The Grassmannian complex and Goncharov's motivic complex in weight 4

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To Spencer Bloch's 80th birthday, in admiration

Abstract. For a field F and a given integer $n > 1$, Goncharov [**12**] has given a motivic complex $\Gamma_F(n)$, generalising the Bloch-Suslin complex ([**1**], [**18**]) for $n = 2$, which he expects to rationally compute the weight n motivic cohomology of $\operatorname{Spec} F$, hence its algebraic K-groups in Adams weight n, and he was also led to—conjecturally quasiisomorphic—'thickened' complexes thereof.

These complexes involve tensor products of higher polylogarithm groups, the latter having been linked to the geometry of certain configurations in Goncharov's proof of Zagier's Polylogarithm Conjecture for weight 3, and an analogous picture has long been envisioned by Goncharov for higher weight [**12**].

We provide a partial morphism in weight 4 by giving three out of four maps for configurations in general position. We moreover give partial results for the leftmost square.

1. Introduction

Let F be a field (typically $F = \mathbf{C}$, a number field F or a rational function field $F(t_1, \dots, t_r)$ over the latter). We will consider configurations of $2n$ points in $\mathbb{P}^{n-1}(F)$ in general position (i.e. any $k \leqslant n$ points are linearly independent) as studied by Suslin and Goncharov. Specifically, for $n = 4$ we focus on the configurations of up to eight points in $\mathbb{P}^3(F)$. We define $\tilde{C}_N(4)$ as the free abelian group generated by configurations of N points in $\mathbb{P}^3(F)$ in general position, and let $C_N(4) = \tilde{C}_N(4)/\mathrm{PGL}_4(F)$ be the group of coinvariants under the action of $G = \mathrm{PGL}_4(F)$, which acts by projective transformations. There is a natural differential map $\partial_N : C_N(4) \to C_{N-1}(4)$ defined by alternatingly omitting one of the points.

Our goal is to relate the resulting complex to Goncharov's so-called motivic complex $\Gamma_F(4)$ (e.g. [**12**], [**15**]) which is defined with the help of certain single-valued variants $\mathcal{L}_n(z)$ of the classical polylogarithms $\mathrm{Li}_n(z) = \sum_{n>0} z^k/k^n$. The idea is to use higher polylogarithm groups[1] $B_n(F) = \mathbf{Z}[F]/\langle\text{all functional equations of } \mathcal{L}_n\rangle$, together with their tensor (or wedge) products, relating them by certain coboundary maps. The latter are ultimately inspired by a cobracket in an associated 'motivic Lie

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[1]Goncharov [**12**] calls them "higher Bloch groups", which alas clashes with other authors (e.g. Suslin who coined the notion "Bloch group", and Zagier [**19**]) for a much smaller subgroup thereof. Some authors call $B_2(F)$ the "pre-Bloch group". We follow Cathelineau's convention [**2**].

coalgebra' originally envisaged by Beilinson, which is known to exist for a number field F [**7**]—to form a complex that computes certain (graded pieces of) algebraic K-groups and hence the associated motivic cohomology groups.

There are some intricate combinatorial problems that make it rather hard to produce a morphism that relates the two complexes, where the most right hand map is rather straightforward. We define two of the other three maps involved in such a way that the corresponding squares are indeed commutative.

A correct definition of the fourth map making the remaining square on the left commutative should give a "quadruple ratio" akin to Goncharov's triple ratio in the weight 3 case and would be key in a particularly satisfactory proof of said conjecture. We are able to give an—alas only partial—solution of this problem in that we can make the diagram commutative for certain degenerate configurations but not for generic ones. In collaboration with Charlton and Radchenko [**4**] we were subsequently able to give an explicit (albeit complicated) quadruple ratio, invoking several hundred orbits under the action of a rather large group.

Moreover, we show in §3 that the coboundary map in the associated Chevalley-Eilenberg Lie coalgebra complex annihilates (a dual of) $f_7(4)\circ d_8(\mathbf{v})$ for each generator $\mathbf{v}$ in $C_8(4)$, by showing its vanishing under a specific eight-fold antisymmetrisation on its symbol. We also check that this combination satisfies a Chen type integrability condition and hence is expressible as the coboundary of a combination of weight 4 multiple polylogarithms, albeit not explicitly.

According to Dan [**5**], one can reduce any weight 4 multiple polylogarithm to an explicit combination of the function $\mathrm{Li}_{3,1}(x,y)=\sum_{0<k<\ell} x^k y^\ell/k^3\ell$ (or rather its cousin $I_{3,1}(z_1,z_2)=\mathrm{Li}_{3,1}(z_2/z_1,1/z_2)$ arising from its integral representation) and Li_4. Furthermore, he reduces any combination of $I_{3,1}$-terms with vanishing $\bigwedge^2 B_2(F)$–component to a sum of combinations of the form $\sum_{i=1}^5 I_{3,1}\big(V_i(x,y),z\big)$ where the $V_i(x,y)$ denote the terms in the five term relation for the dilogarithm. Finally, a long-standing conjecture of Goncharov held that one can write any of the latter five term combinations explicitly in terms of Li_4 alone, and this was solved by this author in [**10**], Thm 17.

Overall the above provides the existence of a map $C_8(4)$ to $B_4(F)$ but does not give an explicit form for it.

A similar morphism of complexes, again without an explicit map on the very left, has been given by Goncharov and Rudenko in their inspiring paper [**17**], §7+9, where they use Goncharov's motivic correlators and a surprising connection to cluster algebras allowing them to give much more structure to the main result of [**10**] which, in conjunction with many other insights, permits them to give a proof of Zagier's Polylogarithm Conjecture in weight 4.

2. Towards a morphism of complexes

Notation. For a field F, we will be looking at many cross ratios (of four points on the projective line over F), triple ratios (of six points in a projective plane), projected cross ratios and projected triple ratios as well as 4×4-determinants where the columns arise from vectors in affine 4-space F^4.

We adopt Goncharov's notation from [**15**]. Configurations in $C_N(4)$ are ordered sets of N vectors (by abuse of notation we will also sometimes call then points) $v_i\in F^4$ in general position (i.e. each four of them give a non-zero determinant), viewed up to the diagonal action of the general linear group $GL_4(F)$.

In particular, an expression consisting of four indices $(i_1\, i_2\, i_3\, i_4)$ is shorthand for $\Delta(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}) = |v_{i_1}\, v_{i_2}\, v_{i_3}\, v_{i_4}|$, i.e. to the determinant of the 4×4-matrix whose columns are given by the points v_{i_k}.

All the expressions in the following are homogeneous in each vector (i.e. they will occur in any given expression as often in the numerator as in the denominator), so we can view the vectors also as points in $\mathbf{P}^3(F)$.

Furthermore, an ordered set of four points x_i in $\mathbf{P}^1(F)$ in general position has a well-known invariant, their cross-ratio r_2, where our convention is

$$\mathrm{r}_2(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}\,,$$

which we also abbreviate further by $(1\,2\,3\,4)$ where the x_i are understood.

For an ordered set of six points P_j $(j = 1, \dots, 6)$ in $\mathbf{P}^2(F)$ in general position Goncharov, using a clever simplification by Zagier, has invented the triple ratio, which is the antisymmetrisation under the symmetric group $\mathcal{S}_6$ of the following expression r_3', again denoting Δ by $|\cdot|$ for short,

$$\mathrm{r}_3'(P_1, P_2, P_3, P_4, P_5, P_6) = \frac{|P_1P_2P_4| \cdot |P_2P_3P_5| \cdot |P_3P_1P_6|}{|P_1P_2P_5| \cdot |P_2P_3P_6| \cdot |P_3P_1P_4|}\,,$$

which we also abbreviate by $(P_1P_2P_3P_4P_5P_6)$, or even, provided the context is clear, simply by the index vector $(1\,2\,3\,4\,5\,6)$.

The only other abbreviations we are using are

1) the projected cross ratio of six points in F^4, denoted by sequences of six indices separated by a bar where we project four points (written to the right of the bar) from two points (written to the left of the bar) onto any generic plane in F^4, so with the above shorthand we have (one recognises the formula for the cross ratio after simply fixing the two vectors indicated by i_1 and i_2)

$$(i_1\, i_2 \,|\, i_3\, i_4\, i_5\, i_6) = \frac{|i_1i_2i_3i_5|\,|i_1i_2i_4i_6|}{|i_1i_2i_3i_6|\,|i_1i_2i_4i_5|}\,;$$

2) the projected triple ratio term of seven points in F^4, denoted by sequences of seven indices separated by a bar where we project six points (written to the right of the bar) from a seventh point (written to the left of the bar) onto any generic hyperplane in F^4, so with the above shorthand we have (one recognises the formula for the triple ratio term after simply fixing the vector indicated by i_1)

$$(i_1 \,|\, i_2\; i_3\, i_4\, i_5\, i_6\, i_7) = \frac{|i_1i_2i_3i_5|\,|i_1i_3i_4i_6|\,|i_1i_4i_2i_7|}{|i_1i_2i_3i_6|\,|i_1i_3i_4i_7|\,|i_1i_4i_2i_5|}\,.$$

Finally, a subscript $(\dots)_n$ for $n = 2, 3$ is shorthand for $\{(\dots)\}_n$ indicating a generator in $B_n(F) = \mathbf{Z}[F]/R_n(F)$, where $R_n(F)$ is a formal version of the group of functional equations of $\mathcal{L}_n$ (see e.g. [**12**], p.214, 217).

By $\mathrm{Alt}_n\big(f(v_1, \dots, v_n)\big) = \sum_{\sigma\in\mathcal{S}_n} (-1)^{|\sigma|} f(v_{\sigma(1)}, \dots, v_{\sigma(n)})$, for a function f on n vectors in F^4, we understand the alternation under the symmetric group $\mathcal{S}_n$. Note that we adopt the convention that we do not divide by $n!$.

2.1. Defining maps from configurations to Goncharov's polylogarithmic motivic complex. With the notations as in the introduction we can draw the diagram as follows, where the lower left hand group $\mathcal{G}_4(F)$ is defined as a quotient of $\mathbf{Z}[F] \oplus \bigwedge^2 \mathbf{Z}[F]$ by a certain group of relations that plays no role in the following.

$$\begin{array}{ccccccc}
C_8(4) & \xrightarrow{d_8} & C_7(4) & \xrightarrow{d_7} & C_6(4) & \xrightarrow{d_6} & C_5(4) \\
{\scriptstyle f_8(4)}\downarrow & & {\scriptstyle f_7(4)}\downarrow & & {\scriptstyle f_6(4)}\downarrow & & {\scriptstyle f_5(4)}\downarrow \\
\mathcal{G}_4(F) & \xrightarrow{\partial_8} & \begin{array}{c} B_3(F)\otimes F^\times \\ \oplus \bigwedge^2 B_2(F) \end{array} & \xrightarrow{\partial_7} & B_2(F)\otimes \bigwedge^2 F^\times & \xrightarrow{\partial_6} & \bigwedge^4 F^\times .
\end{array}$$

Here the map ∂_6 is defined on generators as $\{a\}_2 \otimes b \wedge c \mapsto a \wedge (1-a) \wedge b \wedge c$, ∂_7 is given on respective generators by $\{a\}_3 \otimes b \mapsto \{a\}_2 \otimes a \wedge b$ as well as by $\{a\}_2 \wedge \{b\}_2 \mapsto \{a\}_2 \wedge (b \wedge (1-b)) - \{b\}_2 \wedge (a \wedge (1-a))$, and ∂_8 sends a generator $\{a\}$ to $\{a\}_3 \otimes a$ and $\{a,b\}$ to $\{a\}_2 \wedge \{b\}_2$.

Ignoring torsion. Note that some of the maps involved have denominators (with only small primes involved), so we should strictly speaking tensor the groups by $\mathbf{Z}[1/N]$ for a certain small N (in our formulas below the largest denominator is 7 and $N = 7!$ should certainly suffice). For ease of notation we take this as understood.

The vertical maps in the diagram are defined as follows:

2.1.1. *The map $f_5(4)$.* The right hand map is simply (note the sign) given by

$$f_5(4) : (1, \ldots, 5) \mapsto -\mathrm{Alt}_5\Big((1234) \wedge (1235) \wedge (1245) \wedge (1345)\Big) .$$

2.1.2. *The map $f_6(4)$.* The map $f_6(4)$ can be given as a linear combination of 3 orbits, with coefficients 2, 2 and 1, respectively.

More precisely, we have

$$f_6(4) : (1, \ldots, 6) \longmapsto \mathrm{Alt}_6 \begin{pmatrix} 2\ (12|3456)_2 \otimes (1234) \wedge (1356) \\ +1\ (12|3456)_2 \otimes (1345) \wedge (2345) \\ +2\ (12|3456)_2 \otimes (1234) \wedge (1345) \end{pmatrix} .$$

Note that this combination is not unique—see Remark 2.2 below.

2.1.3. *The map $f_7(4)$.* Again there are ambiguities in choosing orbits under the alternation, the possibly most convenient one being the following. We define

$$f_{22}\big((1,2,3,4,5,6,7)\big) = \mathrm{Alt}_7\left((12|3456)_2 \wedge (34|1257)_2\right)$$

and then put

$$f_7(4) : (1, \ldots, 7) \longmapsto \frac{3}{7} f_{22} + 2 f_{31} ,$$

where

$$f_{31}\big((1,2,3,4,5,6,7)\big) = \mathrm{Alt}_7\big((1|234567)_3 \otimes (2356)\big).$$

2.1.4. *The map $f_8(4)$.* The definition for this map obtained so far for certain *degenerate* configurations are somewhat complicated: For a specific configuration with three parameters the computer found a matching Li_4–expression with 1260 terms (and integer coefficients of modulus $\leqslant 318$).

2.1.5. *A compatibility condition for $f_8(4)$.* A first condition to consider at this stage is that $f_7(4) \circ d_8$ for each generator $\mathbf{v} = (v_1, \ldots, v_8)$ vanishes under the coboundary map, by checking that its associated symbol vanishes under a specific eight-fold antisymmetry, where the symbol $\mathcal{S}$ replaces $\{a\}_2$ by $a \wedge (1-a)$ and $\{a\}_3$ by $\big(a \wedge (1-a)\big) \otimes a$. It turns out that it is equivalent but somewhat less cumbersome to pass to the dual picture where d_8 is replaced by the other Grassmannian differential

$d'_8 : C_8(4) \to C_7(3)$, $(v_1, \ldots, v_8) \mapsto \sum_{j=1}^{8} (-1)^j (v_j | v_1, \ldots, \widehat{v_j}, \ldots, v_8)$ (i.e. which amounts to alternatingly sum the projections of the v_i $(i \neq j)$ along v_j from one of the eight points v_j of a configuration in $C_8(4)$ giving a configuration of 7 points in $\mathbf{P}^2(F)$ each).

A second property to check is that the combination $f_7(3) \circ d'_8(\mathbf{v})$ is actually integrable in the sense of Chen. In §3 below we indicate a—rather computational—proof that $f_7(3) \circ d'_8(\mathbf{v})$ satisfies this integrability. An independent computer check of this result has also been obtained by Radchenko (private communication). This integrability ensures that a combination in $\mathbf{Z}[F] \oplus \bigwedge^2 \mathbf{Z}[F]$ (corresponding to a multiple polylogarithm combination in weight 4) maps to $f_7(3) \circ d'_8(\mathbf{v})$, and then the main result from [**10**] guarantees, in view of Dan's Théorème 4 [**5**], that a combination in the smaller group $\mathbf{Z}[F]$ (corresponding to a *classical* polylogarithm combination in weight 4) suffices, and hence that we can find a map $f_8(4)$ with the prescribed properties, albeit the procedure does not give an explicit candidate.

2.2. Relating the maps $f_m(4)$.

2.2.1. *Relating* $f_5(4)$ *and* $f_6(4)$. Considering the composite maps $C_6(4) \to B_2(F) \otimes \bigwedge^2 F^\times$, we find the following statement showing that the right hand square of (2.1) commutes.

THEOREM 2.1. $\partial_6 \circ f_6(4) = f_5(4) \circ d_6$.

PROOF. We will introduce a convenient notation by associating to a term $(abcd)$ its "complement/dual" inside (123456), so e.g. (1346) $\leftrightsquigarrow$ (25), (5436) $\leftrightsquigarrow$ (12) etc. Note that we are allowed to reorder the indices (e.g. we can replace the 4×4-determinant $(bacd)$ by $(abcd)$ up to invoking a possible sign which we can safely ignore, as it only affects 2-torsion).

1. *The first term:* $\mathrm{Alt}_6\big(2\ (12|3456)_2 \otimes (1234) \wedge (1356)\big)$.
For the first of the three terms under the Alt_6-operator, we consider the boundary of $(12|3456)_2$, i.e. $(12|3456) \wedge (12|3546) \in F^\times \wedge F^\times$, which can be written as

$$\big((1235) - (1236) + (1246) - (1245)\big) \wedge \big((1234) - (1236) + (1256) - (1254)\big)\,.$$

After wedging this expression with $(1234) \wedge (1356)$ on the right and passing to the "complement notation" as indicated above, we get the following contributions (for convenience we also replace wedges by bars), all of which are understood to be under the Alt_6 operator

$$\begin{array}{l} +46|56|56|24 \;+\; 46|34|56|24 \;-\; 46|45|56|24 \;-\; 46|36|56|24 \\ -45|56|56|24 \;-\; 45|34|56|24 \;+\; 45|45|56|24 \;+\; 45|36|56|24 \\ +35|56|56|24 \;+\; 35|34|56|24 \;-\; 35|45|56|24 \;-\; 35|36|56|24 \\ -36|56|56|24 \;-\; 36|34|56|24 \;+\; 36|45|56|24 \;+\; 36|36|56|24\,. \end{array}$$

Now the terms in the first column of this tableau can be ignored as we have a factor $56 \wedge 56$ in each. The first and last of the red terms vanish for similar reasons while the remaining two red terms cancel due to the antisymmetry property of the wedge $(45|36|\cdots + 36|45|\cdots = 0)$. So we are left with eight terms

$$\begin{aligned} &+ 46|34|56|24 \; - \; 46|45|56|24 \; - \; 46|36|56|24 \\ &- 45|34|56|24 \\ &+ 35|34|56|24 \; - \; 35|45|56|24 \; - \; 35|36|56|24 \\ &- 36|34|56|24\,, \end{aligned}$$

which we bring into a slightly more convenient form by permuting the four factors, possibly invoking a sign which we mark below in red

$$\begin{aligned} &+ 24|34|46|65 \; - \; 24|45|46|65 \; + \; 24|46|36|65 \\ &- 24|34|45|56 \\ &+ 24|43|35|56 \; - \; 24|45|53|56 \; + \; 24|35|36|56 \\ &- 24|34|36|56\,. \end{aligned}$$

Under the alternation Alt_6 some of the terms cancel while others are identified. More precisely, leftover terms 3, 4 and 6 are identified with leftover term 1 using the permutations $(4\,6)(2\,5)$, $(5\,6)$, and $(4\,6\,2\,5)$, respectively (all occurring with the same sign), and leftover terms 8 is identified via $(5\,6)$ with term 5. Term 2 is fixed under the odd permutation (13), while term 7 is fixed under the odd permutation (24); hence the Alt_6–orbits of the latter two vanish.

Overall, the boundary of the first term (i.e. of $\mathrm{Alt}_6\big((12|3456)_2 \otimes (1234) \wedge (1235)\big)$) becomes

$$4\big(24|34|46|65\big) + 2\big(24|43|35|56\big)\,.$$

2. *The second term:* $\mathrm{Alt}_6\big(+1\ (12|3456)_2 \otimes (1345) \wedge (2345)\big)$.
For the second term, we find similarly

$$\begin{aligned} &+46|56|56|26 \; + \; 46|34|56|26 \; - \; 46|45|56|26 \; - \; 46|36|56|26 \\ &-45|56|56|26 \; - \; 45|34|56|26 \; + \; 45|45|56|26 \; + \; 45|36|56|26 \\ &+35|56|56|26 \; + \; 35|34|56|26 \; - \; 35|45|56|26 \; - \; 35|36|56|26 \\ &-36|56|56|26 \; - \; 36|34|56|26 \; + \; 36|45|56|26 \; + \; 36|36|56|26\,, \end{aligned}$$

giving after suitable permutation of factors (again, eight of the terms cancel for reasons of antisymmetry)

$$\begin{aligned} &- 34|46|65|26 \; + \; 45|46|65|26 \; - \; 26|36|46|56 \\ &+ 34|45|56|62 \\ &- 43|35|56|62 \; - \; 35|45|56|26 \; - \; 35|36|56|26 \\ &+ 34|36|56|26\,. \end{aligned}$$

In the latter expression, leftover terms 4 and 5 are identified via $(3\,4)$, while terms 1 and 6 cancel, and terms 2 and 7 are each fixed under an odd permutation, hence their alternation vanishes. So we obtain for the boundary of the second term, i.e. of $\mathrm{Alt}_6\big((12|3456)_2 \otimes (1234) \wedge (3456)\big)$, the contribution

$$\big(34|36|56|26\big) + 2\big(34|45|56|62\big) - \big(26|36|46|56\big)\,,$$

or equivalently, applying the odd permutation $(3\,6\,4\,5)$ to the first term and $(6\,5\,3\,2)$ to the second one, we can write the result as

$$\big(65|64|34|24\big) + 2\big(24|43|35|56\big) - \big(26|36|46|56\big)\,.$$

3. *The third term:* $\mathrm{Alt}_6\big(+2\ (12|3456)_2 \otimes (1234) \wedge (1345)\big)$.
Finally, for the third term we obtain

$$\begin{array}{l} +46|56|26|16 \ + \ 46|34|26|16 \ - \ 46|45|26|16 \ - \ 46|36|26|16 \\ -45|56|26|16 \ - \ 45|34|26|16 \ + \ 45|45|26|16 \ + \ 45|36|26|16 \\ +35|56|26|16 \ + \ 35|34|26|16 \ - \ 35|45|26|16 \ - \ 35|36|26|16 \\ -36|56|26|16 \ - \ 36|34|26|16 \ + \ 36|45|26|16 \ + \ 36|36|26|16\,, \end{array}$$

and in slightly adapted form

$$\begin{array}{l} -16|26|46|56 \ - \ 34|46|26|16 + 45|46|26|16 \ - \ 16|26|36|46 \\ -45|56|26|16 \ - \ 45|34|26|16 \\ +35|56|26|16 + 35|34|26|16 - 35|45|26|16 \ - \ 35|36|26|16 \\ -16|26|56|36 \ + \ 34|36|26|16. \end{array}$$

This time the terms in the first column are non-zero, and we have three different types of terms, all of which arise with compatible sign. Leftover terms 6, 8 and 9 are fixed under an odd permutation each, hence do not contribute. Terms 1, 4 and 11 combine to multiplicity 3, while the remaining six terms combine to multiplicity 6, so we obtain for the boundary of the third term the contribution

$$-6\big(34|46|26|16\big) - 3\big(16|26|46|56\big)\,.$$

4. *Combining the three contributions.*
Now it remains to compare the terms under the alternation. Putting

$$[a,b,c] = \mathrm{Alt}_6\big(a\big(24|34|46|65\big) + b\big(24|43|35|56\big) + c\big(26|36|46|56\big)\big)\,,$$

we find $[4,2,0]$, $[-1,-2,-1]$ and $[-6,0,-3]$ for the terms in 1., 2. and 3., respectively.

Noting that $2[4,2,0] + 2[-1,-2,1] + [-6,0,-3] = [0,0,-1]$, and taking the corresponding complement notation (e.g. 26 is replaced by 1345 etc.) we obtain for the LHS $-\mathrm{Alt}_6((1345) \wedge (1245) \wedge (1235) \wedge (1234)$, showing that the diagram commutes. □

There is an equivalent way to write the map $f_6(4)$ using the following claim.

Remark 2.2. *The following combination in $B_2(F) \otimes \bigwedge^2 F^\times$ has vanishing boundary, i.e.*

$$\partial_6 \mathrm{Alt}_6\Bigg((12|3456)_2 \otimes \Big(3\big((1345)\wedge(1234)\big) + (1345)\wedge(2345) + (1345)\wedge(3456)\Big)\Bigg) = 0\,.$$

Proof. From a similar analysis as used in the proof of Theorem 2.1 , using the same shorthands, we find that alternating the term $(12|3456)_2 \otimes (1345) \wedge (3456)$ can be expressed as $[3,-6,0]$. Now use from the proof of that theorem that the alternations of the terms $3(12|3456)_2 \otimes (1345) \wedge (1234)$ and $(12|3456)_2 \otimes (1345) \wedge (2345)$ can be written in the same shorthand as $3\,[1,2,1]$ and $[-6,0,-3]$, respectively. □

2.2.2. *Relating $f_6(4)$ and $f_7(4)$ (the middle square of (2.1)).* We have the following compatibility of maps $C_7(4) \to B_3(F) \otimes F^\times$, showing that the centre square of (2.1) commutes.

THEOREM 2.3. *$f_6(4) \circ d = \partial \circ f_7(4)$.*

PROOF. *LHS.* We first compute the left hand side for an arbitrary configuration of seven vectors, using the definition of $f_6(4)$, to get

$$\begin{aligned} f_6(4) \circ d\big((123467)\big) &= \quad 2\,\mathrm{Alt}_7(12|3456)_2 \otimes (1234) \wedge (1345) \\ &\quad + 2\,\mathrm{Alt}_7(12|3456)_2 \otimes (1234) \wedge (1356) \\ &\quad + \ \mathrm{Alt}_7(12|3456)_2 \otimes (1345) \wedge (2345)\,. \end{aligned}$$

We rewrite below the three contributions under the Alt–sign using the permutations $(2\,5\,4)$, $(2\,3)$ and $(3\,1\,4\,2\,5)$, respectively, to obtain

$$\begin{aligned} f_6(4) \circ d\big((123467)\big) &= \quad 2\,\mathrm{Alt}_7(15|3246)_2 \otimes (1532) \wedge (1324) \\ &\quad + 2\,\mathrm{Alt}_7(13|2456)_2 \otimes (1234) \wedge (1256) \\ &\quad + \ \mathrm{Alt}_7(45|1236)_2 \otimes (4123) \wedge (5123)\,. \end{aligned}$$

Note that the terms in the respective wedge products come in two types: they either overlap in two entries (the second sum) or in three entries (first and last sum). Hence it is useful to collect the ones of the same type; in fact we also apply for later convenience the permutation $(1\,3)(6\,7)$ to the first sum. Moreover, we note the stabilisers that will fix the expressions or turn them into their negatives: for the last sum, which we can write as

$$\mathrm{Alt}_7(45|1237)_2 \otimes (1235) \wedge (1234)$$

(note that we swapped the terms in the wedge product but also applied the odd permutation $(6\,7)$, hence keep the sign). This expression is symmetric in the indices $4, 5$ and antisymmetric in the three indices $1, 2, 3$ as well as in the indices $6, 7$; so we can think of regrouping terms in the sum in this manner and combine first and third sum to

$$\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(1\,3),(6\,7)\rangle}\Big(\big((2(35|1247) + (45|1237)\big) \otimes 1235 \wedge 1234\Big) \ + \ \text{symm.}$$

where "+ symm." denotes the sum over the different Alt_7–translates of that expression. Similarly, the second sum can be written as

$$\mathrm{Alt}_{(3\,5)(4\,6)}\mathrm{Alt}_{\langle(1\,2),(3\,4),(5\,6)\rangle}\big((13|2456) \otimes (1256) \wedge (1234)\big) \ + \ \text{symm.}$$

where this time the antisymmetrization is over a group of order 4, generated by the two involutions (given in the indices of Alt) whose cycle forms are $(3\,5)(4\,6)$ and $(1\,2)(3\,4)(5\,6)$, respectively. It will suffice to match these regrouped terms in order to prove commutativity of the square.

In summary, the left hand side can be written as

$$\begin{aligned} &\Big(\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(1\,3),(6\,7)\rangle}\Big(\big((2(35|1247) + (45|1237)\big) \otimes (1235) \wedge (1234)\Big) \ + \ \text{symm.}\Big) \\ &\qquad + \Big(\mathrm{Alt}_{(3\,5)(4\,6)}\mathrm{Alt}_{\langle(1\,2),(3\,4),(5\,6)\rangle}\Big((13|2456) \otimes (1256) \wedge (1234)\Big) \ + \ \text{symm.}\Big). \end{aligned}$$

RHS. We will now analyse the two expressions arising from taking the boundary of the two terms in $f_7(4)$.

Factoring the first term $\partial f_{31}\big((1234567)\big)$ we get $(1|234567) = \frac{|1235|\cdot|1346|\cdot|1427|}{|1236|\cdot|1347|\cdot|1425|}$, where the factors are 4×4–determinants and so we find

Lemma 2.4.

$$\begin{aligned}\partial\mathrm{Alt}_7\Big((1|234567)_3\otimes(2356)\Big) = \mathrm{Alt}_7(1|234567)_2\otimes\Big(&(1235)\wedge(2356)-(1236)\wedge(2356)\\ &+(1346)\wedge(2356)-(1347)\wedge(2356)\\ &+(1427)\wedge(2356)-(1425)\wedge(2356)\Big).\end{aligned}$$

(Proof is straightforward.)

Grouping again by types of "overlaps" of indices of the two tensor factors on the right, we find that the first two terms $(1235)\wedge(2356)$ and $(1236)\wedge(2356)$ have three overlapping indices ($\{2,3,5\}$ and $\{2,3,6\}$, respectively) and are combined into the first line below, terms #3 and #6 have overlap 2 and are combined into the second line below, and finally terms #4 and #5 have overlap 1 and are combined into the third line below. In summary, we can write the first term on the RHS ($=\partial\circ\mathrm{Alt}_7\big((1|234567)_3\otimes(2356)\big)$) as

$$\begin{aligned}&2\,\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(1\,3),(6\,7)\rangle}\Big((5|236147)_2+(5|236417)_2\Big)\otimes(1235)\wedge(1234)+\text{symm.}\\ &-2\,\mathrm{Alt}_{(3\,5)(4\,6)}\mathrm{Alt}_{(1\,2)(3\,4)(5\,6)}\Big((3|614527)_2+(5|236147)_2\Big)\otimes(1256)\wedge(1234)+\text{symm.}\\ &-4\,\mathrm{Alt}_7\Big((1|234567)_2\otimes(1347)\wedge(2356)\Big).\end{aligned}$$

It turns out that the last line (i.e. the contribution with overlap 1) vanishes.

Proposition 2.5. *We have*

$$\mathrm{Alt}_7\Big((1|234567)_2\otimes(1347)\wedge(2356)\Big)=0\,.$$

Proof. We expand the expression in a similar way to what we did for the above terms into

$$\mathrm{Alt}_{(1\,2)(4\,5)(6\,7)}\mathrm{Alt}_{\langle(1\,4),(4\,7),(2\,5),(5\,6)\rangle}\Big((1|234567)_2\otimes(1347)\wedge(2356)\Big)\ +\ \text{symm.}$$

and then are reduced to show that the individual sums already vanish. For this we use (4.3) below. □

Similarly, the second term $\partial\circ\frac{3}{7}\mathrm{Alt}_7\Big((12|3456)_2\otimes(34|1257)_2\Big)$ of the RHS is by definition

$$\frac{3}{7}\,\mathrm{Alt}_7\Big((12|3456)_2\otimes(34|1257)\wedge(34|1527)\ -\ (34|1257)_2\otimes(12|3456)\wedge(12|3546)\Big)$$

and we can rewrite it as follows.

Proposition 2.6. *The expression* $-\partial\circ\frac{3}{7}\mathrm{Alt}_7\Big((12|3456)_2\otimes(34|1257)_2\Big)$ *equals*

$$2\,\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(1\,3),(6\,7)\rangle}\Big(\big((45|1237)_2+2(46|1237)_2\big)\otimes(1254)\wedge(1234)\Big)+\ \text{symm.}$$

Proof. The differential ∂_7 of $\{x\}_2\wedge\{y\}_2\in\bigwedge^2 B_2(F)$ is defined as follows: $\{x\}_2\otimes y\wedge(1-y)\ -\ \{y\}_2\otimes x\wedge(1-x)\in B_2(F)\otimes\bigwedge^2 F^\times$, so up to the involution

$(1\,3)(2\,4)(6\,7)$ we obtain twice the same terms, so using that $1-\mathrm{r}_2(x_1,x_2,x_3,x_4)=\mathrm{r}_2(x_1,x_3,x_2,x_4)$ we obtain

$$-\partial\circ\frac{3}{7}\mathrm{Alt}_7\Big((12|3456)_2\otimes(34|1257)_2\Big)=2\,(34|1257)_2\otimes(12|3456)\wedge(12|3546)$$

and the latter can be written as

$$2\,\mathrm{Alt}_7(34|1257)_2\otimes\begin{pmatrix}(1235)\wedge(1234)\\+(1235)\wedge(1256)\\-(1235)\wedge(1236)\\-(1235)\wedge(1254)\\+(1246)\wedge(1234)\\+(1246)\wedge(1256)\\-(1246)\wedge(1236)\\-(1246)\wedge(1254)\\-(1236)\wedge(1234)\\-(1236)\wedge(1256)\\-(1245)\wedge(1234)\\-(1245)\wedge(1256)\end{pmatrix}=2\,\mathrm{Alt}_7\begin{pmatrix}(34|1257)_2\\+(56|1237)_2\\+(36|1257)_2\\+(54|1237)_2\\+(34|1267)_2\\+(64|1257)_2\\+(54|1267)_2\\+(63|1257)_2\\+(34|1267)_2\\+(46|1257)_2\\+(53|1247)_2\\+(64|1237)_2\end{pmatrix}\otimes(1235)\wedge(1234)$$

where we apply the following permutations, respectively: #2: $(3\,5)(4\,6)$; #3: $(4\,6)$; #4: $(3\,5)$; #5: $(3\,4)(5\,6)$; #6: $(3\,6)$; #7: $(6\,3\,5)$; #8: $(6\,4\,3)$; #9: $(5\,6)$; #10: $(6\,3)$; #11: $(3\,4)$; #12: $(6\,5\,3)$.

Note that terms #1 and #11 give the same orbit as do terms #2 and #12, terms #3 and #8, terms #5 and #9 as well as terms #6 and #10, so we we are left with only seven different terms as follows

$$2\,\mathrm{Alt}_7\begin{pmatrix}2(34|1257)_2\\+2(56|1237)_2\\+2(36|1257)_2\\+(54|1237)_2\\+2(34|1267)_2\\+2(64|1257)_2\\+(54|1267)_2\end{pmatrix}\otimes(1235)\wedge(1234).$$

Now we need to use a couple of functional equations to write each of these summands (or rather their $\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(2\,3),(6\,7)\rangle}$–orbits) in terms of a generating orbit set. It turns out that we need three such orbits, we choose as representatives $(45|1237)_2$, $(46|1237)_2$ and $(46|1257)_2$. As a shorthand, we will then express each of the six summands as a linear combination of the given three generators and identify each with the corresponding coefficient vector. E.g. $(34|1257)_2 \rightsquigarrow [\frac{1}{3},\frac{1}{3},-1]$ denotes

$$\mathrm{SymAlt}(34|1257)_2=\mathrm{SymAlt}\big(\frac{1}{3}(45|1237)_2+\frac{1}{3}(46|1237)_2-(46|1257)_2\big),$$

where SymAlt is shorthand for $\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(2\,3),(6\,7)\rangle}$. Then we find the respective correspondences

$$\begin{array}{rcl} 2(34|1257)_2 & \rightsquigarrow & [\frac{2}{3},\frac{2}{3},-2] \\ +2(56|1237)_2 & \rightsquigarrow & [0,2,0] \\ +2(36|1257)_2 & \rightsquigarrow & [0,\frac{2}{3},0] \\ +(54|1237)_2 & \rightsquigarrow & [0,\frac{4}{3},0] \\ +2(34|1267)_2 & \rightsquigarrow & [1,0,0] \\ +2(64|1257)_2 & \rightsquigarrow & [\frac{2}{3},0,0] \\ +(54|1267)_2 & \rightsquigarrow & [0,0,2] \end{array}$$

which arise from the following functional equations (valid after applying SymAlt)

$$\begin{array}{rcl} 3(45|1267)_2 & = & 2(45|1237)_2\,, \\ 3(34|1267)_2 & = & 2(46|1237)_2\,, \\ 3(36|1257)_2 & = & (46|1237)_2\,, \\ 3(34|1257)_2 & = & (45|1237)_2+(46|1237)_2-3(46|1257)_2\,, \\ 3(35|1267)_2 & = & 2(46|1237)_2\,. \end{array}$$

Adding up the right hand vectors gives $-\frac{7}{3}[1,2,0]$ which then immediately translates into the claim of the proposition. □

In order to finish the proof of the theorem, we need to compare the three different types of contributions according to the number of overlapping indices (either one, two or three) in the respective rightmost two wedge factors.

(1) For #overlaps= 1 there is no contribution from the RHS, so the contribution from the LHS has to vanish. Indeed we have

Lemma 2.7.

$$\mathrm{Alt}_{(1\,2)(4\,5)(6\,7)}\mathrm{Alt}_{\langle(1\,4),(4\,7),(2\,5)(5\,6)\rangle}(1|234567)_2=0\,.$$

(Proof see below, FE 4.3.)

(2) For #overlaps= 2 there are contributions from the LHS and from the first term of the RHS. Demanding that their difference vanishes amounts to the following statement.

Lemma 2.8.

$$\mathrm{Alt}_{(3\,5)(4\,6)}\mathrm{Alt}_{\langle(1\,2),(3\,4),(5\,6)\rangle}\big((13|2456)_2+(3|614527)_2+(5|236147)_2\big)=0\,.$$

(Proof see below, FE 4.4.)

(3) For #overlaps= 3 there are contributions from the LHS and both terms on the RHS. Demanding that their difference vanishes amounts to the following statement.

Lemma 2.9.

$$\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(2\,3),(6\,7)\rangle}\big((35|1247)_2-(46|1237)_2 \\ -(5|236147)_2-(5|236417)_2\big)=0\,.$$

(Proof see below, FE 4.5.)

Invoking these three relations we obtain that each of the contributions on the LHS equals the corresponding contribution of the RHS, which proves the theorem. □

Remark 2.10. *Given that there is some ambiguity involved in the choice of* $f_7(4)$*, it is conceivable that there is a more suitable candidate for it. In particular, in Goncharov's analysis of the morphism* (2.1) *using Aomoto polylogarithms there are other orbits involved in his definition of* $f_7(4)$*. Moreover, it would be interesting to compare our morphism of complexes to a very related one in* [**17**]*, §7, which invokes insight from cluster algebras.*

3. Vanishing of the $\bigwedge^2 B_2$–component and an integrability condition

3.1. The $\bigwedge^2 B_2$–component. For the final (leftmost) square of the main diagram, we give the following partial results. For a generator $\mathbf{v} = (v_1, \dots, v_8) \in C_8(4)$ the image under $f_7(4) \circ d_8$ is expected to be representable as the image under ∂_8 of a formal "depth 1" combination $\sum_i n_i[x_i]$, and hence the a priori larger complex $\mathcal{G}_4(F) \xrightarrow{\partial_8} B_3(F) \otimes F^\times \oplus \bigwedge^2 B_2(F) \xrightarrow{\partial_7} \dots$ should be quasiisomorphic to $B_4(F) \xrightarrow{\partial_8} B_3(F) \otimes F^\times \xrightarrow{\partial_7}$ (same). Goncharov ([**12**] and [**15**]) shows that one can replace any individual $\{a\}_2 \wedge \{b\}_2$ by a combination of terms in $B_3(F) \otimes F^\times$ with the same ∂_7-image, hence we can indeed land in $B_3(F) \otimes F^\times$ instead. But he also shows ([**11**], Thm. 4.7) that there is no natural map, in terms of formulas, from $\bigwedge^2 B_2(F)$ to $B_3(F) \otimes F^\times$.

In order to check that the outcome of $\xi = f_7(4) \circ d_8(\mathbf{v})$ indeed lies in $\partial_8(B_4(F))$, we first check that it has no genuine contribution to depth 2 by showing that the coboundary δ of ξ in the associated motivic Lie coalgebra is trivial, which in our case boils down to a check—already profitably used in spectacular fashion by Goncharov, Spradlin, Vergu and Volovich in [**16**] to drastically reduce a very complicated expression found by Del Duca, Duhr and Smirnov [**6**] for a central integral in $N = 4$ supersymmetric Yang Mills theory—that ξ vanishes under a specific eight-fold symmetry, due to Goncharov, which is outlined below. A second property needed is that the resulting expression in $B_3(F) \otimes F^\times$ is actually integrable in the sense of Chen. This guarantees that it arises as the ∂_8-image of an element of $\mathcal{G}_4(F)$. Finally, a theorem of (now Rumanian President) Dan ([**5**], Théorème 4), together with the main result of [**10**], implies that an element in $\partial_8 \mathcal{G}_4(F)$ with trivial coboundary as above arises from a depth 1 term in weight 4, i.e. it already lies in the smaller group $\partial_8 B_4(F)$, albeit it does not give an explicit expression.

Preparations. When dealing with $C_8(4)$, and in particular when dealing with determinants of 4×4-matrices, duality simply replaces any 4-element subset of $\{1, 2, \dots, 8\}$ of indices by the complementary set of indices, while preserving the multiplication and division of determinants. We still ignore torsion (in particular 2-torsion) in the following, so we can neglect signs of these determinants and hence write sequences of four indices representing such determinants in the natural ascending order.

We also note that we drop the checks that our expressions are indeed invariant when multiplying any individual vector with a non-zero scalar, and instead refer to Goncharov who has elegantly provided very similar checks in [**12**] and [**15**].

Finally, we will refer to the eightfold antisymmetrisation which mimics the coboundary map in the Chevalley-Eilenberg complex in weight 4 as "applying δ".

3.1.1. *The part from f_{31}.* We now consider the term corresponding to the second contribution f_{31} of the dual ξ' of ξ. In particular we will deal with 3×3–determinants in this dual situation which makes the task slightly less cumbersome. This amounts to analysing the symbol attached to the alternating sum Alt_7 of the following expression where each number $1,\dots,7$ stands for an associated point in $\mathbf{P}^2(F)$

$$(123456)_3 \otimes |147|\,,$$

and where $(\dots)_3$ is a shorthand for the triple ratio (consisting of 6! terms) of the six points, viewed as an element of the higher Bloch group $B_3(F)$. For a triple ratio term a, the numerator of $(1-a)$ factors into a 3×3-determinant and a 6×6-determinant, indicating that we have to consider two types of factors here.

In order to check the coboundary vanishing of ξ', we map all the expressions into $(F^\times)^{\otimes 4}$ and introduce the following 'symbol' $\mathcal{S}$: $\mathcal{S}$ maps $\{a\}_3 \otimes b$ to the antisymmetrisation of $(1-a)\otimes a\otimes a\otimes b$ with respect to the first two slots, i.e. it associates $(1-a)\otimes a\otimes a\otimes b \;-\; a\otimes(1-a)\otimes a\otimes b)$, and it maps $\{a\}_2\wedge\{b\}_2$ to $\big(a\wedge(1-a)\big)\wedge(b\wedge(1-b))$, obvious shorthand for a sum of eight elementary tensors.

Type 1: In the decomposition of $1-x$, where x is one of Goncharov's triple ratios, there are 'new factors' arising (certain 6×6-determinants). For each such 'new factor' the contribution to $\mathcal{S}\circ f_7(3)\circ d_8'(\mathbf{v})$ is zero. In order to see this, note that anti-symmetrising with respect to the group of order eight generated by the transpositions $(1\,4)$, $(2\,5)$ and $(3\,6)$ fixes the new factor as well as the right hand tensor factor $|147|$, and the corresponding eight terms add up to zero in the same way as for Goncharov-Zagier's (cf. the comment in [**11**] before Theorem 3.10) 840-term relation for the trilogarithm.

Type 2: All the other factors are 3×3–determinants. We are left with considering the Alt_7–alternation of

$$\Big(\frac{|123|}{|125|\,|236|\,|314|}\wedge\frac{|124|\,|235|\,|316|}{|125|\,|236|\,|314|}\Big)\otimes\frac{|124|\,|235|\,|316|}{|125|\,|236|\,|314|}\otimes|147|\,.$$

Ignoring the terms with Type-1-factors, each tensor factor of any of the remaining terms, after expansion, becomes simply a single 3×3–determinant.

After expanding in this way we are dealing with $36(=(3+3)\times(3+3))$ terms containing the leftover numerator $|123|$ of the leftmost tensor factor, together with $54(=3\times 3\times(3+3))$ terms containing instead one of the denominator factors of that same tensor factor. (Note that, due to the antisymmetry in the first two slots, we can ignore terms involving only denominator factors from the first two slots.)

Among the Alt_7–orbits of the resulting 90 expressions there are only 34 terms which are actually non-zero. We will consider those in more detail. Two of the orbits occur with multiplicity 5, four others occur with multiplicity 2, the remaining 16 ones only occur once.

For completeness' sake we reproduce representatives for all the leftover orbits, together with their multiplicity (we drop the determinant bars and replace tensor or wedge signs by a comma for ease of notation), while introducing reference labels in square brackets

$$\begin{array}{ll}
5(123,124,135,246)[*], & +5(123,124,135,356)[\mathrm{X}], \\
+2(123,124,125,146)[2\mathrm{A}], & -2(123,124,125,156)[-2\mathrm{B}], \\
+2(123,124,135,146)[2\mathrm{E}], & +2(123,124,135,156)[*], \\
+(123,124,135,126)[\mathrm{A}], & +(123,124,135,136)[\mathrm{B}], \\
+(123,124,145,126)[\mathrm{A}], & +(123,124,145,146)[\mathrm{B}], \\
+(123,124,145,236)[*], & +(123,124,145,456)[\mathrm{X}], \\
+(123,145,124,126)[\mathrm{A}], & +(123,145,124,146)[-\mathrm{A}], \\
+(123,145,124,236)[\mathrm{F}], & +(123,145,124,456)[-\mathrm{F}], \\
-(123,145,246,137)[\mathrm{G}], & -(123,145,246,157)[-\mathrm{G}], \\
-(123,145,246,237)[\mathrm{H}], & -(123,145,246,457)[-\mathrm{H}], \\
-(123,145,246,267)[\mathrm{I}], & -(123,145,246,467)[-\mathrm{I}] \,.
\end{array}$$

Each of these elementary tensors $a\otimes b\otimes c\otimes d$ represents an eightfold combination arising from alternating the first two slots, alternating the last two slots, and furthermore alternating under the swap (slot 1$\leftrightarrow$ slot 3, slot 2$\leftrightarrow$ slot 4).

Three orbits among the above 34 ones, marked with a $[*]$, vanish under δ: apply the cycle $(1\,2)(3\,4)(5\,6)$ to both the multiplicity 5 orbits $(123,124,135,246)$ and the multiplicity 1 orbit $(123,145,145,236)$, and apply the cycle $(2\,5)(4\,6)$ to the multiplicity 2 orbit $(123,124,135,156)$.

Under δ some of the remaining $26(=34-5-2-1)$ orbits agree, possibly up to sign only, and we are left with 8 orbit types only, denoted by roman letter (A, B, E, F, G, H, I, X) as follows, where we indicate the contributions from multiplicities by a superscript. The three orbits marked F, G and H in the three last lines above cancel pairwise (use the permutation $(2\,4)(3\,5)$ in each case).

Type A: $(+^2)$, $(+)$, $(+)$, $(+)$, $(-)$, overall multiplicity $4(=2+1+1+1-1)$;
Type B: $(-^2)$, $(+)$, $(+)$, overall multiplicity 0;
Type E: $(+^2)$, overall multiplicity 2;
Type F, G, H, I: $(+)$, $(-)$, overall 0;
Type X: $(+^5)$, $(+)$, overall multiplicity 6.

Finally, since Type A and Type E consist of expressions whose four factors contain a common index, they vanish once we compose with the boundary map d' as the latter provides each factor with a second common index and the transposition swapping these two indices is an odd permutation fixing the expression.

Upshot: The only type that gives a non-zero contribution to $\delta\circ\mathcal{S}\circ f_{31}\circ d_8'(\mathbf{v})$ is type X, and it occurs with coefficient 6.

3.1.2. *The part from f_{22}.* We consider the term corresponding to the dual of the second contribution f_{22}. This amounts to analysing the Alt_7–alternation of

$$\left(\frac{|123|\,|145|}{|125|\,|143|}\wedge\frac{|124|\,|135|}{|125|\,|134|}\right)\wedge\left(\frac{|215|\,|267|}{|217|\,|265|}\wedge\frac{|216|\,|257|}{|217|\,|256|}\right).$$

which gives us $12^2=144$ terms (again, we do not need to consider contributions if the two leftmost factors—or the two rightmost factors—both arise from the denominator).

Under Alt_7 there are 89 non-zero orbits leftover, which are grouped into orbits of multiplicities 9 (2 such), 7 (5 such), 5 (2 such), 2 (7 such) and 1 (12 such).

$$\begin{array}{cc}
9(123,124,456,145),[-\mathrm{X}] & 9(123,124,356,135),[-\mathrm{X}]\\
7(123,124,156,157),[\mathrm{D}] & -7(123,124,156,125),[\mathrm{B}]\\
7(123,124,145,146),[\mathrm{B}] & 7(123,124,135,136),[\mathrm{B}]\\
7(123,124,125,156),[\mathrm{B}] & -5(123,124,145,456),[-\mathrm{X}]\\
-5(123,124,135,356),[-\mathrm{X}] & -2(123,124,156,257)[*],\\
2(123,124,156,145)[*], & 2(123,124,156,135)[*],\\
-2(123,124,145,246)[*], & 2(123,124,145,156)[*],\\
-2(123,124,135,236)[*], & 2(123,124,135,156)[*],\\
123,124,456,157,[\mathrm{I}] & 123,124,356,157,[\mathrm{I}]\\
123,124,345,146,[\mathrm{C}] & 123,124,345,136,[\mathrm{C}]\\
-(123,124,156,457),[-\mathrm{I}] & -(123,124,156,357),[-\mathrm{I}]\\
-(123,124,145,346),[-\mathrm{C}] & -(123,124,145,126),[-\mathrm{A}]\\
-(123,124,135,346),[-\mathrm{C}] & -(123,124,135,126),[-\mathrm{A}]\\
123,124,125,146,[\mathrm{A}] & 123,124,125,136,[\mathrm{A}]\,.
\end{array}$$

After applying δ, precisely the seven multiplicity 2 orbits (marked by $[*]$) vanish.

Furthermore, under δ some of the remaining $75(= 89 - 2\cdot 7)$ orbits agree, possibly up to sign only, and we are left with 6 orbit types only (four of which agree with orbit types for f_{31}).

Type A: $(+)$, $(+)$, $(-)$, $(-)$, overall multiplicity 0;
Type B: $(+^7)$, $(+^7)$, $(+^7)$ $(+^7)$, overall multiplicity 28;
Type C, I: $(+)$, $(+)$, $(-)$, $(-)$, overall multiplicity 0;
Type D, $(+^7)$, overall 7;
Type X: $(-^9)$, $(-^9)$, $(-^5)$, $(-^5)$, overall multiplicity -28.

Finally, since Type B and Type D consist of expressions whose four factors contain a common index, they vanish under d' as above.

Upshot: The only type that will contribute is type X, and it occurs with coefficient -28, so combining with the result above it is now clear how to cancel the type X contributions. "Dualising the indices" as indicated above we get the following theorem. Note that the linear combination given is 12 times the map $f_7(4)$ above.

THEOREM 3.1. *The following linear combination vanishes under $\delta\circ\mathcal{S}$:*

$$\mathrm{Alt}_8\left(28\Big((1|234567)_3\otimes(1258)\Big)\ +\ 6\Big((81|2345)_2\wedge(82|1567)_2\Big)\right).$$

3.2. Integrability. We now consider the above orbits after applying d'_8, which essentially amounts to adding an eighth common index to the determinant indices of each of the four factors. Note that all the resulting orbits, with the possible exception of type X orbits, either cancel each other or they vanish under the combined antisymmetrisation under Alt_8 and under swapping the first two or the last two tensor factors. Using the criterion for integrability as adapted from Chen, as e.g. given in [**9**] (3.17), one can check that also the type X orbit is indeed integrable to a weight 4 multiple polylogarithm.

For the f_{31}–part, we need to apply the map sending $\{a\}_3\otimes b\in B_3(F)\otimes F^\times$ to $\big((1-a)\otimes a\ -\ a\otimes(1-a)\big)\otimes\big(d\log a\wedge d\log b\big)$ and check that the image of the type X orbit vanishes.

For the f_{22}–part, we have to apply δ to $\{a\}_2 \wedge \{b\}_2$ and further map it (up to overall sign) to the tensor product of $a \wedge b$ with the sum of four symmetric tensors, with rational functions as coefficients, given by

$$\frac{1}{ab}\big((1-a)\odot(1-b)\big)+\frac{1}{a(1-b)}\big((1-a)\odot b\big)+\frac{1}{(1-a)b}\big(a\odot(1-b)\big)+\frac{1}{(1-a)(1-b)}(a\odot b)\,.$$

Here we have used the notation $x \odot y = x \otimes y + y \otimes x$ for the symmetric tensor. The package [**8**] can be used to check that the corresponding antisymmetrised expressions vanish: for the former one only needs to check the integrability with respect to the last two tensor factors as the other possibilities vanish, anyway, while for the latter only the two middle factors have to be tested. The cancellations are rather non-trivial and rely on (differences of) projected Plücker relations.

Dualising, we finally get the result that $f_7(4) \circ d$ maps any configuration in $C_8(4)$ to an integrable expression.

Comments: The results from the above paragraph imply that we can attach to each configuration in $C_8(4)$ a weight 4 multiple polylogarithm. As already indicated in the introduction, Dan [**5**] showed that any multiple polylogarithmic expression in weight 4 can be explicitly reduced[2] to one in I_{31} and Li_4. He also showed that the exactness of the complex $0 \to B_4(F)_{\mathbf{Q}} \to \mathcal{H}_4(F)_{\mathbf{Q}} \to^{\delta_4} \bigwedge^2 B_2(F)_{\mathbf{Q}} \to 0$, where $\mathcal{H}_4(F)_{\mathbf{Q}}$ denotes the vector space of formal multiple polylogarithms in weight 4 and δ_4 an associated coboundary map, would follow provided one can show a conjecture of Goncharov stating that $\sum_{i=1}^5 I_{31}(V(x,y),z)$, where the $V_i(x,y)$ denote the five term relation for the dilogarithm, can be expressed in terms of Li_4 only. A solution to this conjecture was given in [**10**], Thm 17. Therefore we conclude that we can attach to each configuration in $C_8(4)$ a linear combination of Li_4 terms, i.e. a map $f_8(4)$ completing the left hand square. While an explicit version for it is still elusive, with Charlton and Radchenko we give a complicated map in [**4**], a "quadruple ratio" (consisting of 368 orbits under the symmetric group $\mathcal{S}_8$), which constitutes a close cousin. A similar (non-explicit) result has been given by Goncharov and Rudenko in [**17**] in their proof of Zagier's Conjecture in weight 4. Moreover, with Radchenko we have obtained partial results pertaining to certain *degenerate* configurations which allow us to define $f_8(4)$ explicitly in those situations. In particular, for a specific configuration in $C_8(4)$ with three parameters a, b, c, consisting of the four standard basis vectors $\mathbf{e}_i$ $(i = 1, \ldots, 4)$ together with $(1,-1,0,0)$, $(0,1,-1,0)$, $(0,0,1,-1)$ and $(-b,c,-1,a)$ the computer found a matching Li_4–expression with 1260 terms (and integer coefficients of modulus $\leqslant 318$).

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[2]A couple of misprints in his formula were originally corrected in collaboration with Duhr, and treated more thoroughly by Charlton in his thesis [**3**], Thm. 5.2.5 and Rem. 5.2.6, and in [**4**].

4. Appendix: Functional equations.

In this section we collect and prove a couple of functional equations for the dilogarithm needed in the proof that the centre square of (2.1) commutes.

LEMMA 4.1. *There are the following symmetries among the terms* $(\dots)_2$*:*

(1) $(12|3456)_2$ *is symmetric with respect to the permutation* $(1\,2)$ *and anti-symmetric with respect to permutations of 3,4,5,6, so e.g.* $(12|3456)_2 = -(12|4356)_2 = (12|4536)_2$ *etc.*
(2) $(1|234567)_2 = (1|342675)_2 = (1|423756)_2 = -(1|324657)_2 = -(1|432765)_2 = -(1|243576)_2$.
(3) *The (projected) triple ratio* $(1|234567)$ *is a product of two (doubly) projected cross ratios in the following three ways:*
$$(1|234567) = \frac{(12|3457)}{(13|2467)} = \frac{(13|4265)}{(14|3275)} = \frac{(14|2376)}{(12|4356)}.$$
(4) *We have a five term relation involving two triple ratios and two projected cross ratios:*
$$(1|234567)_2 - (1|237564)_2 = -\Big(\frac{(12|3574)}{(13|2674)}\Big)_2 + (13|2764)_2 - (12|3754)_2.$$
[*Proof: Write* $x = (13|2764)$ *and* $y = (12|3754)$*, then the five terms are expressed as follows:* $(1|234567) = \frac{(12|3457)}{(13|2467)} = \frac{1-x^{-1}}{1-y^{-1}}$,
$(1|237564) = \frac{(12|3754)}{(13|2764)} = \frac{y}{x}$ *and* $\frac{(12|3574)}{(13|2674)} = \frac{1-y}{1-x}$*, so the above reduces to the five term relation in the form*
$$\Big(\frac{1-x^{-1}}{1-y^{-1}}\Big)_2 - \Big(\frac{y}{x}\Big)_2 = -\Big(\frac{1-y}{1-x}\Big)_2 + (x)_2 - (y)_2 .]$$

Furthermore, we note that the terms of the form $(a\,b\,c\,d)$ *occurring in a tensor or wedge factor are a shorthand for a* 4×4*–determinant and hence are, up to 2-torsion, invariant under permutation.*

FE 4.2. *Variants of the five term relation.*
$$2\,\mathrm{Alt}_{\langle(6\,7),(1\,2\,3)\rangle}(56|1237)_2 = 3\mathrm{Alt}_{\langle(6\,7),(1\,2\,3)\rangle}(52|3716)_2. \tag{4.1}$$

$$\mathrm{Alt}_{(6\,7)}(5|236147)_2 = -(53|2746)_2 + (52|3716)_2. \tag{4.2}$$

FE 4.3.
$$\mathrm{Alt}_{(1\,2)(4\,5)(6\,7)}\mathrm{Alt}_{\langle(1\,4),(4\,7),(2\,5)(5\,6)\rangle}(1|234567)_2 = 0. \tag{4.3}$$
(We skip the proof which is similar to but easier than the ones for FE 4.4 and FE 4.5 below.)

FE 4.4.
$$\mathrm{Alt}_{(3\,5)(4\,6)}\mathrm{Alt}_{\langle(1\,2),(3\,4),(5\,6)\rangle}\big((13|2456)_2 + (3|614527)_2 + (5|236147)_2\big) = 0.$$

PROOF. The second term $(3|614527)_2$ can be written via Lemma 4.1 (2), as $(3|146275)_2 = \Big(\frac{31|4625}{34|1675}\Big)_2$ and invoking Lemma 4.1 (4) we find
$$\mathrm{Alt}_{(5\,6)}(3|146275)_2 = -\Big(\frac{31|4256}{34|1756}\Big)_2 + (34|1576)_2 - (31|4526)_2.$$

Similarly, the third term $(5|236147)_2$ can be rewritten under the alternation sign as $-(3|254167)_2$ (use the permutation $(3\,5)(4\,6)$) and hence via Lemma 4.1 (2) as $-(3|425716)_2$ and we get an analogous five term relation

$$-\mathrm{Alt}_{(5\,6)}(3|425716)_2 = \left(\frac{34|2765}{32|4165}\right)_2 - (32|4615)_2 + (34|2675)_2\,.$$

Now the first terms on the right in the above two five term equations turn out to be negatives of each other, hence they cancel in the sum. (One can also check that Alt-orbit of that first term on the right vanishes as it is invariant under the odd permutation $(1\,2)$: write it out as a product of determinants to get $\frac{|3126|\cdot|3475|}{|3125|\cdot|3476|}$.)

Therefore the original sum can be replaced by the sum

$$\mathrm{Alt}_{(3\,5)(4\,6)}\ \mathrm{Alt}_{\langle(1\,2),(3\,4),(5\,6)\rangle}$$
$$\Big((13|2456)_2 \quad + \quad \frac{1}{2}\big((34|1576)_2 - (31|4526)_2\big) + \frac{1}{2}\big(-(32|4615)_2 + (34|2675)_2\big)\Big).$$

But $\mathrm{Alt}_{(1\,2)}(34|1576)_2 = -\mathrm{Alt}_{(1\,2)}(34|2576)_2 = \mathrm{Alt}_{(1\,2)}(34|2675)_2$ (for the second equality we use the symmetries in Lemma 4.1 (1) and similarly $\mathrm{Alt}_{(1\,2)}(31|4526)_2 = \mathrm{Alt}_{(1\,2)}(32|4615)_2$, so we can combine the five summands to three, all with coefficient ± 1, in fact to

$$\big((13|2456)_2 + (34|1576)_2 - (31|4526)_2\big)$$

but the first and last of these cancel in view of Lemma 4.1 (1) while the middle one is invariant under the odd permutation $(3\,4)$ and hence its Alt-orbit vanishes. In summary, the original sum indeed vanishes as claimed. ☐

FE 4.5.

$$\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(2\,3),(6\,7)\rangle}\big((35|1247)_2 - (46|1237)_2$$
$$- (5|236147)_2 - (5|236417)_2\big) = 0\,.$$

Proof. Adding (4.2) to its variant where 1 and 4 are swapped we get

$$-(5|236147)_2-(5|236417)_2 = -\frac{1}{2}(53|2746)_2+\frac{1}{2}(52|3716)_2-\frac{1}{2}(53|2716)_2+\frac{1}{2}(52|3746)_2$$

and so, alternating with respect to $(2\,3)$ we find

$$\mathrm{Alt}_{(2\,3)}\big(-(5|236147)_2 - (5|236417)_2\big) = \mathrm{Alt}_{(2\,3)}\big(-(53|2746)_2 - (53|2716)_2\big)\,.$$

Furthermore, antisymmetrising with respect to $(4\,5)$ and then invoking (4.1) we find

$$\mathrm{Alt}_{(4\,5)}\big(-(46|1237)_2\big) = \mathrm{Alt}_{(4\,5)}\big((56|1237)_2\big) = \frac{3}{2}\mathrm{Alt}_{(4\,5)}(53|2716)_2\,,$$

so we can now write the combination in question

$$\begin{aligned} &\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(2\,3),(6\,7)\rangle}\big((35|1247)_2 - (46|1237)_2 \\ &\qquad\qquad - (5|236147)_2 - (5|236417)_2\big) \\ = \quad &\mathrm{Sym}_{(4\,5)}\mathrm{Alt}_{\langle(1\,2),(2\,3),(6\,7)\rangle}\big((35|1247)_2 + \frac{3}{2}(53|2716)_2 \\ &\qquad\qquad - (53|2746)_2 - (53|2716)_2\big)\,. \end{aligned}$$

We combine the second and fourth term to $\frac{1}{2}(53|2716)_2$ and use that the operator $\mathrm{Alt}_{\langle(1\,2),(2\,3),(6\,7)\rangle}$ antisymmetrises over $(1\,2)$ and also over $(6\,7)$ so under the alternation sign we can replace $(35|1247)_2$ by $\frac{1}{2}(35|1247)_2 - \frac{1}{2}(35|1246)_2$ and $-(53|2746)_2$

by $-\frac{1}{2}(53|2746)_2 + \frac{1}{2}(53|1746)_2$. Hence we obtain that the expression under the alternation sign reduces to a standard five term relation (with fixed projection points 3 and 5) as follows, with the obvious new ad hoc notation $(53|27164)$,

$$0 = \text{`}\partial\big((53|27164)\big)\text{'} = (53|7164)_2 - (53|2164)_2 + (53|2764)_2 - (53|2714)_2 + (53|2716)_2\,,$$

thereby proving the statement. □

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