k, P_2)}{h^{1-s+\beta} k^{1-s+\alpha}} (h, k)^{1-2s+\alpha+\beta}\, e\left(\frac{mn\bar{H}}{K}\right) \tag{69}$$

where $H = h/(h, k)$ and $K = k/(h, k)$. The other term is slightly less complicated and may be treated the same way as this one will be. Replacing $mn$ by $n$ and arranging the sums over $h$ and $k$ according to the g.c.d. of $h$ and $k$ we get

$$\mathcal{M}(\alpha, \beta, s) = \sum_{g \leqq y} 1/g \sum_{N, U, V} \mathcal{M}(N, U, V, \alpha, \beta, g, s) \tag{70}$$

where

$$\mathcal{M}(N, U, V, \alpha, \beta, g, s) = \sum_{n \sim N} \frac{\delta(n)}{n^s} \sum_{u \sim U} \sum_{\substack{v \sim V \\ (u,v)=1}} \frac{b(ug, P_1)\, b(vg, P_2)}{u^{1-s+\beta} v^{1-s+\alpha}}\, e\left(\frac{n\bar{u}}{v}\right) \tag{71}$$

with

$$\delta(n) = \sum_{d \mid n} d^{\alpha+\beta} \tag{72}$$

and where the notation $x \sim X$ means $X < x \leqq 2X$, and the sums on $U$ and $V$ have $\ll \log y$ terms with $U, V \ll y/g$ and the sum on $N$ is for $N = 2^J$, $J = 0, 1, 2, \ldots$. Now $Z$ is a sum of terms of the shape

$$Z(N, U, V) = \int_{L_\lambda} \int_{(c)} G(\alpha+\beta, v, s_1, \Delta, s)\, \mathcal{M}(N, U, V, \alpha, \beta, g, s)\, ds\, dv. \tag{73}$$

If $UV \geqq TN$, then we move the $s$ path of integration to $s = \eta + it$; otherwise we leave it at $s = c + it = 1 + \eta + it$. (Recall that $\eta > 0$ is fixed, to be chosen at the end of the proof in terms of $\varepsilon$.) In moving the path of integration, we cross a pole at $s = 1$ with residue

$$\Gamma(1-\alpha)\, (2\pi)^{\alpha-1} \cos(\pi/2(2+\alpha))\, \mathcal{M}(N, U, V, \alpha, \beta, g, 1) \int_{L_\lambda} v^{s_1} \exp\left(-\Delta^2 \frac{\log^2 v}{4}\right) \frac{dv}{v} \ll_\delta T^{-10} \tag{74}$$

by (66). To complete the proof we require two lemmas.

**Lemma 7.** *Let $G$ be as in (68) with the usual conventions about $s_1$, $\Delta$, and $\alpha$. Suppose that $c=\eta$ or $c=1+\eta$ where $\eta>0$ is a small fixed number. Let $\lambda=1/T$. Then*

$$\int_{L_\lambda}\int_{(c)}(1+|s|)\,|G(\alpha, v, s_1, \Delta, s)\,ds\,dv| \ll_{\varepsilon,\eta} \Delta^{-c-5/2}\,T^{5/2+\eta+\varepsilon}$$

*for any $\varepsilon>0$, uniformly in $\alpha$ and $s_1$.*

The proof is exactly the same as that of Lemma 5 of [2].

**Lemma 8.** *Let $\mathcal{M}(N, U, V, \alpha, \beta, g, s)$ be as in (71). Suppose that $y \leqq T^{8/13}$; $1 \leqq U$, $V \leqq y$; $\eta>0$ and $s=c+it$ with $c=\eta$ if $UV \geqq TN$ and $c=1+\eta$ if $UV<TN$. Then*

$$\mathcal{M}(N, U, V, \alpha, \beta, g, s) \ll_{\varepsilon,\eta} (1+|s|)\,(TN)^{\varepsilon}\, y^{2\eta}\, T^{c} N^{-\eta}(T^{-1/2} y^{7/8} + T^{-1} y^{7/4})$$

*uniformly for $a$, $b \ll 1$, all $t$, and all $g \ll y/V$.*

Before giving the proof of Lemma 8 we complete the proof of Lemma 2. We have by Lemmas 7 and 8

$$(75)\qquad \begin{aligned} Z &= \sum_{g\leqq y}\frac{1}{g}\sum_{N,U,V} Z(N, U, V)\\ &\ll_{\varepsilon,\eta} \Delta^{-7/2}\,T^{5/2+2\varepsilon} y^{2\eta+\varepsilon}(T^{1/2}y^{7/8}+y^{7/4}) \sum_{N,U,V} N^{\varepsilon-\eta} + \sum_{\substack{N,U,V\\ NT\leqq UV}} T^{-1}\\ &\ll_{\varepsilon} \Delta^{-7/2}\,T^{5/2+2\varepsilon} y^{3\varepsilon}(T^{1/2}y^{7/8}+y^{7/4}) \end{aligned}$$

on taking $\eta=\varepsilon/2$. Since $\theta<7/4$, this is $o_\varepsilon(1)$ as $T\to\infty$ if $\delta$ and $\varepsilon$ are sufficiently small.

*Proof of Lemma* 8. Initially we use the fact that the variable $g$ can be separated from $u$ and from $v$ since, for example,

$$P\left(\frac{\log y/(ug)}{\log y}\right)$$

is a sum of $\ll 1$ terms of the shape a constant times

$$\left(1-\frac{\log u+\log g}{\log y}\right)^{k}$$

which is itself a sum of $\ll 1$ terms of the shape

$$\left(\frac{\log u}{\log y}\right)^{k_1}\left(\frac{\log g}{\log y}\right)^{k_2}.$$

Also, $\mu(ug)=\mu(u)\,\mu(g)$ if $(u,g)=1$ and $\mu(ug)=0$ if $(u,g)>1$. Thus, $\mathcal{M}(N, U, V, \alpha, \beta, g, s)$ is a sum of $\ll 1$ terms that are themselves

$$\ll N^{-c}(UV)^{1-c}\,|S|$$

where

$$S=\sum_{n\sim N} r(n) \sum_{\substack{u\sim U\\(u,vg)=1}} \sum_{v\sim V} \mu(u)\, r^*(u)\, r(v)\, e\left(\frac{n\bar{u}}{v}\right). \tag{76}$$

Here the functions $r$ may be different at each occurrence; but they all may be described as follows: $r(\ )$ depends on its argument as well as $g, s, \alpha, \beta, N, U$, and $V$ and $r(n)\ll_\varepsilon n^\varepsilon$ for any $\varepsilon>0$ uniformly in $g, s, \alpha, \beta, N, U$, and $V$. In addition, $r^*$ is an $r$ function which is smooth in its dependency on $u$, satisfying

$$\frac{d}{du} r^*(u) \ll (1+|s|)\, u^{-1} r(u) \tag{77}$$

for some $r(u)$, and having the property of separability, i.e.,

$$r^*(ab)=r^*(a)\, r^*(b) \tag{78}$$

where the $r^*$'s here are not necessarily the same at each occurrence.

We now use Vaughan's identity to get a new expression for $\mu(n)$; equating coefficients on both sides of the identity $1/\zeta=1/\zeta(1-\zeta M)^2+2M-\zeta M^2$ where

$$M=M(s)=\sum_{n\leqq W} \mu(n)\, n^{-s}, \quad W=U^{1/4} \tag{79}$$

we find that

$$\mu(u)=c_1(u)+c_2(u)+c_3(u) \tag{80}$$

where

$$c_1(u)=\sum_{\substack{\alpha\beta\gamma=u\\ \alpha\geqq W,\, \beta\geqq W}} \mu(\gamma)\, c_4(\alpha)\, c_4(\beta) \tag{81}$$

with

$$c_4(\alpha)=-\sum_{\substack{d_1 d_2=\alpha\\ d_1\leqq W}} \mu(d_1); \tag{82}$$

$$c_2(u)=\begin{cases} 2\mu(u) & \text{if } u\leqq W,\\ 0 & \text{if } u>W; \end{cases} \tag{83}$$

$$c_3(u)=-\sum_{\substack{\alpha\beta\gamma=u\\ \alpha\leqq W\\ \beta\leqq W}} \mu(\alpha)\, \mu(\beta). \tag{84}$$

This leads to $S = S_1 + S_2 + S_3$ where

(85)
$$S_i = \sum_{n \sim N} r(n) \sum_{\substack{u \sim U \\ (u, vg)=1}} \sum_{v \sim V} c_i(u)\, r^*(u)\, r(v)\, e\left(\frac{n\bar{u}}{v}\right)$$

for $i = 1$, 2, and 3. We treat each of these sums in a slightly different way. We note also that it suffices to show that for any $\varepsilon > 0$,

(86)
$$S_i \ll_\varepsilon (yN)^\varepsilon \max(TN, UV)\, y^{7/8}\, T^{-1/2};$$

the lemma then follows with a different value of $\varepsilon$.

We start with $S_2$ which is trivially estimated by

(87)
$$\begin{aligned} S_2 &\ll_\varepsilon (yN)^\varepsilon NWV \ll_\varepsilon (yN)^\varepsilon (TN)\, T^{-1} U^{1/4} V \\ &\ll_\varepsilon (yN)^\varepsilon (TN)\, T^{-1} y^{5/4}. \end{aligned}$$

Thus $S_2$ satisfies (86) since $y \leqq T^{4/3}$. Next we consider $S_1$. Grouping together $\gamma$ and the larger of $\alpha$ and $\beta$ in (81) into a variable $b$ and calling the other variable $a$, we see that $S_1$ can be split into $\ll_\varepsilon y^\varepsilon$ sums of the shape

(88)
$$S_1' = \sum_{n \sim N} r(n) \sum_{\substack{a \sim A \\ b \sim B \\ (ab, v)=1}} \sum_{v \sim V} r(a)\, r(b)\, r(v)\, e\left(\frac{n\bar{a}\bar{b}}{v}\right)$$

where $U \ll AB \ll U$ and $W \leqq A \leqq B$. Now we have the following lemma, which is a case of Lemma 1 of Deshouillers and Iwaniec [7].

**Lemma 9.** *Suppose that* $|c(a, n)| \leqq 1$ *and* $U \ll AB \ll U$. *Then for any* $\varepsilon > 0$

$$\sum_{\substack{v \sim V \\ (b,v)=1}} \sum_{b \sim B} \left| \sum_{n \sim N} \sum_{\substack{a \sim A \\ (a,v)=1}} c(a, n)\, e\left(\frac{n\bar{a}\bar{b}}{v}\right) \right|$$

$$\ll_\varepsilon (NUV)^{1/2+\varepsilon} \{(UVA^{-1})^{1/2} + (A+N)^{1/4} [UVA^{-1}(N+A)(V+A^2) + NU^2]^{1/4}\}.$$

Using the fact that $x^a + y^a \ll (x+y)^a \ll x^a + y^a$ for $a \geqq 0$ and $x, y \geqq 1$ it is not hard to see that the right hand side of the relation in the proposition is

(89)
$$\ll_\varepsilon (Ny)^\varepsilon \Big( \sum_{(a,n,u,v) \in E} A^a N^n U^u V^v \Big)^{1/4}$$

for any $\varepsilon > 0$ where

(90)
$$\begin{aligned} E = \{&(-2, 2, 4, 4), (-1, 4, 3, 4), (1, 4, 3, 3), (1, 2, 3, 4), \\ &(3, 2, 3, 3), (0, 4, 4, 2), (1, 3, 4, 2)\}. \end{aligned}$$

Clearly, the fact that our coefficients $r(n)$ satisfy

$$r(n) \ll_\varepsilon n^\varepsilon$$

for any $\varepsilon > 0$ does not affect the use of the bound (89) for $S_2'$. We now show how to bound $A^a N^n U^u V^v$ for $(a, n, u, v) \in E$, and $U^{1/4} \ll A \ll U^{1/2}$.

We have two cases to deal with: $a \geqq 0$ and $a < 0$. If $a \geqq 0$ then

$$\begin{aligned}(91)\qquad A^a N^n U^u V^v &\ll (TN)^n (UV)^{4-n} T^{-n} U^{a/2+n+u-4} V^{n+v-4}\\ &\ll (\max\{TN, UV\})^4 T^{-n} y^{2n+u+v-8+a/2}\end{aligned}$$

since for all $(a, n, u, v) \in E$, $n+u \geqq 4$ and $n+v \geqq 4$ so that we may use $U, V \ll y$. Now we have to show that

$$(92)\qquad T^{-n} y^{2n+u+v-8+a/2} \ll T^{-2} y^{7/2}$$

for all $(a, n, u, v) \in E$ with $a \geqq 0$. The terms we get for the left side are

$$T^{-4} y^{13/2},\quad T^{-2} y^{7/2},\quad T^{-2} y^{7/2},\quad T^{-4} y^6,\quad \text{and}\quad T^{-3} y^{9/2}.$$

Clearly (92) holds since $y \ll T^{2/3}$. For $a < 0$ we use

$$\begin{aligned}(93)\qquad A^a N^n U^u V^v &\ll (TN)^n (UV)^{4-n} T^{-n} U^{a/4+n+u-4} V^{n+v-4}\\ &\ll (\max\{TN, UV\})^4 T^{-n} y^{2n+u+v-8+a/4}\end{aligned}$$

since for all $(a, n, u, v) \in E$ with $a < 0$ we have $a/4 + n + u - 4 \geqq 0$. Now we have to show that

$$(94)\qquad T^{-n} y^{2n+u+v-8+a/4} \ll T^{-2} y^{7/2}$$

for both $(a, n, u, v) \in E$ with $a < 0$. The terms we get for the left side are

$$(95)\qquad T^{-2} y^{7/2} \quad \text{and} \quad T^{-4} y^{27/4}$$

and so (94) holds since $y \ll T^{8/13}$. It follows now that

$$(96)\qquad S_1' \ll_\varepsilon \max(TN, UV)\,(yN)^\varepsilon T^{-1/2} y^{7/8}.$$

Finally, we consider $S_3$. Grouping together $\alpha$ and $\beta$ into a variable $a$ and replacing $\gamma$ by $b$ we see that $S_3$ can be split into $\ll_\varepsilon y^\varepsilon$ sums of the shape

$$(97)\qquad S_3' = \sum_{v \sim V} r(v) \sum_{\substack{b \sim B\\ (b, vg)=1}} r^*(b) \sum_{n \sim N} r(n) \sum_{\substack{a \sim A\\ (a, v)=1}} r(a)\, e\left(\frac{n\bar{a}\bar{b}}{v}\right)$$

where $U \ll AB \ll U$ and $A \ll W^2 = U^{1/2}$. Now if $A \gg U^{1/4}$, then the treatment is exactly as with $S_2$ above using Lemma 9. If $A \ll U^{1/4}$, then we sum over $b$ first using Weil's bound for the Kloosterman sum. Thus, Weil's bound implies that

$$\sum_{\substack{B \leqq b < B+x \\ (b, vg)=1}} e\left(\frac{l\bar{b}}{v}\right) \ll_\varepsilon v^{1/2} (vg)^\varepsilon (l, v) (1 + Bv^{-1}) \tag{98}$$

so that by a summation by parts (using the bound in (77) for $\frac{d}{dx} r^*(x)$) we get

$$\begin{aligned} S_3' &\ll_\varepsilon (1+|s|)(yN)^\varepsilon A V^{1/2} (1 + BV^{-1}) \sum_{n \sim N} \sum_{v \sim V} (n, v) \\ &\ll_\varepsilon (1+|s|)(yN)^\varepsilon A N V^{1/2} (V + B) \\ &\ll_\varepsilon (1+|s|)(yN)^\varepsilon (A N V^{3/2} + U N V^{1/2}). \end{aligned} \tag{99}$$

If $A \ll U^{1/4}$, then

$$\begin{aligned} A N V^{3/2} + U N V^{1/2} &\ll N y^{7/4} \\ &\ll \max\{TN, UV\} T^{-1} y^{7/4}. \end{aligned}$$

Thus, in any event

$$S_3 \ll_\varepsilon (1+|s|) \max\{TN, UV\} (yN)^\varepsilon (T^{-1} y^{7/4} + T^{-1/2} y^{7/8}).$$

This completes the proof of the lemma and the theorems.

**Note added in proof.** We can improve Theorem 1 slightly. With $R = 1.28$ and

$$\begin{aligned} Q(x) = 0.492 &+ 0.602\,(1-2x) - 0.08\,(1-2x)^3 \\ &- 0.06\,(1-2x)^5 + 0.046\,(1-2x)^7 \end{aligned}$$

we have $\kappa \geqq 0.4088$.

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