A finite regulator map for the Bloch group

Herbert Gangl and Maxim Kontsevich

Abstract. For a number field F and an N-th root of unity q, put $k_N = F(q)$ and $K_N = \sqrt[N]{F}$. We give two homomorphisms mapping an element $\sum_i [x_i]$ in the Bloch group of F to the quotient group $K_N^\times / k_N^\times (K_N^\times)^N$. The first one is $\prod_i f(x_i;q)$ with $f(a;q) = \prod_{j=1}^N (1-aq^j)^j$, whereas the second assigns $\mathrm{Trace}(M_N)^N / \det(M_N)$, where the $N\times N$-matrix M_N arises as a product of Morita equivalences relating central simple algebras associated to the terms x_i.

1. Motivation[1]

For odd $N \in \mathbb{N}$, choose an N-th root $y = 2^{1/N}$ of 2 and a primitive N-th root of 1, denoted $\zeta = \zeta_N$. Experiments show that we have the following conjectural identity

$$\prod_{j=1}^{N-1} (\zeta^j y - 1)^j \overset{?}{=} \Big(\frac{1}{N} \sum_{k=1}^{N} \zeta^{k^2} \sum_{i=1}^{N} \prod_{j=i}^{N-1} (\zeta^j y - 1) \Big)^N . \tag{1.1}$$

This identity is related to the well-known evaluation of the dilogarithm function $\mathrm{Li}_2(z) = \sum_{m>0} z^m/m^2$ at the point $\frac{1}{2}$, given by

$$\mathrm{Li}_2\Big(\frac{1}{2}\Big) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2)\,.$$

There are only six further non-trivial evaluations of the dilogarithm known (cf. [**6**])^2, half of which involving the golden ratio $\rho = \frac{\sqrt{5}-1}{2}$, stating that

$$\mathrm{Li}_2(\rho) = \frac{\pi^2}{10} - \log^2(\rho)\,,$$

and the latter is related to an identity for ρ similar to (1.1).

The purpose of this note is to explain how such canonical choices for these N-th roots arise naturally. This leads to a suggestion of a map which might be called “étale dilogarithm”.

2020 *Mathematics Subject Classification.* Primary 11R70, 12G05, 57K16.

[1] *Disclaimer:* This paper was essentially written in the early 2000s. It ought to be read in conjunction with the problem stated by the second named author in 2006 [**3**], and some of its contents have been superseded by related work of Calegari, Garoufalidis and Zagier [**2**]. Speculations on the asymptotics, being outdated, have been removed. For a recent far more general prediction see [**4**], §8.

[2] The coefficient $\frac{1}{2}$ of $\log(\frac{1+\sqrt{5}}{2})$ in the evaluation for $\mathrm{Li}_2(\frac{-1-\sqrt{5}}{2})$ in loc.cit. should read -1.

2. Outline of results

Let F be a number field. It was envisaged by Bloch [**1**] and proved by Suslin [**5**] that the "indecomposable" algebraic K-group $\mathrm{K}_3^{\mathrm{ind}}(F)$, defined as the quotient of Quillen's K_3-group $\mathrm{K}_3(F)$ by Milnor's one $\mathrm{K}_3^{\mathrm{M}}(F)$, has a rather explicit avatar, at least after tensoring with $\mathbb{Q}$, as the Bloch group

$$B(F) = \ker \partial_2 / \langle \text{specialisations of the 5-term relation} \rangle$$

with $\partial_2 : \mathbb{Z}[F] \to \bigwedge^2 F^\times$, where $\bigwedge^2 A = A \otimes A / \langle a \otimes b + b \otimes a \rangle$, defined on generators by $[x] \mapsto (1-x) \wedge x$, and (one of the variants of) the 5-term relation is given by

$$[x] + [y] + \Big[\frac{1-y}{x}\Big] + \Big[\frac{x+y-1}{xy}\Big] + \Big[\frac{1-x}{y}\Big]. \tag{2.1}$$

More precisely, a theorem of Suslin shows that $B(F)$ differs from $\mathrm{K}_3^{\mathrm{ind}}(F)$ by a torsion group which for a number field F consists essentially of its roots of unity, a finite subgroup of $F^\times$, so one does not lose much by restricting oneself to $B(F)$.

We describe, for any $N \in \mathbb{N}$, a map

$$L_2^{\text{ét}} = L_2^{\text{ét},N} : B(F)/\text{torsion} \to K_N^\times / k_N^\times (K_N^\times)^N ,$$

where $k_N = F(\sqrt[N]{1})$ and $K_N = \sqrt[N]{F}$ (typically a huge Kummer extension).

Using this fact, we attach to each element $\xi = \sum_i [x_i] \in \mathbb{Z}[F]$ an expression $L_2^{\text{ét}}(\xi)$ in K_N in such a way that, given $\xi \in \ker \partial_2$, together with a compatible choice of N-th roots of the x_i and $1 - x_i$, we expect that $L_2^{\text{ét}}(\xi)$ is essentially an N-th power in K_N. We want to think of the expression $L_2^{\text{ét}}(\xi)^{1/N}$ as an "étale regulator value" in motivic weight 2, hence related to the dilogarithm.

2.1. Our results.

The map factors through a quotient $K_N^\times / k_N^\times (K_N^\times)^N$, because the five term relation for the dilogarithm is mapped to $(K_N^\times)^N$ (Theorem 3.1 below).

In fact, we get two maps which should be closely related.

1. The more straightforward map simply takes $\xi = \sum_{i \in I} [x_i]$ and attaches, for given N and a "compatible choice" of N-th roots $x_i^{1/N}$, together with $q = 1^{1/N}$ a primitive N-th root of 1, the expression $\Phi(\xi, q)$ where we define
$$\Phi(\xi, q) = \prod_{i \in I} \prod_{j=1}^{N} (1 - x_i^{1/N} q^j)^j .$$
2. The second map involves Azumaya algebras. In fact, as we are working over a field, these can be realised as central simple algebras. For any $\xi = \sum_i [x_i]$, we construct a sequence of algebras and Morita equivalences between them, where each of the latter corresponds to precisely one of the terms $[x_i]$.
3. If $\xi \in \ker \partial_2$ then the assignment of algebras is subject to the requirement that the final algebra in this sequence of isomorphic ones *agrees* with the one we started out with. The composition of the Morita equivalences involved provides a matrix M_N (of size $N \times N$) and we then put $\xi_N = \mathrm{Trace}(M_N)^N / \det(M_N)$. We show that, for the five term relation, the latter quotient ξ_N is an N-th power (cf. the proof of Theorem 3.1).
4. We illustrate the construction for a non-trivial element in the Bloch group for the field $\mathbb{Q}(\sqrt{-7})$.

We note that Calegari, Garoufalidis and Zagier [**2**] have independently studied a very similar map to $\Phi(\xi, q)$, and they proved a beautiful application to "Nahm's Conjecture" relating torsion elements in $B(F)$ to modular q-hypergeometric series.

3. A category C of algebras reflecting generators and relations in K_2 of a field

3.1. The category C. We will work with the following tensor (i.e. symmetric monoidal) category C.

Its **objects** $Ob(C)$ are given as follows:
for each pair $(x, y) \in \mathbb{C}^\times \times \mathbb{C}^\times$ there is an object $\mathcal{E}_{x,y} \in Ob(C)$;
for each sequence $((x_1, y_1), \ldots, (x_m, y_m))$ $(m \geq 0)$ there is an object

$$\mathcal{E}_{x_1,y_1} \otimes \cdots \otimes \mathcal{E}_{x_m,y_m} .$$

Its **Hom sets** are freely generated by the following three types mimicking the generators and relations in the Milnor K-group K_2 of a suitable field.

(1) For any $m \geq 0$ there is a certain group Γ_m, given as an extension of the symplectic group $Sp(2m, \mathbb{Z})$ by $\mathbb{Z}^{2m}$, i.e.

$$1 \longrightarrow \mathbb{Z}^{2m} \longrightarrow \Gamma_m \xrightarrow{pr_m} Sp(2m, \mathbb{Z}) \longrightarrow 1 ,$$

with the following property: $\forall \gamma \in \Gamma_m$, $\forall x_1, y_1, \ldots, x_m, y_m \in \mathbb{C}^\times$, there is an isomorphism

$$i_{\gamma;x_1,y_1,\ldots,x_m,y_m} : \bigotimes_{i=1}^{m} \mathcal{E}_{x_i,y_i} \xrightarrow{\sim} \bigotimes_{i=1}^{m} \mathcal{E}_{x'_i,y'_i} ,$$

where $pr_m(\gamma) \in Sp(2m, \mathbb{Z})$ maps $(x_1, y_1, \ldots, x_m, y_m) \in (\mathbb{C}^\times)^{2m} = \mathbb{Z}^{2m} \otimes \mathbb{C}^\times$ to $(x'_1, y'_1, \ldots, x'_m, y'_m) \in (\mathbb{C}^\times)^{2m}$.

The extension group Γ_m is realised as the group of automorphisms of the algebra

$$\mathbb{C}(q)\langle \alpha_1^\pm, \ldots, \alpha_{2m}^\pm \rangle \big/ I$$

over the function field $\mathbb{C}(q)$, where the ideal I is generated by the symplectic relations $\alpha_{2i-1}\alpha_{2i} = q\alpha_{2i}\alpha_{2i-1}$ $(i = 1, \ldots, m)$, together with the commutation relations $\alpha_j\alpha_k = \alpha_k\alpha_j$ for all the other pairs (j, k). More precisely, we associate to $(a_i)_i \in \mathbb{Z}^{2m}$ and $(c_{ij})_{i,j} \in Sp(2m, \mathbb{Z})$ the automorphism defined by mapping (we use the notation $\rightarrow$ to indicate the ordered product of the non-commuting variables α_j)

$$\alpha_i \mapsto q^{a_i} \overrightarrow{\prod_{j=1,\ldots,2m}} \alpha_j^{c_{ij}} , \quad (i = 1, \ldots, 2m) .$$

(2) We impose an "antisymmetry" isomorphism, as well as its dual relation

$$\forall x, y \in \mathbb{C}^\times \qquad \begin{aligned} \tau_{x,y} &: \mathcal{E}_{x,y} \otimes \mathcal{E}_{y,x} \longrightarrow \mathbb{1} \\ \tau_{x,y}^\vee &: \mathbb{1} \longrightarrow \mathcal{E}_{x,y} \otimes \mathcal{E}_{y,x} . \end{aligned}$$

(3) Finally, we impose specific isomorphisms reflecting "Steinberg relations"

$$\forall x, y \in \mathbb{C}^\times , \quad x \neq 1, \qquad \sigma_{x,y} : \mathcal{E}_{x,y} \xrightarrow{\sim} \mathcal{E}_{x,(1-x)y} .$$

3.2. Mapping C to the category of (finitely generated) associative algebras. The above tensor category C can now be "realised", for any integer $N \geq 1$ and any choice of primitive N-th root of unity $q \in \mathbb{C}^\times$, in the category of finitely generated associative $\mathbb{C}$–algebras $\mathcal{A}$. We recall that the objects of $\mathcal{A}$ are given by

$$\begin{aligned} Ob(\mathcal{A}) : \quad & \text{a finitely generated associative } \mathbb{C}\text{–algebra } A \\ & (\text{e.g. } A \cong Mat(k \times k, \mathbb{C}), \ \text{for some } k \geq 1)\,, \end{aligned}$$

and its Hom sets are

$$\begin{aligned} Hom_{\mathcal{A}}(A,B) \ = \ & \{ \text{ isomorphism classes of pairs } (M,m), \\ & \text{where } M \in A\text{-}Mod\text{-}B \text{ is an } (A,B)\text{-bimodule, together with some} \\ & \text{distinguished element } m \in M \text{ realising Morita equivalence}\}. \end{aligned}$$

So if $A \xrightarrow{\sim} Mat(k_1 \times k_1, \mathbb{C})$ and $B \xrightarrow{\sim} Mat(k_2 \times k_2, \mathbb{C})$, then $M \cong \mathbb{C}^{k_1} \otimes \mathbb{C}^{k_2}$ (the isomorphism is defined up to a scalar in $\mathbb{C}^\times$), and m is a $k_1 \times k_2$-matrix (again up to multiplying by a constant in $\mathbb{C}^\times$).

Also, the tensor $\otimes$ in $\mathcal{A}$ is the usual tensor product.

Now $\mathcal{A}$ contains a full subcategory $\mathcal{S}$ (Algebras, isomorphisms) whose objects are isomorphic to *matrix algebras* and whose [iso]morphisms are given as follows: in the case where $k_1 = k_2$ and m is invertible, a homomorphism class $[(M,m)] \in Hom_{\mathcal{A}}(A,B)$ gives rise to an isomorphism $A \to B$ where $a \in A \mapsto b \in B$ if $am = mb \in M$.

We map our category C above to $\mathcal{S} \subset \mathcal{A}$, for each N and q, via the following $\otimes$-functor:

$$F_q : C \longrightarrow \mathcal{S} \subset \mathcal{A}\,,$$

defined on objects by

$$\mathcal{E}_{x,y} \mapsto A_{x,y;q} := \mathbb{C}(q)\langle \xi, \eta \rangle / \langle \xi^N = x\,,\ \eta^N = y\,,\ \eta\xi = q\xi\eta \rangle\,,$$

and on the three types of morphisms, respectively, as follows.

(1) For each $\gamma \in \Gamma_m$, putting $(x'_1, y'_1, \ldots, x'_m, y'_m) = pr_m(\gamma)(x_1, y_1, \ldots, x_m, y_m)$, we have an isomorphism

$$\gamma_{m;x_1,y_1,\ldots,x_m,y_m} \ : A_{x_1,y_1;q} \otimes \cdots \otimes A_{x_m,y_m;q} \ \cong A_{x'_1,y'_1;q} \otimes \cdots \otimes A_{x'_m,y'_m;q}\,.$$

(2) For any $x, y \in \mathbb{C}^\times$ we have a canonical isomorphism

$$A_{x,y;q} \xrightarrow{\sim} A^{\mathrm{op}}_{y,x;q}\,.$$

For any algebra A there are bimodules

$$M \in A \otimes A^{\mathrm{op}} - mod - \mathbb{C}\,, \quad m \in M\,, \qquad M' \in \mathbb{C} - mod - A^{\mathrm{op}} \otimes A\,, \quad m' \in M'\,,$$

such that

$$M = M' \cong A\,, \qquad m = m' = \mathbb{1}\,.$$

(3) For any $x, y \in \mathbb{C}^\times$ we have an isomorphism of algebras

$$\begin{aligned} A_{x,y;q} \ &\xrightarrow{\sim}\ A_{x,(1-x)y;q}\,, \\ \xi, \eta \ &\longrightarrow\ \xi, (1-\xi)\eta\,. \end{aligned}$$

3.3. Two equivalence relations on the morphisms in $\mathcal{C}$.

3.3.1. *The first equivalence relation* $\sim$. Using the $\otimes$-functor F_q above, we impose the following equivalence relation on morphisms in $\mathcal{C}$: if there are two isomorphisms

$$\varphi,\ \psi : \otimes_i \mathcal{E}_{x_i,y_i} \ \longmapsto\ \otimes_i \mathcal{E}_{x'_i,y'_i}\,,$$

then we define

$$\varphi \sim \psi \quad \text{if} \quad F_q(\varphi) = F_q(\psi) \quad \forall N, q\,.$$

On the one hand, we get a (surjective) map to the second K-group of $\mathbb{C}$:

$$\begin{array}{rcl} Ob(\mathcal{C}/\sim) & \longrightarrow & \mathrm{K}_2(\mathbb{C}) \\ \otimes_i \mathcal{E}_{x_i,y_i} & \longmapsto & \sum\limits_i x_i \wedge y_i \in \bigwedge^2 \mathbb{C}^\times / J\,, \end{array}$$

where J denotes the subgroup generated by the $z \wedge (1-z)$ for $z \in \mathbb{C}^\times \setminus \{1\}$. Here we use the fact that we can also write a presentation of $\mathrm{K}_2(\mathbb{C})$ via wedge products rather than tensor products if we put $\bigwedge^2 \mathbb{C}^\times = \mathbb{C}^\times \otimes \mathbb{C}^\times / \langle x \otimes y + y \otimes x \rangle$.

Notice that all morphisms in $\mathcal{C}/\sim$ fiber over $\mathrm{K}_2(\mathbb{C})$.

On the other hand, if we more carefully keep track of the contributions to the wedge product, then for any pair of equivalent (sequences of) morphisms (φ, ψ) we attach an element in the Bloch group of $\mathbb{C}$. Indeed, we can view the composition $\psi^{-1} \circ \varphi$ as an endomorphism χ of $\otimes_i \mathcal{E}_{x_i,y_i}$, and F_q maps it to J, hence we can find $z_j \in \mathbb{C}$ such that χ can be realised as a composition of isomorphisms of type (3), replacing $\mathcal{E}_{z_j,w_j}$ by $\mathcal{E}_{z_j,(1-z_j)w_j}$, and those of type (2) swapping the arguments. Their difference has to vanish under ∂_2, i.e.

$$\sum_j [z_j] \in \ker\Big(\partial_2 : \mathbb{Z}[\mathbb{C}] \longrightarrow \bigwedge^2 \mathbb{C}^\times\Big).$$

3.3.2. *The second equivalence relation* $\approx$. But we also impose a stronger equivalence relation

$$\varphi \approx \psi \ \text{ provided } \ \varphi \sim \psi \ \textit{and} \text{ the associated element in the Bloch group is zero}\,.$$

There is a natural (projection) map

$$\mathcal{C}/\approx\ \longrightarrow\ \mathcal{C}/\sim\,,$$

and the Hom set $Hom_{\mathcal{C}/\approx}$ for the stronger relation fibres over the Hom set $Hom_{\mathcal{C}/\sim}$ of the weaker one, i.e. we can think of the former as a principal homogeneous space over the latter with fibre being the Bloch group of $\mathbb{C}$.

3.4. Explicit matrices for the Azumaya algebras. Now given x_i and y_i ($i = 1, \dots, m$), we choose, for fixed $N > 0$ and q an N-th root of unity, N-th roots $\sqrt[N]{x_i}$, $\sqrt[N]{y_i}$, compatibly with multiplicative relations, i.e. $\prod_i x_i^{a_i} y_i^{b_i} = 1$ entails $\prod_i \sqrt[N]{x_i}^{a_i} \sqrt[N]{y_i}^{b_i} = 1$. Then we identify

$$A_{x_i,y_i;q} \xrightarrow{\sim} \text{Matrix algebra } Mat(N \times N, \mathbb{C}) \text{ with}$$

$$\xi_i = x_i^{1/N} \begin{pmatrix} 1 & & & \\ & q & & \\ & & q^2 & \\ & & & \ddots \end{pmatrix}, \qquad \eta_i = y_i^{1/N} \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \end{pmatrix}.$$

In this case, we can associate to the isomorphism class $[(Mat(N\times N,\mathbb{C}),m)]$ certain explicit matrices, e.g. for a morphism of type (3) above we have

$$M_N = (1-x_i^{1/N})\begin{pmatrix} 1 & & & \\ & (1-qx_i^{1/N}) & & \\ & & (1-qx_i^{1/N})(1-q^2x_i^{1/N}) & \\ & & & \ddots \end{pmatrix}(q^{ij})_{i,j}\,,$$

where the matrix $(q^{ij})_{i,j}$ is a Vandermonde matrix.

Conjecture: *If two morphisms in C are equivalent under $\approx$ then the products of the explicit matrices above are equal (up to an N-th power).*

3.5. Trace and determinant.

Finally we can attach, to any $\varphi\in End_C(\otimes\,\mathcal{E}_{x_i,y_i})$ and an N-th root of unity q, a matrix M_N attached in a similar way to the above, i.e., multiplying the associated matrices for the individual moves of type (3) in a realization of an endomorphism of some $\otimes_i\mathcal{E}_{x_i,y_i}$, and hence we associate numerical invariants, the trace $\mathrm{Tr}(M_N)\in K_N$ as well as the determinant $\mathrm{Det}(M_N)$.

3.6. Well-definedness of the map.

We first establish that the relations in the Bloch group map to zero in the quotient. More precisely, with the above notation we can show the following.

Theorem 3.1. *Let F be a number field, $N\geq 1$ and let q be any choice of a primitive N-th root of unity. For the five term relation* (2.1)*, written as $\sum_i[t_i]$, the trace* $\mathrm{Tr}(M_N)$ *of the associated matrix M_N is a number $Z_N\in F(q,\{t_i^{1/N},(1-t)_i^{1/N}\}_i)$, and* $\mathrm{Det}(M_N)=(NZ_N)^N$.

Proof. For each $x,y\in F^\times$ we consider the cyclic algebra $A_{x,y;q}$ generated over F by ξ, η, subject to the relations $\xi^N=x$, $\eta^N=y$ and $\eta\xi=q\xi\eta$ (q commuting with both ξ and η).

If furthermore $y\neq 1$ then there is an algebra isomorphism

$$A_{x,y;q}\overset{\widetilde{\rightarrow}}{\longrightarrow}A_{y,\frac{1-y}{x};q}\,, \tag{3.1}$$

given by

$$(\xi,\eta)\longmapsto(\eta,\xi^{-1}(1-\eta)\cdot q^{-1})=:(\tilde\xi,\tilde\eta)\,, \tag{3.2}$$

whose inverse is $(\tilde\xi,\tilde\eta)\mapsto((1-\tilde\xi)\tilde\eta^{-1}\cdot q^{-1},\tilde\xi)$.

In keeping with the above conventions, we put $\tilde x=y$ and $\tilde y=\frac{1-y}{x}$ and then identify $A_{x,y;q}$ with $A_{\tilde x,\tilde y;q}$ by checking that the latter has generators $\tilde\xi$, $\tilde\eta$ subject to $\tilde\xi^N=\tilde x$, $\tilde\eta^N=\tilde y$ and $\tilde\eta\tilde\xi=q\tilde\xi\tilde\eta$. Indeed, $\tilde\xi^N=\eta^N=y=\tilde x$ and

$$\begin{aligned}\tilde\eta^N &= \underbrace{\xi^{-1}(1-\eta)\cdots\xi^{-1}(1-\eta)}_{N\text{ factors}}\cdot q^{-N}=\xi^{-N}(1-q^{N-1}\eta)\cdots(1-q^{-1}\eta)(1-\eta)\\ &= \xi^{-N}(1-\eta^N)=x^{-1}(1-y)=\tilde y\,.\end{aligned}$$

Here we have used that $\xi^{-1}f(\eta)=f(q\eta)\xi^{-1}$ and $f(\eta)\xi^{-1}=\xi^{-1}f(q^{-1}\eta)$ for any polynomial $f(\eta)$ in η. Finally, since $\xi^{-1}\eta=q\eta\xi^{-1}$, we have

$$\tilde\eta\tilde\xi=\xi^{-1}(1-\eta)\eta\cdot q^{-1}=\xi^{-1}\eta(1-\eta)\cdot q^{-1}=q\,\eta\,\xi^{-1}(1-\eta)\cdot q^{-1}=q\tilde\xi\tilde\eta\,.$$

The automorphism of $\mathbb{Q}(x,y)$ given by $(x,y)\mapsto(y,\frac{1-y}{x})$ has order 5 and one obtains an automorphism of $A_{x,y;q}$ by composing the respective five isomorphisms of the type in (3.1). By choosing the N-th roots of $(t_i)_i=(x,y,\frac{1-y}{x},\frac{x+y-1}{xy},\frac{1-x}{y})$ compatibly, we find that this composition equals the identity.

Lemma 3.2. *Choose* $\tau_i = \sqrt[N]{t_i}$ *such that* $\tau_{i-1}\tau_{i+1} = (1-\tau_i)q^{-1}$ (i mod 5). *Then the product of the corresponding five algebra isomorphisms* $\varphi_i : A_{t_{i-1},t_i;q} \overset{\sim}{\longrightarrow} A_{t_i,t_{i+1};q}$ *equals the identity.*

Proof. Writing the terms τ_i in sequence, there are four neighbouring terms that are immediate from (3.2) and its inverse, giving

$$(\tau_{i-2}, \tau_{i-1}, \tau_i, \tau_{i+1}) = ((1-\xi)\eta^{-1}q^{-1},\ \xi,\ \eta,\ \xi^{-1}(1-\eta)q^{-1}).$$

It remains to check that $\tau_{i-3} = \tau_{i+2}$. Indeed,

$$\begin{aligned} \tau_{i-3} &= (1-(1-\xi)\eta^{-1}q^{-1})\,\xi^{-1}q^{-1} = (1 - q^{-1}\eta^{-1} + q^{-1}\xi\eta^{-1})\,\xi^{-1}\eta^{-1} \\ &= q^{-1}(\xi^{-1} - q^{-1}\eta^{-1}\xi^{-1} + q^{-1}\xi\,\underbrace{\eta^{-1}\xi^{-1}}_{=q\xi^{-1}\eta^{-1}}), \end{aligned}$$

which indeed agrees with

$$\begin{aligned} \tau_{i+2} &= \eta^{-1}(1-\xi^{-1}(1-\eta)q^{-1})\,q^{-1} = \eta^{-1}(1 - q^{-1}\xi^{-1} + q^{-1}\xi^{-1}\eta)\,q^{-1} \\ &= q^{-1}(\eta^{-1} - q^{-1}\eta^{-1}\xi^{-1} + \underbrace{q^{-1}\eta^{-1}\xi^{-1}}_{=\xi^{-1}\eta^{-1}}\,\eta)\,. \qquad \square \end{aligned}$$

Since we want to attach a numerical invariant to a given 5-term relation, we consider matrix representations for the associated algebras. A typical one is suggested by the commutation relation $\eta\xi = q\xi\eta$. Define

$$\rho_{x,y;q} : A_{x,y;q} \to Mat(N \times N, F') = End(V_{x,y;q})$$

for some extension field F' of F and some N-dimensional F'-vector space $V_{x,y;q}$, by choosing a basis $(e_j)_j \simeq \mathbb{Z}/N\mathbb{Z}$ and, after a further choice of N-th roots $x^{1/N}$, $y^{1/N}$, we can take the endomorphism

$$\xi : e_j \mapsto q^j x^{1/N} e_j\,, \qquad \eta : e_j \mapsto y^{1/N} e_{j+1}$$

(one checks immediately the commutation relation above, as well as $\xi^N = x{\cdot}Id$, $\eta^N = y{\cdot}Id$).

We can think of the five rational functions t_i arising from the 5-cycle above as coordinates on the moduli space $\mathcal{M}_{0,5}$ of 5 points on $\mathbb{P}^1$ up to the standard action of $PGL_2(K)$. (Note that the constraints on the t_i are given by $1 - t_i = t_{i-1}t_{i+1}$ (with i mod 5).)

More generally, in the same way we can impose coordinates on the variety $\sqrt[N]{\mathcal{M}_{0,5}}$ with coordinate functionals s_i (i mod 5) such that $s_i \in F^\times \setminus \mu_N$ (μ_N denoting the N-th roots of unity) and subject to the constraints $s_{i-1}^N s_{i+1}^N = 1 - s_i^N$. (One should think that s_i^N corresponds to t_i above.) Thus we can view $\sqrt[N]{\mathcal{M}_{0,5}}$ as an N^5-fold covering of $\mathcal{M}_{0,5}$, with Galois group $(\mathbb{Z}/N\mathbb{Z})^5$ acting by $s_i \mapsto q^{a_i}s_i$, $a_i \in \mathbb{Z}/N\mathbb{Z}$ ($i = 1, \dots, 5$). For each choice of such $(s_i)_i$ we obtain a representation $\rho_{s_i,s_{i+1}}$ in the above way, where the basis vectors $e_j^{(i,i+1)} \in V_{i,i+1}$ are acted upon as follows (j mod N)

$$\tau_i : e_j^{(i,i+1)} \longmapsto q^j s_i e_j^{(i,i+1)}\,, \qquad \tau_{i+1} : e_j^{(i,i+1)} \longmapsto s_{i+1} e_{j+1}^{(i,i+1)}\,.$$

Now identify explicitly each isomorphism $\phi_i : V_{i-1,i} \overset{\sim}{\longrightarrow} V_{i,i+1}$ since then the composition $\prod_{i \bmod 5} \phi_i$ automatically acts as a scalar (matrix) $Z = Z(s_1, \dots, s_5)$.

Denote the basis vectors $e_j^{(i-1,i)}$ on the left by ℓ_j and the $e_j^{(i,i+1)}$ on the right by r_j. With the notations above, we find on the left

$$\xi : \ell_j \longmapsto q^j x^{1/N} \ell_j\,, \qquad \eta : \ell_j \longmapsto y^{1/N} \ell_{j+1}\,,$$

while on the right we get (cf. above)

$$\tilde{\xi} = \eta : r_j \longmapsto q^j y^{1/N} r_j\,, \qquad \tilde{\eta} = q^{-1}\xi^{-1}(1-\eta) : r_j \longmapsto \Big(\frac{1-y}{x}\Big)^{1/N} r_{j+1}\,.$$

It is clear that r_j is an eigenvector of η with eigenvalue $q^j y^{1/N}$. But the same is true for $\sum_i q^{-ij}\ell_i$. (For the proof, shift the summation index and use $q^{-(i-1)j} = q^j \cdot q^{-ij}$.) Therefore we make the Ansatz

$$r_j = c_j \cdot \sum_i q^{-ij}\ell_i =: c_j \cdot F_j$$

for some constant $c_j \in \sqrt[N]{F}$; here F_j should be thought of as a “Fourier transform”. Then we use the transformation property under $q^{-1}\xi^{-1}(1-\eta) : c_j F_j \longmapsto c_{j+1}F_{j+1}$ which we know produces a factor $(\frac{1-y}{x})^{1/N}$. This implies, after normalizing $c_0 = 1$, the constants successively as $c_{-1} = q^{-1}(1-y)^{-1/N}\cdot(1-y^{1/N})$, $c_{-2} = q^{-2}(1-y)^{-2/N}\cdot(1-y^{1/N})(1-q^{-1}y^{1/N})$, etc., and hence we can rewrite the product as

$$\prod_{j\in\mathbb{Z}/N\mathbb{Z}} c_j = \big(q(1-y)^{1/N}\big)^{-\binom{N}{2}} \cdot (1-y^{1/N})^{N-1}(1-q^{-1}y^{1/N})^{N-2}\cdots(1-q^{-(N-2)}y^{1/N}))^1\,.$$

On the one hand, the product over the $N-1$ righthand factors is reexpressed, after rearranging the factors backwards, as

$$\begin{aligned} & (1-y^{1/N})^{N-1}(1-q^{-1}y^{1/N})^{N-2}\cdots(1-q^{-(N-3)}y^{1/N}))^2(1-q^{-(N-2)}y^{1/N}))^1 \\ = \; & (1-q^2y^{1/N}))^1(1-q^3y^{1/N}))^2\cdots(1-q^{N-1}y^{1/N}))^{N-2}(1-q^Ny^{1/N}))^{N-1} \\ = \; & \frac{(1-q^1y^{1/N}))(1-q^2y^{1/N}))^2(1-q^3y^{1/N}))^3\cdots(1-q^{N-1}y^{1/N}))^{N-1}}{(1-q^1y^{1/N}))^1(1-q^2y^{1/N}))^1(1-q^3y^{1/N}))^1\cdots(1-q^{N-1}y^{1/N}))^1} \cdot \frac{(1-q^Ny^{1/N}))^N}{(1-q^Ny^{1/N}))^1}\,, \end{aligned}$$

where the denominator now simplifies to $\prod_j(1-q^jy^{1/N}) = 1-y$ and we can rewrite the left numerator as $f(y^{1/N};q)$, where

$$f(t;q) := \prod_{j=1}^{N-1}(1-q^jt)^j\,.$$

(Equivalently, we can let the product run also over $j=0$, at least for $t \neq 1$, and for $t=1$ it is then still consistent with the usual convention $0^0 = 1$.)

We further note that

$$q^{\frac{N(N-1)}{2}} = (-1)^{N+1}\,, \tag{3.3}$$

and that the two exponents $\binom{N}{2}$ and N of the factor $(1-y)$ add up to $\binom{N+1}{2}$.

In summary, we obtain

$$\prod_{j\in\mathbb{Z}/N\mathbb{Z}} c_j = (-1)^{N+1}\cdot\frac{(1-y^{1/N})^N}{\big((1-y)^{1/N}\big)^{\frac{N(N+1)}{2}}}\cdot f(y^{1/N};q)\,.$$

On the other hand, we can compute the determinant of the product of the Jacobians of all the transformations from “left” (ℓ_j) to “right” (r_j):

$$\frac{\bigwedge_{j=0}^{N-1}\mathrm{d}\,r_j}{\bigwedge_{j=0}^{N-1}\mathrm{d}\,\ell_j} = \Big(\prod_j c_j\Big)\cdot\frac{\bigwedge_{j=0}^{N-1}\mathrm{d}\,F_j}{\bigwedge_{j=0}^{N-1}\mathrm{d}\,\ell_j}\,,$$

the second factor on the right being a Vandermonde determinant, more precisely

$$\prod_{0\le i<j\le N-1}(q^j-q^i) = (q^1-1)^{N-1}(q^2-1)^{N-2}\cdots(q^{N-1}-1)^1\cdot q^{(N-1)(N-2)/2}\,.$$

Here the rightmost factor in turn can be simplified, in view of (3.3), to

$$q^{(N-1)(N-2)/2} = q^{N(N-1)/2} q^{-(N-1)} = (-1)^{N+1} q \,,$$

and the product of the remaining $N-1$ factors equals

$$\frac{(1-q^1)^N (1-q^2)^N \cdots (1-q^{N-1})^N}{(1-q^1)(1-q^2)^2 \cdots (1-q^{N-1})^{N-1}} (-1)^{\frac{N(N-1)}{2}} = \frac{N^N (-1)^{\frac{N(N-1)}{2}}}{f(1;q)} \,.$$

Therefore we get overall

$$\prod_{0 \le i < j \le N-1} (q^j - q^i) = \frac{N^N (-1)^{\frac{N(N-1)}{2}}}{f(1;q)} \cdot (-1)^{N+1} q \,.$$

Together with the above result on $\prod c_j$, we find that, at least up to $f(1;q)$ and a root of unity, the product $\prod_{i=1}^5 f(s_i;q)$ is an N-th power in the underlying field, more precisely, it is (essentially) equal to the N-th power of

$$\frac{\prod_i s_i^{N+1}}{\prod_i (1-s_i)} Z_N^{-1} \,,$$

where

$$N \cdot Z_N = \sum_{j_1,\ldots,j_5 \in \mathbb{Z}/N\mathbb{Z}} q^{\sum_{i \in \mathbb{Z}/5\mathbb{Z}} j_i j_{i+1}} \cdot \prod_{i \in \mathbb{Z}/5\mathbb{Z}} (q s_{i-1} s_{i+1}))^{-j_i} (s_i; q^{-1})_{j_i} \,,$$

with the usual q-Pochhammer notation $(s;q)_a = (1-s)(1-qs)\cdots(1-q^{a-1}s)$ $(a \ge 1)$. □

In view of the theorem, the map $L_2^{\text{ét},N}$ factors through $\mathbb{Z}[F']/\langle\text{5-term relations}\rangle$, where $F' = F(\{t_i^{1/N}\}_i)$, and may be thought of as a "finite/étale regulator map" for $\ker \partial_2$ and hence for $\mathrm{K}_3(F')$.

Remark. It may be interesting to give details on $f(1;q)$, which plays the role of $\mathrm{Li}_2(1)$. Experimentally, one has (for $N > 3$ a prime, while for $N = 3$ a slightly different formula holds: the right hand side needs to be multiplied by q)

$$\prod_{j=1}^{N-1} (1-q^j)^j = \left(\frac{2}{N}\right) N^{\frac{N-1}{2}} \sum_{k=1}^{N-1} \left(\frac{k}{N}\right) q^k$$

with the usual notation for the Kronecker symbol. (Can we think of it as some kind of "finite/étale period"?)

Remark. There is a natural way to produce the map suggested above, by "differentiating the distribution relation" in the following sense: let $\partial_2 : \mathbb{Z}[F] \to \bigwedge^2 F^\times$ be the usual map given on generators by $[t] \mapsto t \wedge (1-t)$, then

$$\partial_2\Big(\frac{1}{N}[t^N] - \sum_{j=0}^{N-1} [\zeta^i t]\Big) = \prod_{j=1}^{N-1} (\zeta \otimes (1-\zeta^j t)^j) = \zeta \otimes \prod_{j=1}^{N-1} (1-\zeta^j t)^j \,.$$

4. Non-vanishing elements in $B(F)$

4.1. An example with $F = \mathbb{Q}(\sqrt{-7})$. We want to associate the above trace to non-trivial elements in the Bloch group. Let α and β be algebraic numbers satisfying

$$1 - \alpha = -\frac{1}{\beta} \,, \qquad 1 - \beta = \alpha^2 \beta \,.$$

This can be realized over the field $F = \mathbb{Q}(\sqrt{-7})$, where $\alpha = \frac{1-\sqrt{-7}}{2}$ (cf. [**6**], §4, or [**7**]).

Then $\rho = 2[\alpha] + [1-\beta]$ lies in the kernel of the above map ∂_2.

The Beilinson regulator of ρ in weight 2 (which we identify here with the Bloch-Wigner dilogarithm for simplicity) is a special L-value, more precisely it equals $\frac{3\sqrt{7}^3}{4\pi^2}\zeta_F(2)$.

Remark: How to characterize Bloch elements which are linear combinations of specialisations of the 5-term relation, as opposed to non-trivial elements in the Bloch group? If we write

$$\Phi(a,q)=\prod_{j=1}^{N}(1-aq^j)^j\,,$$

then we look at the quotient

$$\frac{\Phi(x_i^{1/N}q,q)}{\Phi(x_i^{1/N},q)}=\frac{(1-x_i^{1/N})^N}{(1-x_i)}=\frac{(1-x_i^{1/N})^N}{\prod_j x_j^{a_{ij}}}\,.$$

Now a functional equation/five term relation is characterised by having the product (over i) of the corresponding quotients above being an N-th power.

Hence in a sense an element in $B(F)$ gives a canonical extension field: Any $N>0$ and $\sum_i[x_i]\in\mathbb{Z}[F]$ give rise to $F(\sqrt[N]{1},\{\sqrt[N]{x_i},\ \sqrt[N]{1-x_i}\}_i)$, with compatibly chosen roots for the x_i and $1-x_i$.

Example (cont'd): We consider the above example in detail. The algebras arising from this process (on the left), together with an ordered basis (on the right), are

$$\begin{array}{rcl}
\mathcal{E}_{\alpha,-1} & \leftrightarrow & \quad\xi_1,\ \eta_1\,,\\
\mathcal{E}_{\alpha,-(1-\alpha)} & \leftrightarrow & \quad\xi_2,\ \eta_2\,,\\
\mathcal{E}_{\alpha,1/\beta} & \leftrightarrow & \text{the same}\\
\mathcal{E}_{1/\alpha,\beta} & \leftrightarrow & \quad\xi_3,\ \eta_3\,,\\
\mathcal{E}_{(1-\beta)/\alpha,\beta} & \leftrightarrow & \quad\xi_4,\ \eta_4\,,\\
\mathcal{E}_{\alpha\beta,\beta} & \leftrightarrow & \text{the same}\\
\mathcal{E}_{\alpha,\beta} & \leftrightarrow & \quad\xi_5,\ \eta_5\,,\\
\mathcal{E}_{\alpha,\beta(1-\alpha)} & \leftrightarrow & \quad\xi_6,\ \eta_6\,,\\
\mathcal{E}_{\alpha,-1} & \leftrightarrow & \text{the same}\,.
\end{array}$$

The following transitions highlight the steps where we encounter a new Steinberg symbol $z\wedge(1-z)$ when passing from one algebra to the next,

$$\begin{array}{ccc}
(\xi_1,\ \eta_1) & \overset{\alpha\wedge(1-\alpha)}{\longleftrightarrow} & (\xi_2,\ \eta_2)\,,\\
(\xi_3,\ \eta_3) & \overset{(1-\beta)\wedge\beta}{\longleftrightarrow} & (\xi_4,\ \eta_4)\,,\\
(\xi_5,\ \eta_5) & \overset{\alpha\wedge(1-\alpha)}{\longleftrightarrow} & (\xi_6,\ \eta_6)\,.
\end{array}\tag{4.1}$$

We choose the following N-th roots (for consistency of signs, take N odd): $q=\sqrt[N]{1}$, $q'=\sqrt[N]{-1}=-1$, $\sqrt[N]{\alpha}$ and $\sqrt[N]{\beta}$. Then we need to make sure that we choose the roots of $1-\alpha$ and of $1-\beta$ as follows:

$$\sqrt[N]{1-\alpha}=-\frac{1}{\sqrt[N]{\beta}}\,,$$

$$\sqrt[N]{1-\beta}=(\sqrt[N]{\alpha})^2\,\sqrt[N]{\beta}\,.$$

For the generators ξ_i, η_i, we find the following (starting with $i = 1$), recalling that we have $\xi_i\eta_i = q\eta_i\xi_i$):

$$\xi_1 = \sqrt[N]{\alpha}\begin{pmatrix} 1 & & & \\ & q & & \\ & & q^2 & \\ & & & \ddots \end{pmatrix}, \qquad \eta_1 = q'\begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

For $i > 1$, we get successively the following conditions:

$$\xi_2 = T_{12}\,\xi_1\,T_{12}^{-1}\,, \qquad \eta_2 = T_{12}\,\eta_1(1-\xi_1)\,T_{12}^{-1}\,.$$

$$\xi_3 = T_{23}\,\xi_2^{-1}\,T_{23}^{-1}\,, \qquad \eta_3 = T_{23}\,\eta_2^{-1}\,T_{23}^{-1}\,.$$

$$\xi_4 = T_{34}\,\xi_3(1-\eta_3)\,T_{34}^{-1}\,, \qquad \eta_4 = T_{34}\,\eta_3\,T_{34}^{-1}\,.$$

$$\xi_5 = T_{45}\,\xi_4\eta_4^{-1}\,T_{45}^{-1}\,, \qquad \eta_5 = T_{45}\,\eta_4\,T_{45}^{-1}\,.$$

$$\xi_6 = T_{56}\,\xi_5\,T_{56}^{-1}\,, \qquad \eta_6 = T_{56}\,\eta_5(1-\xi_5)\,T_{56}^{-1}\,.$$

We guess the form of T_{23} as

$$\begin{pmatrix} 1 & & & \\ & & & 1 \\ & & \iddots & \\ & 1 & & \end{pmatrix}.$$

Looking at the eigenvectors and their eigenvalues for T_{12}, say, we find

$$T_{12}^{-1} = \operatorname{diag}\begin{pmatrix} 1 \\ -(1-\sqrt[N]{\alpha})\sqrt[N]{\beta} \\ +(1-\sqrt[N]{\alpha})(1-q\sqrt[N]{\alpha})\sqrt[N]{\beta}^2 \\ \vdots \end{pmatrix} = \operatorname{diag}\Big((-1)^i\sqrt[N]{\beta}^i\prod_{0\le j<i}(1-q^j\sqrt[N]{\alpha})\Big).$$

Moreover, we get the same matrix for the last step, i.e., $T_{56} = T_{12}$. The transformation T_{45}^{-1} is given by cyclic shifts of the first row vector $(q^{\binom{i}{2}})_{i=1,\ldots,n}$, i.e. as $(q^{\binom{i-j}{2}})_{i,j}$. Finally, the arguably most complicated one is T_{34}^{-1}, where in the (i, j)-entry ($0 \le i, j < N$) we get

$$\frac{q^{\binom{i-j+1}{2}}\sqrt[N]{\beta}^{i-j}}{\prod_{k=1}^{i-j}(q^k-\sqrt[N]{\alpha}^2\sqrt[N]{\beta})}\,.$$

Now we take the product of all the five transformation matrices involved, i.e.,

$$M_N := T_{56}T_{45}T_{34}T_{23}T_{12}\,,$$

and consider its trace and determinant. The resulting $\xi_N = \operatorname{Trace}(M_N)^N/\det(M_N)$ ought to be related, in a sense, to the corresponding "étale regulator" for the Bloch element $2[\alpha] + [1-\beta]$ where the three contributions stem from (4.1).

A numerical example. For $N = 5$ in the above example, we can recognise the numerical value $2\xi_5 \doteq -62.8929250152668144\ldots + 100.462207922407031\ldots\cdot i$ as an algebraic integer of degree 8 and of norm $(2^4\cdot 331)^5$.

Furthermore, numerical checks appear to relate the limit $\lim_{N\to\infty} \log|\xi_N|^{1/N^2}$ with the Beilinson-Borel regulator in weight 2 for the field $\mathbb{Q}(\sqrt{-7})$. For a recent account of a much more general prediction see [**4**], §8.

Acknowledgments. The first named author is grateful to the IHES in Bures-sur-Yvette and the MPI for Mathematics in Bonn for their hospitality and ideal working conditions, with thanks to Qizhen Sang for spotting inconsistencies in a previous draft.

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Mathematical Sciences & Computer Science Building, Upper Mountjoy, Durham University, DH1 3LE
Email address: `herbert.gangl@durham.ac.uk`

Institut des Hautes Etudes Scientifiques, 35 Route de Chartres, 91440 Bures-sur-Yvette
Email address: `maxim@ihes.fr`