

A 60-fold cross ratio product identity

Herbert Gangl

ABSTRACT. We extend the well-known Plücker identity for individual cross ratios to products of cross ratios. In particular, we exhibit an expression ξ in four variables such that both ξ and $1 - \xi$ are products of 60 cross ratios.

To Masanobu Kaneko, in friendship, admiration, and gratitude

1. Motivation

Every mathematician is familiar with the cross ratio of four points in a projective line \mathbb{P}_F^1 over a field F , its essentially unique projective invariant. It is often given as

$$\text{cr}(x, y, z, w) = \frac{x - z}{x - w} \frac{y - w}{y - z},$$

an ambiguity arising from permuting the variables, and one notices the symmetries

$$\text{cr}(x, y, z, w) = \text{cr}(y, x, w, z) = \text{cr}(z, w, x, y) = \text{cr}(w, z, y, x).$$

Writing $\lambda = \text{cr}(x, y, z, w)$, any permutation of the four variables gives either of the six values

$$(1.1) \quad \lambda, \frac{1}{\lambda}, 1 - \frac{1}{\lambda}, \frac{\lambda}{\lambda - 1}, \frac{1}{1 - \lambda}, 1 - \lambda.$$

In particular, $f = \text{cr}$ satisfies the symmetry (swapping roles of the two middle arguments)

$$(1.2) \quad 1 - f(x, y, z, w) = f(x, z, y, w),$$

amounting to one of the famous Plücker identities.

Products of cross ratios play an important role as the arguments in functional equations of polylogarithms $\text{Li}_n(z) = \sum_{k \geq 1} z^k / k^n$ and, hence, in explicit avatars of algebraic K -groups ([8],[10]). Important functional equations were found, typically in two variables, for $n = 2$ by Spence, Abel and many others, and for $n = 3, 4, 5$ by Kummer ([7]). For $n = 3$ Goncharov provided a crucial new one [5], and we gave the first ones for $n = 6, 7$, the current record, in [4], see also [9], [2], [3]. No non-trivial such equations are known beyond weight $n = 7$, and in order to find such one would need to exhibit many expressions ξ such that both ξ and $1 - \xi$ are highly factorisable into a comparably small set of factors.

In this short note we will exhibit highly factorisable expressions $f(x, y, z, w)$ in four variables, in fact products of cross ratios in rational functions of the individual variables, which satisfy the *same* identity (1.2).

2. Examples and a general statement

EXAMPLE 2.1. It is easily checked that we have the identity (1.2) for the following functions in 4 variables

- (i) $f(x, y, z, w) = \text{cr}(x, y, z, w) \text{cr}(x, y, 1 - z, 1 - w)$,
- (ii) $f(x, y, z, w) = \text{cr}(x, y, z, w) \text{cr}(x, y, \frac{1}{z}, \frac{1}{w})$, and
- (iii) $f(x, y, z, w) = \text{cr}(x, y, z, w) \text{cr}(x, y, 1 - \frac{1}{z}, 1 - \frac{1}{w}) \text{cr}(x, y, \frac{1}{1-z}, \frac{1}{1-w})$.

These examples are special cases of the following more general result.

THEOREM 2.2. *Let H be any finite subgroup of $\text{PGL}_2(\mathbb{C})$, viewed as a group of Möbius transformations in $\text{Aut}(\mathbb{C}(t))$. Then we have the identity*

$$(2.1) \quad 1 - \prod_{\sigma \in H} \text{cr}(x, y, \sigma(z), \sigma(w)) = \prod_{\sigma \in H} \text{cr}(x, z, \sigma(y), \sigma(w)).$$

Note that the right hand side expression in the Theorem has the exact same form as the product on the left, except that the roles of y and z have been swapped.

PROOF. We will consider poles and zeros of the two rational functions on either side.

1) The left hand side has poles (involving two of the variables), all arising with multiplicity 1, for

- (i) for $x = \sigma(w)$, $\forall \sigma \in H$ and
- (ii) for $y = \sigma(z)$, $\forall \sigma \in H$,

whereas the right hand side has poles, all arising with multiplicity 1, for

- (i) for $x = \sigma(w)$, $\forall \sigma \in H$ and
- (ii) for $z = \sigma(y)$, $\forall \sigma \in H$.

In view of H being a group we have agreement in (ii) which proves the claim about poles.

2) The zeros on the left hand side of 2.1 are given by the zeros of the numerator

$$(2.2) \quad \prod_{\sigma} (x - \sigma(z))(y - \sigma(w)) - \prod_{\sigma} (x - \sigma(w))(y - \sigma(z)).$$

Whenever $x = \tau(y)$ for some $\tau \in H$, it becomes

$$\prod_{\sigma} (\tau(y) - \sigma(z))(y - \sigma(w)) - \prod_{\sigma} (\tau(y) - \sigma(w))(y - \sigma(z)),$$

which we rewrite as

$$(2.3) \quad \prod_{\sigma} (y - \sigma(z))(y - \sigma(w)) \left(\prod_{\sigma} \frac{\tau(y) - \sigma(z)}{y - \sigma(z)} - \prod_{\sigma} \frac{\tau(y) - \sigma(w)}{y - \sigma(w)} \right),$$

where all products run through the elements in H . Now we rewrite both summands in the right hand factor, e.g. the left one can be rewritten as

$$\prod_{\sigma} \frac{\tau(y) - \sigma(z)}{y - \tau^{-1}\sigma(z)},$$

since $\tau \in H$.

Then we use that, for a Möbius transformation $\tau(t) = \frac{at+b}{ct+d}$, we get an expression

$$\frac{\tau(y) - \tau(z)}{y - z} = \frac{\det \tau}{(cy + d)(cz + d)},$$

where the variables y and z have been separated. Hence the same separation property holds for the product

$$\prod_{\sigma} \frac{\tau(y) - \sigma(z)}{y - \tau^{-1}\sigma(z)}.$$

But now the denominator part in z of each $\tau(y) - \sigma(z)$ (in the numerator) cancels the denominator part in z of $y - \tau^{-1}(\tau\sigma)(z)$ (in the denominator) when σ runs through H , and hence the product is independent of z . As a consequence, the right hand factor in (2.3) vanishes and we have indeed shown that $\tau(y)$ is a root of the polynomial (2.2) of degree $|H|$ in x . For reasons of degree $\{\tau(y) \mid \tau \in H\}$ constitute all the roots.

Conclusion: We have found that the divisors of the rational functions on the LHS and RHS of (2.2) agree, and hence the functions themselves differ by some non-zero constant factor. It remains to verify that the constant factor is 1, which can be seen by setting $x = z$ as the product on the LHS then vanishes and the RHS also equals 1. \square

3. A 60-fold cross ratio product

In Klein's work on the icosahedron [6] he studied the finite automorphism groups of $\text{PGL}_2(\mathbb{C})$ in terms of fractional linear transformations, and in this context he expressed the symmetry groups of the Platonic solids as such. In particular, the arguably most intriguing cases are the octahedral group, of order 24, and the icosahedral group, of order 60. In order to represent these as Möbius transformations, one passes to rational functions in $\mathbb{Q}(\zeta_8)$ and $\mathbb{Q}(\zeta_5)$ (ζ_n a primitive n -th root of 1), respectively. The former one can be written in terms of three generators of respective order (as projective transformations) 4, 2 and 3 as

$$\text{Octa} = \left\langle \left(\begin{array}{cc} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{array} \right), \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \frac{1}{2} \left(\begin{array}{cc} 1+i & 1-i \\ -1+i & 1+i \end{array} \right) \right\rangle,$$

and the latter one in terms of three generators of respective order 5, 2 and 4 as

$$\text{Icosa} = \left\langle \left(\begin{array}{cc} \zeta_5^2 & 0 \\ 0 & \zeta_5^3 \end{array} \right), \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \frac{1}{\sqrt{5}} \left(\begin{array}{cc} -\zeta_5 + \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{array} \right) \right\rangle.$$

The numbers of their subgroups are well known to be 30 and 59, respectively. Hence we can find, e.g., 57 expressions corresponding to the non-trivial subgroups of A_5 of orders 2 (15), 3 (10), 4 (5), 5 (6), 6 (10), 10 (6) and 12 (5 such), together with their translates under the group action.

REMARK 3.1. Each of those 30 resp. 59 rational expressions in four variables satisfies Theorem 2.2, and the special cases where we take the full group $H = \text{Octa}$ and $H = \text{Icosa}$ lead to products of 24 and 60 cross ratios with the property (1.2). The upshot is that all these rational functions constitute exceptional S -units (i.e. S -units f such that $1 - f$ is also an S -unit) for a comparably small set S of irreducible polynomials, and as such there will be plenty of non-trivial functional equations for polylogarithms (of small weight) among them. It is conceivable that, by suitable specialisation of the variables, say, one might even be able to improve upon the current "highscore" of weight 7 (cf. [4], [2]). Classically, most of the known functional equations have as their arguments rational functions in several variables with \mathbb{Q} -coefficients, with a noteworthy exception of the so-called distribution relations (which hold for any weight and are considered to be trivial) and certain algebraic functional equations, for instance $\sum_i [p_i(t)]$ vanishes under the Bloch-Wigner dilogarithm as $p_i(t)$ runs through the algebraic roots of $x^a(1-x)^b = t$, with $a, b \in \mathbb{N}$, cf. [1], §5.

By contrast, in the above picture we allow $\mathbb{Q}(\zeta_n)$ -coefficients ($n = 5, 8$) which results in a plentiful source of potentially interesting new arguments.

References

- [1] S. Charlton, H. Gangl, and D. Radchenko. On functional equations for Nielsen polylogarithms. *Commun. Number Theory Phys.*, 15(2):363–454, 2021. doi:10.4310/CNTP.2021.v15.n2.a4.
- [2] H. Gangl, *Functional equations for higher logarithms*, *Selecta Math. (N.S.)* **9** (2003), no. 3, 361–377. MR 2006572 (2004m:11105)
- [3] ———, *Multiple polylogarithms in weight 4*, arXiv:1609.05557 (2016).
- [4] ———, *Funktionalgleichungen von Polylogarithmen*, *Bonner Mathematische Schriften* 278, Thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 1995. (In German.) MR 1385924 (97h:19009)
- [5] A. B. Goncharov, *Geometry of configurations, polylogarithms, and motivic cohomology*, *Adv. Math.* **114** (1995), no. 2, 197–318.
- [6] F. Klein, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Leipzig B.G. Teubner, 1884.
- [7] E. E. Kummer, *Ueber die Transcendenten, welche aus wiederholten Integrationen rationaler Formeln entstehen*, *J. Reine Angew. Math.* **21** (1840), 74–90. MR 1578254
- [8] D. Zagier, *Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields*, *Arithmetic algebraic geometry* (Texel, 1989), *Progr. Math.*, vol. 89, Birkhäuser, Boston, MA, 1991, pp. 391–430. MR 1085270 (92f:11161)
- [9] ———, *Special values and functional equations of polylogarithms*, *Structural properties of polylogarithms*, *Math. Surveys Monogr.*, vol. 37, Amer. Math. Soc., Providence, RI, 1991, pp. 377–400. MR 1148388
- [10] D. Zagier and H. Gangl, *Classical and elliptic polylogarithms and special values of L-series*, *The arithmetic and geometry of algebraic cycles* (Banff, AB, 1998), *NATO Sci. Ser. C Math. Phys. Sci.*, vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 561–615. MR 1744961 (2001g:11108)

MATHEMATICAL SCIENCES & COMPUTER SCIENCE BUILDING, UPPER MOUNTJOY, DURHAM UNIVERSITY, DH1 3LE, UK
Email address: herbert.gangl@durham.ac.uk