Chapter 2

Finding the Correct Number is Simplicity Itself

Simplicity, simplicity, simplicity. I say let your affairs be as two or three, and not a hundred or a thousand; instead of a million count half a dozen and keep your accounts on your thumbnail.

Henry David Thoreau, *Walden*.

And calculate the stars.

John Milton, *Paradise Lost*, VIII, 80.

We have seen that positions in Hackenbush and Ski-Jumps are often composed of several non-interacting parts, and that then the proper thing to do is to add up the values of these parts, measured in terms of free moves for Left. We have also seen that halves and quarters of moves can arise. So plainly we'll have to decide exactly what it means to add games together, and work out how to compute their values.

WHICH NUMBERS ARE WHICH?

Let's summarize what we already know, using the notation

$$\{a,b,c,\ldots|d,e,f,\ldots\}$$

for a position in which the options for Left are to positions of values $a,b,c,\ldots$ and those for Right to positions of values $d,e,f,\ldots$. In this notation, the whole numbers are

$$0=\{\ |\ \},\qquad 1=\{0|\ \},\qquad 2=\{1|\ \},\qquad \ldots,\qquad n+1=\{n|\ \},$$

for from a zero position, neither player has a move, and from a position with $n+1$ free moves for Left, he can move so as to leave himself just n moves, whereas Right cannot move at all.

The negative integers are similarly

$$-1=\{\ |0\},\qquad -2=\{\ |-1\},\qquad -3=\{\ |-2\},\qquad \ldots,\qquad -(n+1)=\{\ |-n\}.$$

We also found values involving halves:

$$\tfrac{1}{2}=\{0|1\},\qquad 1\tfrac{1}{2}=\{1|2\},\qquad 2\tfrac{1}{2}=\{2|3\},\qquad \ldots,$$

$$-\tfrac{1}{2}=\{-1|0\},\qquad -1\tfrac{1}{2}=\{-2|-1\},\qquad \ldots\ \text{and so on.}$$

Our proof that $\{0|1\}$ behaves like half a move was contained in the discussion of the Hackenbush position of Fig. 6(a) in Chapter 1.

We also discussed a Hackenbush position (Fig. 8(a) of Chapter 1) whose value was $\{0|\tfrac{1}{2}\}$ and showed that it behaved like one quarter of a move. So we can guess that we have all the equations

$$\{0|1\} = \tfrac{1}{2}, \qquad \{0|\tfrac{1}{2}\} = \tfrac{1}{4}, \qquad \{0|\tfrac{1}{4}\} = \tfrac{1}{8}, \quad \text{and so on},$$

and leave a more precise discussion of what these equations mean until later.

Will there be any game with a position of value $\tfrac{5}{8}$? Yes, of course! All we have to do is add together two positions of values $\tfrac{1}{2}$ and $\tfrac{1}{8}$ as in the Hackenbush position of Fig. 1.

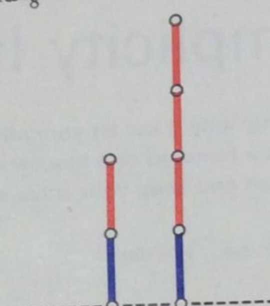


Figure 1. A Blue-Red Hackenbush Position Worth Five-Eighths of a Move.

What are the moves from the position $\tfrac{1}{2}+\tfrac{1}{8}$, that is from the position

$$\tfrac{1}{2} \quad + \quad \tfrac{1}{8}$$

which we write

$$\{0|1\} + \{0|\tfrac{1}{4}\}$$

in the new notation?

Each player can move in either the first or second component, but must then leave the other component untouched, so Left's options are the positions

$$0+\tfrac{1}{8} \quad \text{(if he moves in the first), and}$$

$$\tfrac{1}{2}+0 \quad \text{(if he moves in the second).}$$

He should obviously prefer the latter, which leaves a total value of half a move, rather than one-eighth of a move, to him. Right's options are similarly

$$1+\tfrac{1}{8} \qquad \text{and} \qquad \tfrac{1}{2}+\tfrac{1}{4}$$

of which he should prefer the second, since it leaves Left only three-quarters of a move, rather than one-and-one-eighth. We have shown that the best moves from $\tfrac{5}{8}$ are to $\tfrac{1}{2}$ (Left) and $\tfrac{3}{4}$ (Right), or in our abbreviated notation, we have demonstrated the equation

$$\frac{5}{8} = \left\{\frac{1}{2} \,\middle|\, \frac{3}{4}\right\}.$$

In a precisely similar way, we can add various fractions $1/2^k$ so as to prove that

$$\frac{2p+1}{2^{n+1}} = \left\{\frac{2p}{2^{n+1}} \,\middle|\, \frac{2p+2}{2^{n+1}}\right\} = \left\{\frac{p}{2^n} \,\middle|\, \frac{p+1}{2^n}\right\}$$

or in words, that each fraction with denominator a power of two has as its Left and Right options the two fractions nearest to it on the left and right that have smaller denominator which is again a power of two. For example

$$3\tfrac{57}{128} = \{3\tfrac{56}{128}|3\tfrac{58}{128}\} = \{3\tfrac{7}{16}|3\tfrac{29}{64}\}.$$

SIMPLICITY'S THE ANSWER!

The equations we've just discussed are the easy ones. What number is the game $X = \{1\frac{1}{4}|2\}$? We have already seen in our discussion of Ski-Jumps that we should not necessarily expect the answer to be the mean of $1\frac{1}{4}$ and 2, that is, $1\frac{5}{8}$. Why not? We can test this question by playing the sum

$$X+(-1\tfrac{5}{8}) = \{1\tfrac{1}{4}|2\}+\{-1\tfrac{3}{4}|-1\tfrac{1}{2}\}$$

since we already know that $-1\frac{5}{8} = \{-1\frac{3}{4}|-1\frac{1}{2}\}$. Only if neither player has a winning move in this sum will we have $X = 1\frac{5}{8}$.

The two moves from the component X are certainly losing ones, because $1\frac{5}{8}$ is strictly between $1\frac{1}{4}$ and 2, so that Left's move leaves the total value $1\frac{1}{4}-1\frac{5}{8}$ which is negative, while Right's leaves it $2-1\frac{5}{8}$ which is positive. But Right nevertheless has a good move, namely that from $-1\frac{5}{8}$ to $-1\frac{1}{2}$. Why is this?

The answer is that in the new game

$$X+(-1\tfrac{1}{2}) = \{1\tfrac{1}{4}|2\}+\{-2|-1\}$$

it is still true that neither player will want to move in the component X, *for essentially the same reason as before*, since $1\frac{1}{2}$ still lies strictly between $1\frac{1}{4}$ and 2. So Left's only hope for a reply is to replace $-1\frac{1}{2}$ by -2 which Right can neatly counter by moving from X to 2, leaving a zero position.

So the reason that $\{1\frac{1}{4}|2\}$ is *not* $1\frac{5}{8}$ is that $1\frac{5}{8}$ is not the *simplest* number strictly between $1\frac{1}{4}$ and 2, because it has the Left option $1\frac{1}{2}$ with the same property, and we therefore find ourselves needing to discuss $X+(-1\frac{1}{2})$ before we can evaluate $X+(-1\frac{5}{8})$.

Now $1\frac{1}{2}$ must be the simplest number between $1\frac{1}{4}$ and 2, because the immediately simpler numbers are its options 1 and 2, which don't fit. We shall use this to prove that in fact $X = 1\frac{1}{2}$.

It is still true for the position

$$X+(-1\tfrac{1}{2}) = \{1\tfrac{1}{4}|2\}+\{-2|-1\}$$

that neither player has a good move from the component X, so that we need only consider their moves from $-1\frac{1}{2}$. After Right's move the total is $X+(-1)$, to which Left can reply by moving from the component X so as to leave the positive total $1\frac{1}{4}-1$, because 1 is not *strictly between* $1\frac{1}{4}$ and 2, but *less than* $1\frac{1}{4}$. After Left's move from $-1\frac{1}{2}$, the total is $X+(-2)$ and Right's response is to the zero position $2-2$, because 2 is no longer *strictly between* $1\frac{1}{4}$ and 2, but this time *equal to* 2.

The argument can be used in general to prove the **Simplicity Rule**, which we shall use over and over again:

If there's *any* number that fits, the answer's the *simplest* number that fits.

THE SIMPLICITY RULE

If the options in

$$\{a,b,c,\ldots|d,e,f,\ldots\}$$

are all numbers, we'll say that the number x **fits** just if it's

strictly greater than each of $a,b,c,\ldots$, and
strictly less than each of $d,e,f,\ldots$,

and x will be the **simplest** number that fits, if none of its options fit. For the options of x you should use the particular ones we found in the previous section.

For example, if the best Left move from some game G is to a position of value $2\frac{3}{8}$, and the best Right move to one of value 5, we can show that G itself must have value 3, which we found before in the form $\{2|\ \}$, for in this form 3 has only one option, 2, which does *not* lie strictly between $2\frac{3}{8}$ and 5, while 3 *does*. Note that the simplicity rule still works when one of the players, here Right, has no move from the number c. It also works for games of the form $\{a|\ \}$ or $\{\ |b\}$ in which again one of the two players is deprived of a move. For example, $\{a|\ \}$ is a number c which is greater than a, but has no option with this property. This is in fact the smallest whole number 0 or 1 or 2 or ... which is greater than a. Thus $\{2\frac{1}{2}|\ \} = 3$, $\{-2\frac{1}{2}|\ \} = 0$.

SIMPLEST FORMS FOR NUMBERS

Figure 2 displays most of what we've learnt so far. The central ruler is the ordinary real number line with bigger marks for simpler numbers, while below it are the corresponding Hackenbush strings; the simpler the number, the shorter the string.

The binary tree of numbers appears upside-down above the ruler, although we can't draw all of it on our finite page with finite type—for more details see ONAG, pp. 3–14.† Each fork of the tree is a number whose best options are the nearest numbers left and right of it that are higher up the tree. For example 1 and 2 are the best options for $1\frac{1}{2}$. For $\frac{13}{16}$ we find $\frac{3}{4}$ and $\frac{7}{8}$, so

$$\frac{13}{16} = \left\{\frac{3}{4}\middle|\frac{7}{8}\right\}$$

as a game. (The numbers on the leftmost branch have no Left options and those on the rightmost branch no Right ones.)

The options of a number that we find in this way define its canonical or **simplest form**. Here are the rules for simplest forms:

$$\boxed{\begin{aligned} 0 &= \{\ |\ \} \\ n+1 &= \{n|\ \} \\ -n-1 &= \{\ |-n\} \\ \frac{2p+1}{2^{q+1}} &= \left\{\frac{p}{2^q}\middle|\frac{p+1}{2^q}\right\} \end{aligned}}$$

SIMPLEST FORMS FOR NUMBERS

e.g.

$$79 = \{78|\ \},\ -53 = \{\ |-52\},\ \text{and}\ \frac{47}{64} = \left\{\frac{23}{32}\middle|\frac{24}{32}\right\} = \left\{\frac{23}{32}\middle|\frac{3}{4}\right\}.$$

The simpler the number, the nearer it is to the root (top!) of the tree.

† Throughout the book, ONAG refers to J.H. Conway, "On Numbers and Games", Academic Press, London and New York, 1976

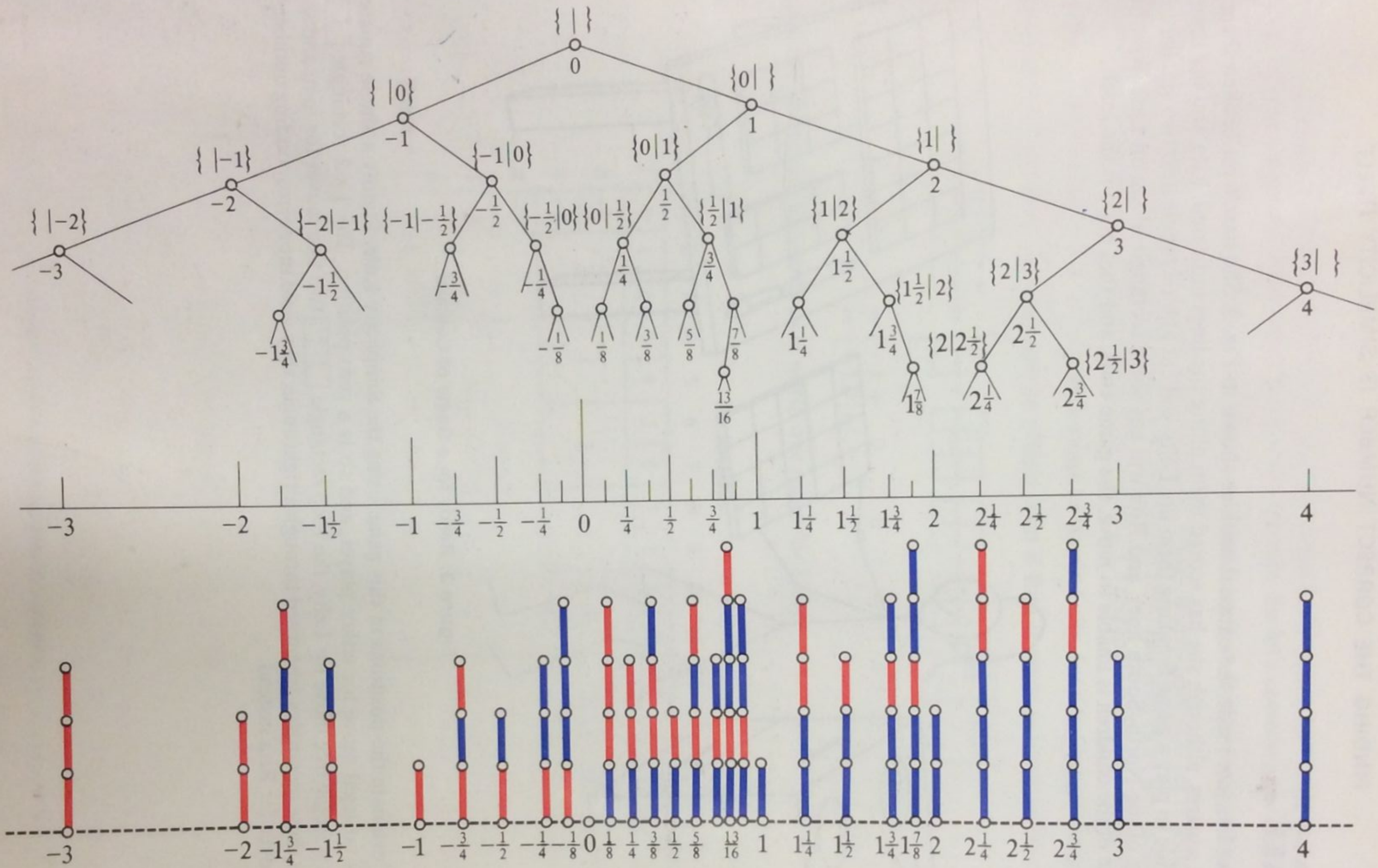


Figure 2. Australian Number Tree, the Real Number Line, and Hackenbush Strings.

CUTCAKE

Mother has just made the oatmeal cookies shown in Fig. 3. She hasn't yet broken them up into little squares, although she has scored them along the lines indicated. Rita and her brother Lefty decide to play a game breaking them up. Lefty will cut any rectangle into two smaller ones along one of the North–South lines, and Rita will cut some rectangle along an East–West line. When one of the children is unable to move, the game ends, and that child is the loser.

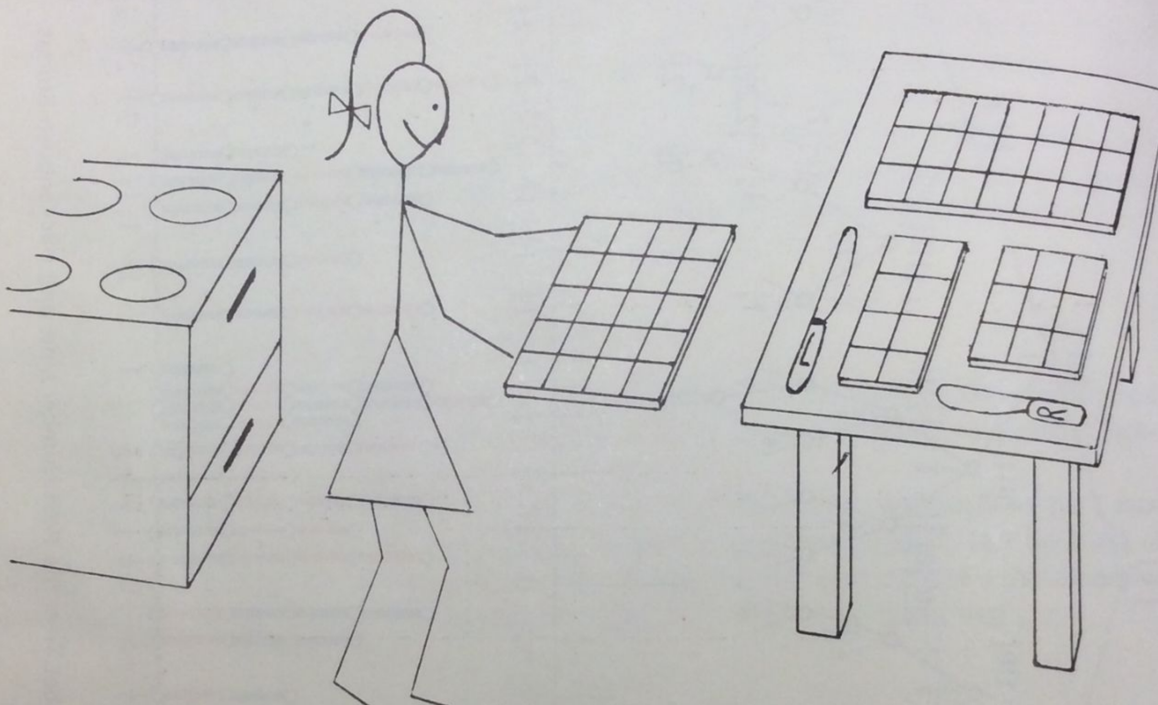
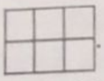
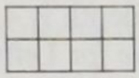
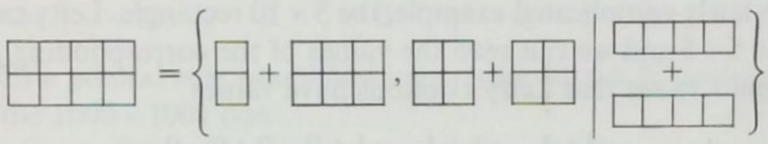


Figure 3. Ready for a Game of Cutcake.

We'll evaluate the positions in this game using the Simplicity Rule. Plainly a single square □ leaves no legal move for either player, and so is a zero position. The 1×2 rectangle □□ gives just a single free move for Lefty, the 1×3 rectangle □□□ two free moves for him, and so on. When these rectangles are turned through a right angle, they yield the corresponding numbers of free moves for Rita instead.

The 2×2 square is the zero position $\{-2|2\}$, for when Lefty starts, he leaves two moves for Rita, and if she starts, she must leave two moves for him. So let's consider the 2×3 rectangle . Since this has more vertical lines than horizontal ones, it should perhaps be a win for Lefty? No! If he starts, he must leave a 2×1 rectangle, which is one move in favor of Rita, together with a 2×2 square, which we can ignore as having value zero. But Rita can't win either, for her only opening move gives Lefty four free moves. So the 2×3 rectangle is the zero position $\{-1|4\}$. But the 2×4 rectangle  *is* long enough to favour Lefty, for if he chops it into two 2×2 squares at his first move, he wins, and he plainly wins if Rita is made to start. In fact, we have



$$= \{-1 + \quad 0 \quad , \quad 0 \; + \; 0 \quad | \; 3 + 3\}$$
$$= \{-1, 0 \,|\, 6\},$$

which the Simplicity Rule tells us is worth one move for Lefty.

<table>
<tr><th>Depth \ Breadth</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>12</th><th>13</th><th>14</th><th>15</th><th>16</th></tr>
<tr><th>1</th><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td></tr>
<tr><th>2</th><td>−1</td><td colspan="2" rowspan="2">0</td><td colspan="2" rowspan="2">1</td><td colspan="2" rowspan="2">2</td><td colspan="2" rowspan="2">3</td><td colspan="2" rowspan="2">4</td><td colspan="2" rowspan="2">5</td><td colspan="3" rowspan="2">6</td></tr>
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<tr><th>5</th><td>−4</td></tr>
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<tr><th>9</th><td>−8</td></tr>
<tr><th>10</th><td>−9</td><td colspan="2" rowspan="2">−4</td></tr>
<tr><th>11</th><td>−10</td></tr>
<tr><th>12</th><td>−11</td><td colspan="2" rowspan="2">−5</td><td colspan="4" rowspan="3">−2</td></tr>
<tr><th>13</th><td>−12</td></tr>
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Table 1. Values of Rectangles in Cutcake.

Using arguments like these, we can draw up a table (Table 1) showing the values of rectangles of various sizes in Cutcake. We see that there is an interesting pattern—the border of the table is divided into 1×1 squares holding a different integer, corresponding to the values of strips of width 1. But then there's a second border of 2×2 squares which is a bit harder to explain. Thus all the four rectangles of breadth 2 or 3 and depth 4 or 5 have the same value, -1, meaning that they count as one free move for Rita. (We already saw that the 2×2 and 2×3 rectangles had the same value, namely 0.) Then the table continues with a third border of 4×4 squares, followed by a fourth of 8×8 squares, and so on. So all rectangles whose depth is 4, 5, 6, or 7, and breadth 8, 9, 10, or 11 have value 1, and behave like a single free move for Lefty, despite their variable shapes.

Let's consider a fairly complicated example, the 5×10 rectangle. Lefty can split 10 into 1+9, 2+8, 3+7, 4+6 or 5+5 and we can read the values of the corresponding rectangles 5×1 and 5×9, etc. from Table 1 to see that Lefty's options have values

$$-4+1, \quad -1+1, \quad -1+0, \quad 0+0, \quad 0+0$$

Rita can split 5 into 1+4 or 2+3 yielding pairs of breadth 10 rectangles of values 9+1 or 4+4. So the 5×10 rectangle has value

$$\{-3,0,-1,0,0|10,8\} = \{0|8\} = 1,$$

and Table 1 is continued in this way.

MAUNDY CAKE

Every Maundy Thursday Lefty and Rita play a different cake-cutting game, in which Lefty's move is to divide one cake into *any* number of *equal* pieces, using only vertical cuts, while

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	1	1	3	1	4	1	7	4	6	1	10	1	8	6	15	1	13
2	−1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4
3	−1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4
4	−3	−1	−1	0	−1	0	−1	1	0	0	−1	1	−1	0	0	3	−1	1
5	−1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4
6	−4	−1	−1	0	−1	0	−1	1	0	0	−1	1	−1	0	0	3	−1	1
7	−1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4
8	−7	−3	−3	−1	−3	−1	−3	0	−1	−1	−3	0	−3	−1	−1	1	−3	0
9	−4	−1	−1	0	−1	0	−1	1	0	0	−1	1	−1	0	0	3	−1	1
10	−6	−1	−1	0	−1	0	−1	1	0	0	−1	1	−1	0	0	3	−1	1
11	−1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4

Table 2. Maundy Cake Values.

Rita does likewise, but with horizontal cuts. Once again the cuts must follow Mother's scorings, so that all dimensions will be whole numbers.

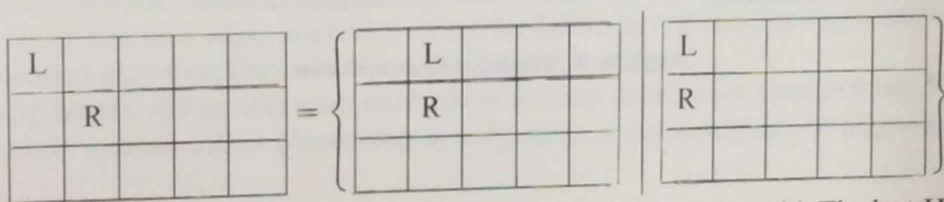
This game was proposed and solved by Patrick Mauhin—can you see the general pattern in his table of values (Table 2)? We worked them out as follows:

$$\begin{aligned}
\begin{matrix}\text{value of}\\ 5\times 12\end{matrix} &= \left\{\begin{matrix}\text{twelve of}\\ 5\times 1\end{matrix}, \begin{matrix}\text{six of}\\ 5\times 2\end{matrix}, \begin{matrix}\text{four of}\\ 5\times 3\end{matrix}, \begin{matrix}\text{three of}\\ 5\times 4\end{matrix}, \begin{matrix}\text{two of}\\ 5\times 6\end{matrix} \;\middle|\; \begin{matrix}\text{five of}\\ 1\times 12\end{matrix}\right\}\\
&= \left\{\begin{matrix}\text{twelve of}\\ -1\end{matrix}, \begin{matrix}\text{six of}\\ 0\end{matrix}, \begin{matrix}\text{four of}\\ 0\end{matrix}, \begin{matrix}\text{three of}\\ 1\end{matrix}, \begin{matrix}\text{two of}\\ 1\end{matrix} \;\middle|\; \begin{matrix}\text{five of}\\ 10\end{matrix}\right\}\\
&= \{\; -12\;,\; 0\;,\; 0\;,\; 3\;,\; 2 \;|\; 50\;\} = 4.
\end{aligned}$$

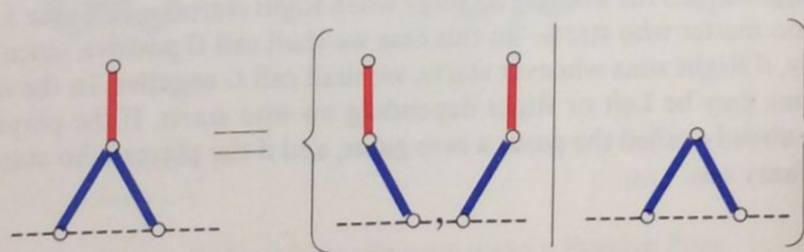
If you haven't guessed a general rule, you'll find ours in the Extras. If you have, try it out on the 999×1000 cake, or the 1000×1001 one.

A FEW MORE APPLICATIONS OF THE SIMPLICITY RULE

The more questionable values for Ski-Jumps and Hackenbush positions are easily understood in terms of the Simplicity Rule. For example the Ski-Jumps position



has value $\{2\frac{1}{2}|4\frac{1}{2}\}$ which the Simplicity Rule requires to be 3, just as we said. The last Hackenbush position



of Fig. 18 in the Extras to Chapter 1 can be seen to have $\{\frac{1}{2},\frac{1}{2}|2\} = 1$ by another application of the Rule. Values of more complicated positions such as the horse of Fig. 4 can be found by repeated applications. We have followed the recommended practice of writing against each edge the value of the position which would result if that edge were deleted. These positions will either be found later in the figure or are sums of the simple positions discussed in Chapter 1.

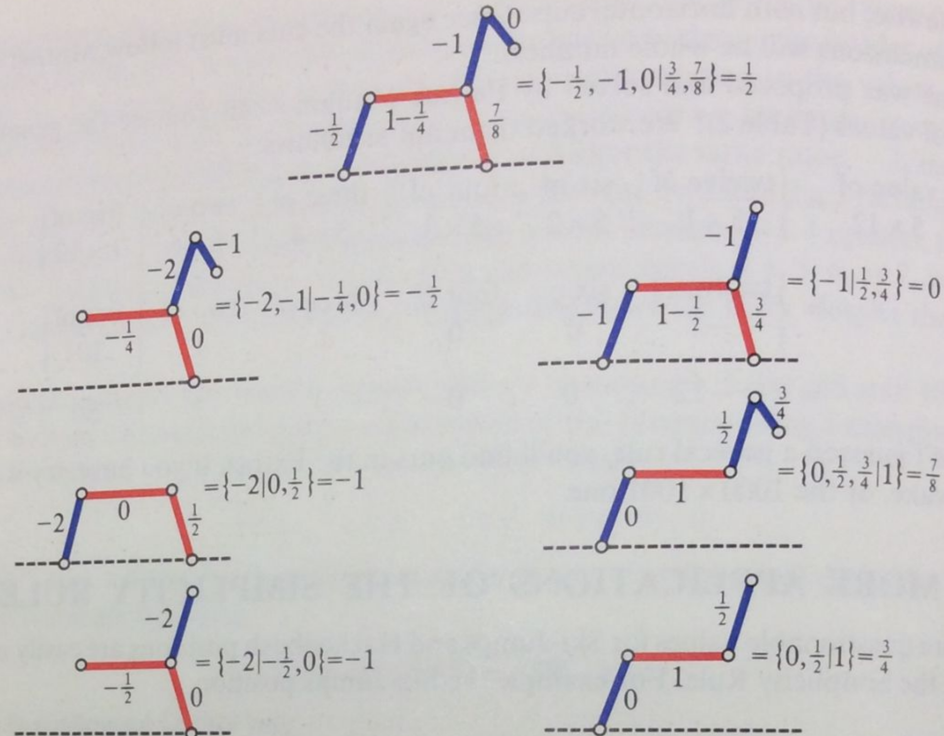


Figure 4. Working Out a Horse.

POSITIVE, NEGATIVE, ZERO AND FUZZY POSITIONS

We can classify all games into four *outcome classes*, which specify who has the winning strategy when Left starts and who has the winning strategy when Right starts, as in Table 3. It may happen that Left can win no matter who starts—in this case we shall call G **positive**, since we are in favor of Left. Conversely, if Right wins whoever starts, we shall call G **negative**. In the other two cases, the player who wins may be Left or Right depending on who starts. If the player who starts is the *loser*, we have already called the game a **zero game**, and if the player who starts is the *winner*, we shall call it a **fuzzy** one.

If Right starts:	If Left starts: Left wins	If Left starts: Right wins
Left wins	positive (L wins)	zero (2 wins)
Right wins	fuzzy (1 wins)	negative (R wins)

Table 3. The Four Possible Outcomes.

A handy way of remembering these four cases is just to describe the player who has the winning strategy—this is either *Left*, *Right*, or the *first*, or the *second* player to move from the start. In symbols, we have

$G > 0$ or G is **positive** if player L (Left) can always win
$G < 0$ or G is **negative** if player R (Right) can always win
$G = 0$ or G is **zero** if player 2 (second) can always win
$G \parallel 0$ or G is **fuzzy** if player 1 (first) can always win.

In Blue-Red Hackenbush we've already seen that a picture with only blue edges is positive (if there are any), and one with only red edges is negative. A picture having no edges is zero, but there are also other zero pictures, for example any picture with as many red edges as blue in which each edge is connected to the ground by its own color, or the rather simple picture of Fig. 6(c) in Chapter 1, which has two blue edges and three red.

There are no *fuzzy* positions in Blue-Red Hackenbush, which makes it rather unusual, because in most games it is some advantage to be the first player. So to get more varied behavior, we introduce a new kind of edge.

HACKENBUSH HOTCHPOTCH

This game is played as before except that there may also be some *green* edges, which either player may chop. But blue edges are still reserved for Left, and red ones for Right and we continue to use the normal play rule, that when you can't move, you lose.

The pretty flower of Fig. 5(a) is an example of a fuzzy position in Hackenbush Hotchpotch, for since its stalk is green, either player may win the game at the first move by chopping this edge.

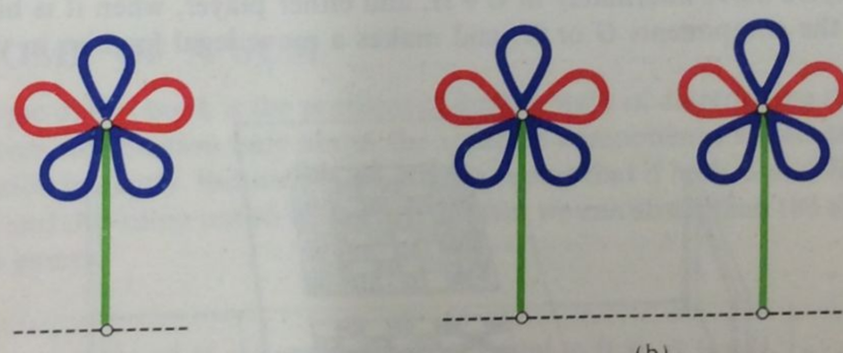


Figure 5. Two Fuzzy Flowers make a Positive Posy.

It might be thought that, like a zero game, a fuzzy game confers no particular advantage on either player, and so should also be said to have value 0. But this would be a misleading convention, because often a fuzzy game can be more in favor of one player than the other, even though either player can win starting first. For example, the flower of Fig. 5(a) has more blue petals than red ones, and this favors Left by just enough to ensure that the sum of two such flowers, as in Fig. 5(b), is positive. For no matter who starts in Fig. 5(b), Left has enough spare moves to arrange that Right is first to take a stalk, whereupon Left wins by taking the other.

In fact a fuzzy game is neither greater than 0, less than 0, nor equal to 0, but rather *confused* with 0. Figure 6 shows a good mental picture, illustrating a fuzzy game G whose place in the number scale is rather indeterminate, being represented by the cloud. Since this covers 0 and stretches some way on either side, we can't tell exactly where G is. It's probably buzzing about under the cloud, so that it seems positive at some times, and negative at others, according to its environment.

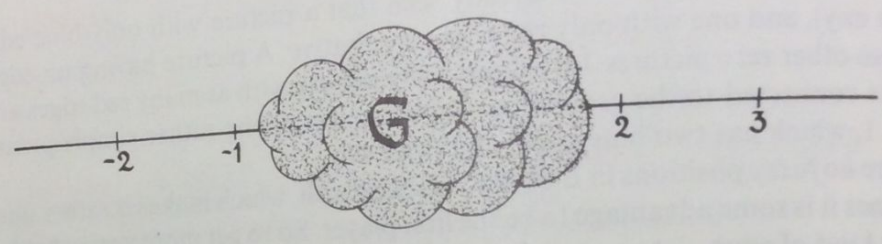


Figure 6. How Big is a Fuzzy Game?

SUMS OF ARBITRARY GAMES

Now that we've learned how to work with numbers and how to find when games are positive, negative, zero, or fuzzy, we should learn what it means to add two games in general. Being very clever, Left and Right may play a sum of *any* pair of games G and H as in Fig. 7. We shall refer to the two games G and H as the **components** of the compound game $G+H$, which is played as follows. The players move alternately in $G+H$, and either player, when it is his turn to move, chooses one of the components G or H, and makes a move legal for him in that component.

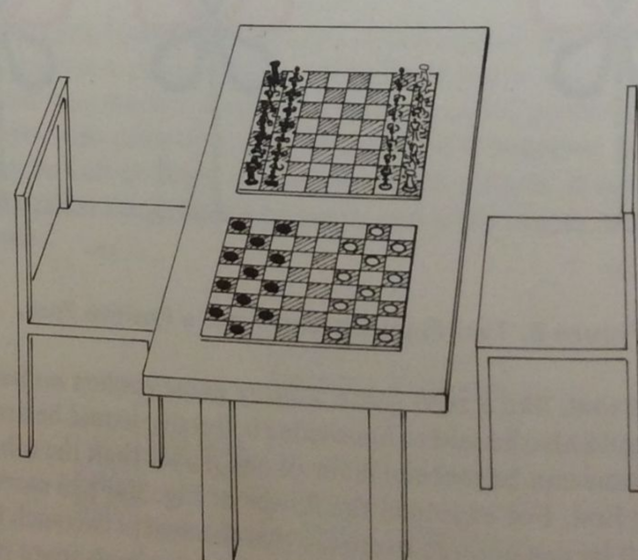


Figure 7. Ready to Play the Sum of Two Games.

♠

The turn then passes to his opponent, who plays in a similar manner. The game ends as usual when some player finds himself unable to move (this will only happen when there is no component in which he has a legal move) and that player loses.

Symbolically we shall write G^L for the typical Left option (i.e., a position Left can move to) from G, and G^R for the typical Right option, so that

$$\boxed{G = \{G^L | G^R\}.}$$

We use this notation even when a player has more than one option, or none at all, so that the symbol G^L need not have a unique value. Thus if $G = \{a,b,c,\ldots | d,e,f,\ldots\}$, G^L means a or b or c or ... and G^R means d or e or f or .... In the game $2 = \{1 | \ \}$, G^L has only the value 1, but G^R has no value. In this notation the definition of sum is written

$$\boxed{G+H = \{G^L+H, G+H^L \mid G^R+H, G+H^R\}}$$

since Left's options from $G+H$ are exactly the sums G^L+H or $G+H^L$ in which he has moved in just one component, and Right's are the similar sums G^R+H, $G+H^R$.

It should be made clear that there is no restriction on the component a player moves in at any time other than his ability to move in that component. You need not follow your opponent's move with another move in the same component, nor need you switch components unless you want to. Indeed in many games (e.g. Blue-Red Hackenbush and Cutcake) a move may produce more than one component.

THE OUTCOME OF A SUM

The major topic of this book is the problem of finding ways of determining the outcome of a sum of games given information only about the separate components, so we cannot expect to answer this question instantly. But we should at least expect that if both G and H are in favor of Left, so is $G+H$ and this turns out to be the case. In fact we can strengthen the assertion a little, by allowing zero games.

If G and H are greater than or equal to 0, so is $G+H$.

What does it mean for G to be greater than or equal to 0? From Table 3, we see that these are just the two cases in which Left has a winning strategy *provided Right starts.* If this is true of G and H, it is also true of $G+H$, for if Right starts, he must make a move in one of G and H, say G, and Left can reply with the responses of his winning strategy in G for as long as Right continues to move in that game. Whenever Right switches to H, Left responds in H with the moves of his winning strategy in that game, and so on. If he plays like this, Left will never be lost for a move in $G+H$, for he can always respond in whatever component Right has just played in, so he cannot lose.

Now we have another principle, which covers some fuzzy games:

> If G is positive or fuzzy, and
> H is positive or zero, then
> $G+H$ is positive or fuzzy.

For we see from Table 3 that the positive or fuzzy games are just those from which Left has a winning strategy *provided Left starts*. So what we have to show is that if Left has a winning strategy in G with Left starting, and one in H with Right starting, he has one in $G+H$ with Left starting.

This is easy. He starts in $G+H$ by making the first move of his winning strategy for G, and then always replies to any of Right's moves with another move in the same component, so that the sequence of moves played in G is begun by Left and that in H by Right. If Left follows his two winning strategies in the two components he will therefore win their sum.

We can summarize these results, and those obtained by interchanging the roles of Left and Right, in symbols.

> If $G \geqslant 0$ and $H \geqslant 0$ then $G+H \geqslant 0$,
> If $G \leqslant 0$ and $H \leqslant 0$ then $G+H \leqslant 0$,
> If $G \mathrel{|\!\triangleright} 0$ and $H \geqslant 0$ then $G+H \mathrel{|\!\triangleright} 0$,
> If $G \mathrel{\triangleleft\!|} 0$ and $H \leqslant 0$ then $G+H \mathrel{\triangleleft\!|} 0$.

Here "$\geqslant$" means "$>$" or "$=$", "$\triangleleft\!|$" means "$<$" *or* "$\parallel$", etc.

In particular if H is a zero game, it may be used in all four lines, and then $G+H$ will have the same outcome as G in all circumstances.

> Adding a zero game never affects the outcome.

We've already seen some of these principles in action in Blue-Red Hackenbush. But now we know that they work for arbitrary games, and did not depend on the fact that the positions we evaluated in Hackenbush turned out to be numbers. Table 4 shows the possibilities for the outcome of $G+H$, given those of G and H.

	$H = 0$	$H > 0$	$H < 0$	$H \parallel 0$
$G = 0$	$G+H = 0$	$G+H > 0$	$G+H < 0$	$G+H \parallel 0$
$G > 0$	$G+H > 0$	$G+H > 0$	$G+H \ ?\ 0$	$G+H \mathrel{|\!\triangleright} 0$
$G < 0$	$G+H < 0$	$G+H \ ?\ 0$	$G+H < 0$	$G+H \mathrel{\triangleleft\!|} 0$
$G \parallel 0$	$G+H \parallel 0$	$G+H \mathrel{|\!\triangleright} 0$	$G+H \mathrel{\triangleleft\!|} 0$	$G+H \ ?\ 0$

Table 4. Outcomes of Sums of Games. The entries $G+H\ ?\ 0$ are unrestricted.

Any Hackenbush picture in which only blue edges touch the ground is positive, for plainly the last move will be Left's. In particular the house of Fig. 8 is positive. But the garden is also positive for it is made from two of the positive flower beds of Fig. 5(b). So the whole picture can be won by Left, no matter who starts.

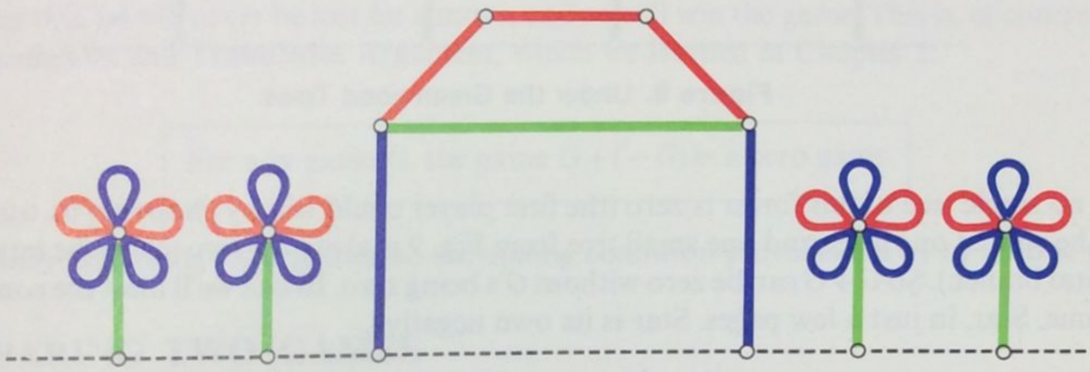


Figure 8. A Positive House and Garden.

THE NEGATIVE OF A GAME

In our examples of Blue-Red Hackenbush we found that whenever we interchanged the colors red and blue throughout, the number representing the value changed sign. This suggests that in general we define the negative of a game by interchanging the roles of Left and Right throughout. So, from no matter what position of G, the moves that once were legal for Left now become legal for Right, and vice versa. If G is the position

$$G = \{A,B,C,\ldots|D,E,F,\ldots\},$$

then $-G$ will be the position

$$-G = \{-D,-E,-F,\ldots|-A,-B,-C,\ldots\}.$$

For the general game $G = \{G^L|G^R\}$ we have

$$\boxed{-G = \{-G^R|-G^L\}.}$$

This definition works even when applied to fuzzy positions. Let's see what it means in practice. The negative of any Hackenbush position is obtained by interchanging the colors red and blue. Any green edges are unaltered. So for example the negative of the flower of Fig. 5(a) is a similar flower, but with three red and two blue petals instead of three blue and two red. A Hackenbush picture made entirely of green edges will therefore be its own negative. This means in particular that the little forest of Fig. 9 is a zero game, for it consists of the sum of two trees and their negatives (which have the same shape).

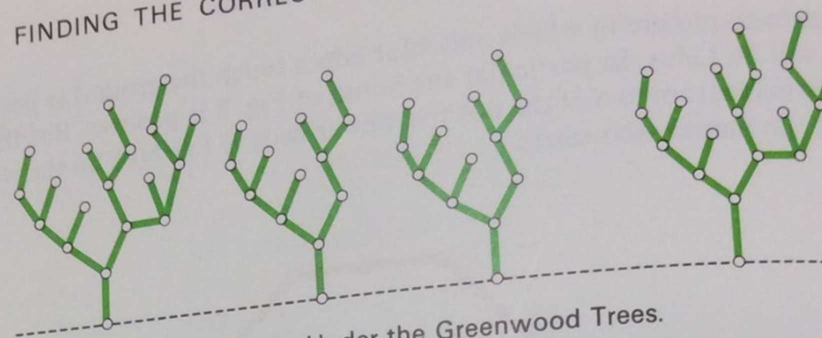


Figure 9. Under the Greenwood Trees.

But no single tree of this forest is zero (the first player could win by chopping its trunk), and in fact the sum of one large and one small tree from Fig. 9 is also non-zero (chop the larger one's horizontal branch). So $G+G$ can be zero without G's being zero. In fact we'll meet the commonest such game, Star, in just a few pages. Star is its own negative.

CANCELLING A GAME WITH ITS NEGATIVE

Is the negative of a game properly defined? Is it really true that the sum of a game and its negative is a zero game? How does the second player win the compound game $G+(-G)$?

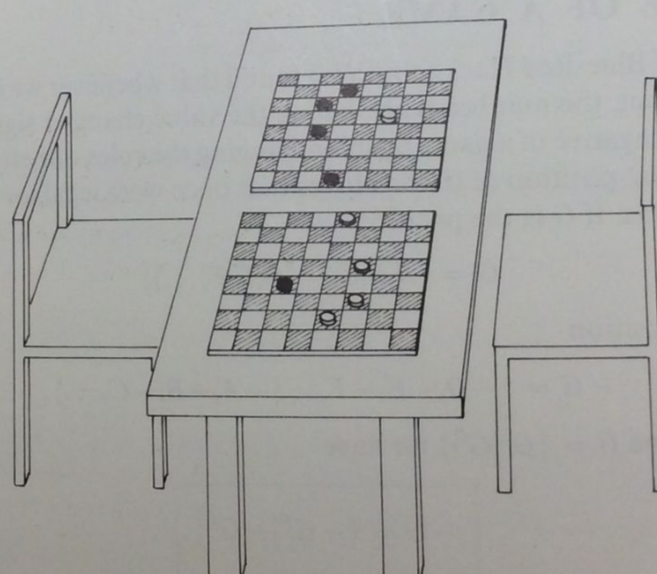


Figure 10. Playing a Game with its Negative.

The answers are fairly obvious. The first player must move in some component—let's suppose he moves from G to H, making the total position $H+(-G)$. Then by the definition of $-G$, the move from $-G$ to $-H$ will be legal for his opponent, who can therefore convert the whole position to $H+(-H)$. The first player might then move to $H+(-K)$, but this the second player can convert to $K+(-K)$, and so on. In other words, the second player can always mimic his opponent's previous move by making an exactly corresponding move in the other component. If he does this, he will never be lost for a move, and so will win the game. This is, of course, simply the Tweedledum and Tweedledee Argument, which we learned in Chapter 1.

For any game G, the game $G+(-G)$ is a zero game.

We are only discussing finite games, so the ending condition prevents draws by infinite play.

COMPARING TWO GAMES

We shall say that G is greater than or equal to H, and write $G \geqslant H$, to mean that G is at least as favorable to Left as H is. What exactly does this mean? We can get a hint from ordinary arithmetic, when $x \geqslant y$ if and only if the number $x-y$ is positive or zero. Let's take this as the definition for games:

$G \geqslant H$ means that $G+(-H) \geqslant 0$.

Then it's easy to see that if $G \geqslant H$ and $H \geqslant K$, we have $G \geqslant K$. For $G+(-K)$ has the same outcome as $G + (H+(-K)) + (-H)$, since $H+(-H)$ is a zero game, and this can be written as the sum of $G+(-H)$ and $H+(-K)$, which are both $\geqslant 0$. Appealing to our results on sums of games, we see that $G+(-K) \geqslant 0$, that is, $G \geqslant K$. In a similar way, from Table 4 we derive Table 5, showing what we can deduce about the order relation between G and K from those between G and H and H and K.

	$H = K$	$H > K$	$H < K$	$H \parallel K$
$G = H$	$G = K$	$G > K$	$G < K$	$G \parallel K$
$G > H$	$G > K$	$G > K$	$G\ ?\ K$	$G \,\|\!\!> K$
$G < H$	$G < K$	$G\ ?\ K$	$G < K$	$G <\!\!\|\, K$
$G \parallel H$	$G \parallel K$	$G \,\|\!\!> K$	$G <\!\!\|\, K$	$G\ ?\ K$

Table 5. What Relation is G to K?

Here $G = H$ means that G and H are *equally favorable* to Left
$G > H$ means that G is *better* than H for Left
$G < H$ means that G is *worse* than H for Left
$G \parallel H$ means that G is *sometimes better, sometimes worse*, than H for Left.

Once again "$\|\!\!>$" means "$>$" or "$\parallel$", etc.

COMPARING HACKENBUSH POSITIONS

The comparisons we made between Blue-Red Hackenbush positions in Chapter 1 are still valid, but more general things can happen when we meet fuzzy positions. Let's discuss the flower of Fig. 5(a). This is fuzzy as it stands. How much do we have to add to it before it becomes positive? It's not too hard to see that adding one free move for Left is already enough, since Left can win no matter who starts, by chopping the flowerstalk if this is still available, and using his free move if not.

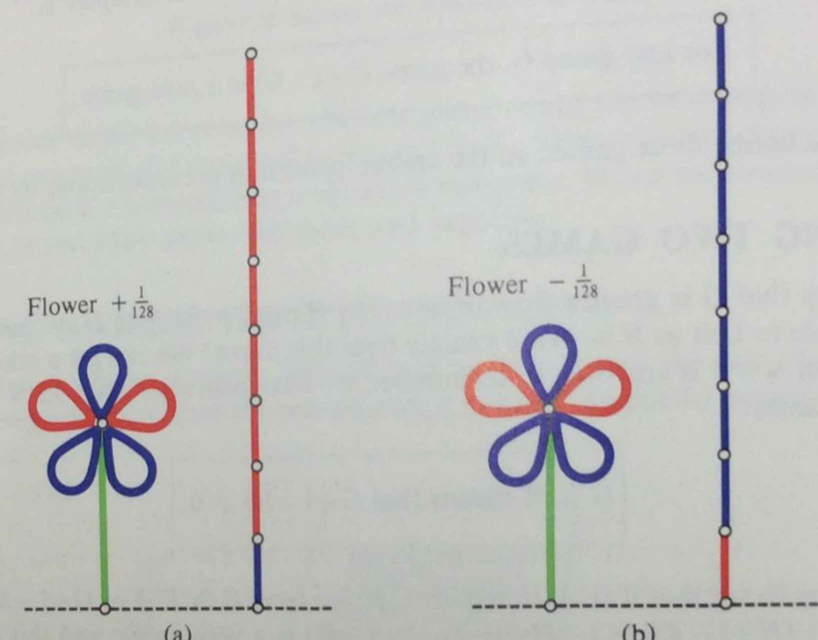


Figure 11. The Flower is Dwarfed by Very Small Hollyhocks of Either Sign.

Is half a move still enough? The answer again turns out to be "yes", and in fact Fig. 11 shows that even a very small fraction of a move is ample. Figure 11(a) adds only $\frac{1}{128}$ of a move to the flower, but it is clear that Left still wins by essentially the same strategy, giving first preference to chopping the flowerstalk, and if the flower has already gone, chopping the blue edge of his allowance. In Fig. 11(b) we have subtracted $\frac{1}{128}$ of a move, and this time Right wins by a similar strategy.

This means that the flower must be very small indeed—we have just proved that

$$-\tfrac{1}{128} < \text{flower} < +\tfrac{1}{128}$$

and of course our argument is actually enough to show that the flower is greater than all negative numbers and less than all positive ones, although still not zero. So the only number its cloud covers is 0 itself (see Fig. 12).

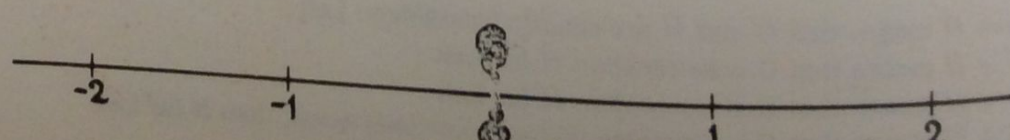


Figure 12. The Cloud Hides the Flower, but Covers only one Number.

The same kind of argument proves a much more general result, that any Hackenbush picture in which all the ground edges are green has a value which lies strictly between all negative and all positive numbers. Right can win when we subtract $\frac{1}{128}$ from such a picture by giving first priority to chopping any ground edge of the picture, and removing his free move allowance only when the rest of the picture has vanished. So the house of Fig. 13 is less than every positive number.

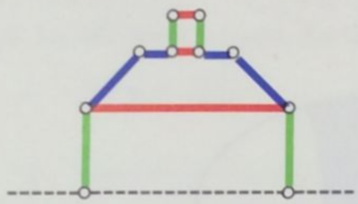


Figure 13. A Small but Positive House.

But Left can win in this picture by itself, so although the house is small, it's quite definitely positive (compare Fig. 5(b)). (The fight is about who first chops one of the walls, for his opponent will win by chopping the other. If Left works down the edges available to him from the chimney, he can make at least 5 moves to Right's at most 4 before a wall need be chopped.)

THE GAME OF COL

Colin Vout has invented the following map-coloring game. Each player, when it is his turn to move, paints one region of the map, Left using the color blue and Right using red. No two regions having a common frontier edge may be painted the same color. Whoever is unable to paint a region loses. Let us suppose that Right has made the first move in the very simple map with three regions shown in Fig. 14(a). What is the value of the resulting position?

The effect of Right's move has been to reserve the central region for Left so that we can think of it as being already *tinted* blue (Fig. 14(b)). In general any unpainted region next to a painted one automatically acquires a tint of the opposite color, indicating that only one player may use it thereafter. In the figures tinting is represented by hatching. Figure 14(c) shows the results of

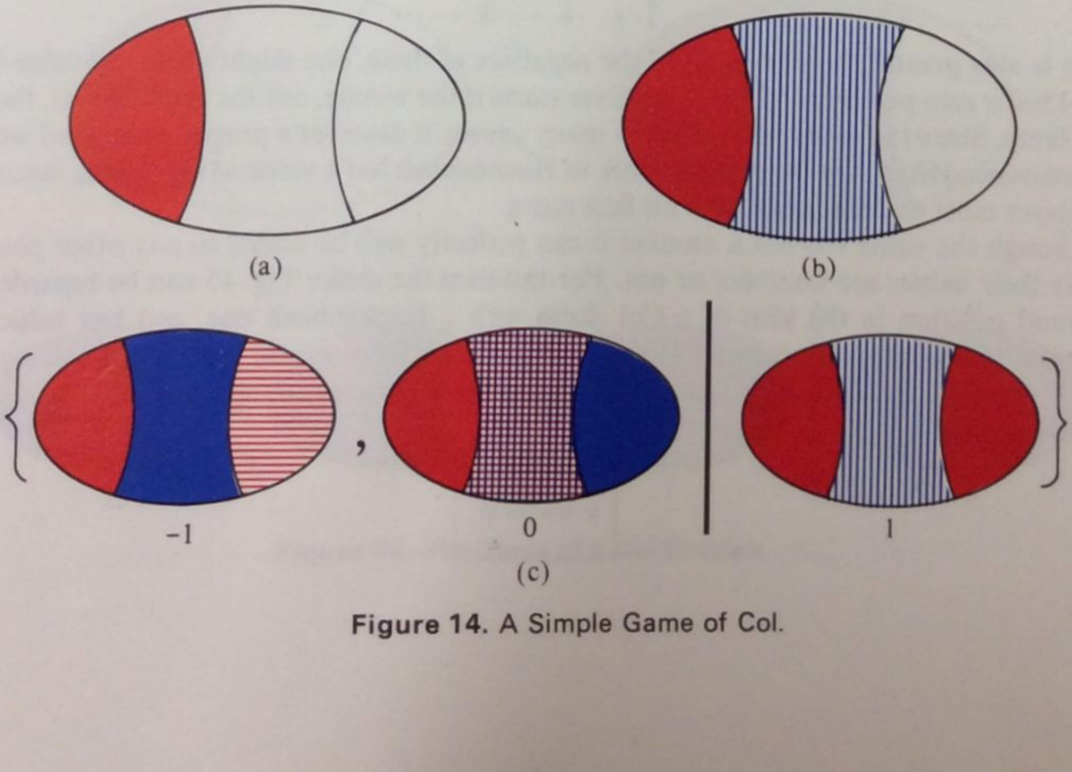
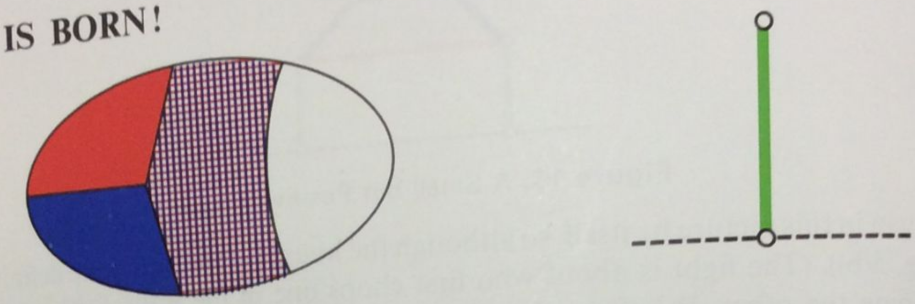


Figure 14. A Simple Game of Col.

each possible move from Fig. 14(b). If Left exercises his first option, there will remain one unpainted region, but this will be tinted red and so have value -1. After his second option, the unpainted region is tinted both red *and* blue, so neither player may use it and the value is zero. Right's only possible move leaves a blue tinted region, value 1. The value of Fig. 14(a) is therefore $\{-1,0|1\} = \frac{1}{2}$.

A STAR IS BORN!



(a) (b)

Figure 15. A Startling Value.

In Fig. 15(a) the only available region is not restricted in any way. Either player may therefore paint it and so move to a position of value zero. The value of Fig. 15(a) is therefore $\{0|0\}$. How should we interpret this? The Simplicity Rule will not help us, for there is no number strictly between 0 and 0, but we should expect the value to be less than or equal to each of

$$\{0|1\},\ \{0|\tfrac{1}{2}\},\ \{0|\tfrac{1}{4}\},\ \ldots,$$

since Right's option 0 is less than or equal to each of

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \ldots$$

In other words the value is less than or equal to each of

$$\tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \ldots$$

Since it is also greater than or equal to the negatives of these, one might guess the value 0. But is Fig. 15(a) a zero position? No! For whoever starts is the winner, not the loser. In fact, the position is fuzzy. Since the value $\{0|0\}$ arises in many games, it deserves a proper name, and we write it $*$, pronounced **Star**. A solitary green stalk in Hackenbush has a value $*$ (Fig. 15(b)), since again each player must end the game with his first move.

Although the value $*$ is not a number it can perfectly well be added to any other positions, whether their values are numbers or not. For instance the entire Fig. 15 can be regarded as a compound position in the sum of a Col game with a Hackenbush one, and has value $*+*$. Who wins this compound position? If you start and paint the region, I shall take the stalk and finish. If you take the stalk, I shall paint the region. In either case the second player wins and so the value is zero!

$$\boxed{*+* = 0.}$$

More generally, we consider positions of value $\{x|x\}$ for any number x. This is strictly greater than every number $y < x$ and strictly less than every number $z > x$, but neither greater than, less than nor equal to x itself. We can also add such values to other values of the same kind or to numbers.

Let us add $\frac{3}{4}$ to $*$, that is $\{\frac{1}{2}|1\} + \{0|0\}$. Left has two options $\frac{1}{2}+*$ (moving from $\frac{3}{4}$) and $\frac{3}{4}+0$ (moving from $*$), and Right has the two options $1+*$, $\frac{3}{4}+0$. Since $* < \frac{1}{4}$, Left's best option is $\frac{3}{4}$, and this is also Right's best option for the same reason. So we have

$$\tfrac{3}{4}+* = \{\tfrac{3}{4}|\tfrac{3}{4}\}$$

and more generally

$$\boxed{\begin{array}{c} x+* = \{x|x\} \\ \text{for any number } x. \end{array}}$$

THE VALUE $x*$

This type of value occurs so often that we'll use an abbreviated notation

$$x* \quad \text{for} \quad x+*$$

just as people write $2\frac{1}{2}$ for $2+\frac{1}{2}$. You must learn not to confuse $x*$ with x times $*$, just as you don't confuse $2\frac{1}{2}$ with 2 times $\frac{1}{2}$.

COL CONTAINS SUCH VALUES

For example, in the position of Fig. 16(a), which has tints as in Fig. 16(b), the players have the options shown in Fig. 16(c). It therefore has the value $\{*,-1,1|1\}$. Since the values $*$ and -1 are both less than 1, this simplifies to $\{1|1\} = 1*$.

You'll find more about Col in the Extras.

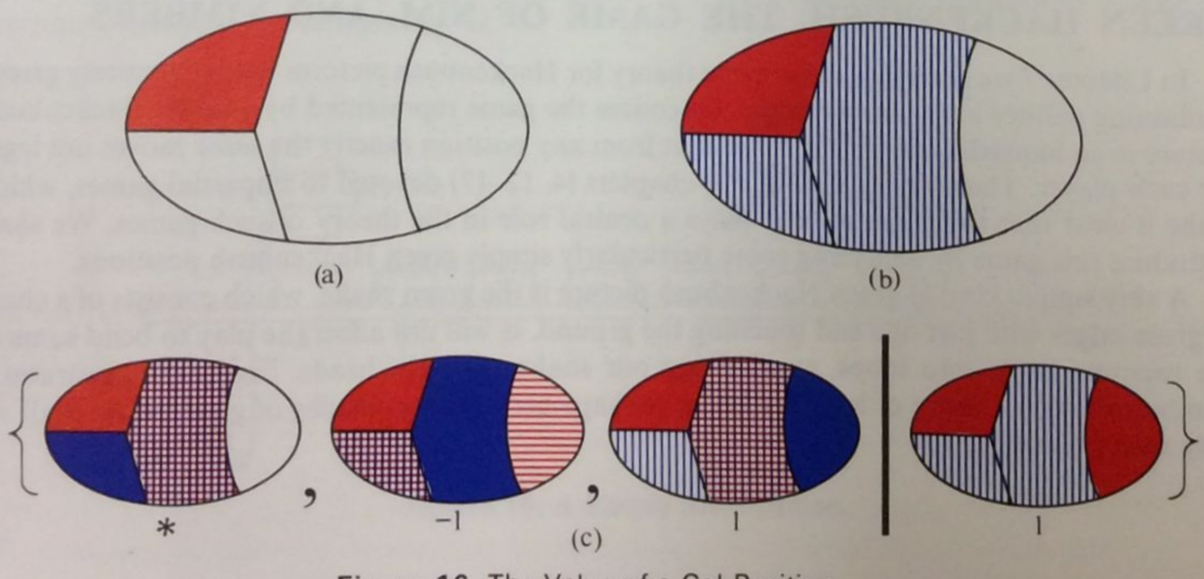
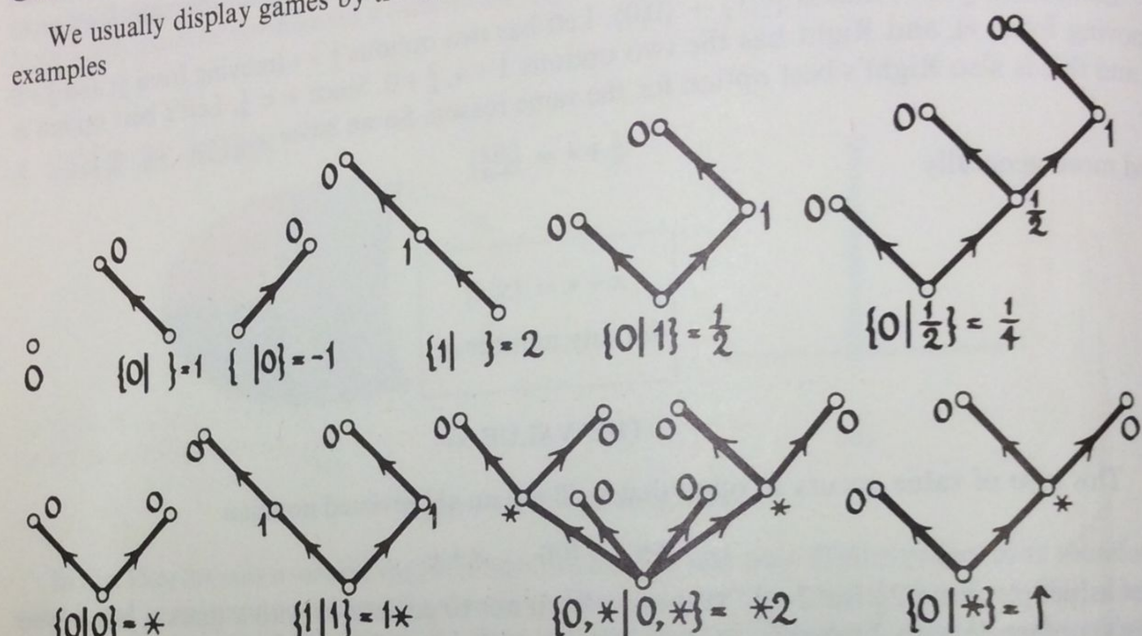


Figure 16. The Value of a Col Position.

GAME TREES

We usually display games by trees, with nodes for positions and edges for moves, as in the examples



Of course we use edges slanting to the left for Left's moves and to the right for Right's. This can help you to see that games that superficially look very different may have the same essential structure (e.g. Figs. 15(a) and (b)). In complicated positions we often combine nodes to avoid repetitions and we sometimes draw the diagrams upside-down as we did for Ski-Jumps and Toads-and-Frogs in Figs. 12 and 16 of Chapter 1.

GREEN HACKENBUSH, THE GAME OF NIM, AND NIMBERS

In Chapter 7 we shall give a complete theory for Hackenbush pictures that are entirely green, containing neither blue nor red edges. Of course the game represented by a green Hackenbush picture is an **impartial** one, in the sense that from any position exactly the same moves are legal for each player. There are several of our chapters (4, 12–17) devoted to impartial games, which make it clear that the game of **Nim** plays a central role in the theory of such games. We shall introduce this game by analyzing some particularly simple green Hackenbush positions.

A very simple kind of green Hackenbush picture is the green snake, which consists of a chain of green edges with just one end touching the ground. It will not affect the play to bend some of the topmost edges into loops, so allowing our snakes to have heads. Figure 17 illustrates a number of snakes, those of length 1 being perhaps better called blades of grass. How shall we play such a game?

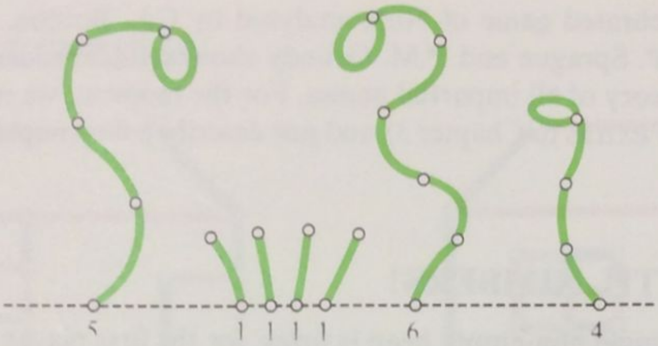


Figure 17. *Latet anguis in herba* (Virgil, Eclogue, III, 93).

Plainly any move will affect just one snake, and will replace that snake by a strictly shorter one. This means that if we write $*n$ for the value of a snake with n edges (counting the head loop, if present), then we have

$$\begin{aligned}
*0 &= \{\ |\ \} = 0,\\
*1 &= \{*0|*0\} = \{0|0\}, \quad \text{the game we called } *,\\
*2 &= \{*0,*1|*0,*1\} = \{0,*|0,*\},\ldots,\\
*n &= \{*0,*1,*2,\ldots,*(n-1)|*0,*1,*2,\ldots,*(n-1)\}.
\end{aligned}$$

These special values are called **nimbers** and you'll hear about them incessantly from now on. The fact that the same options appear on both sides of the | emphasizes the impartiality of the game.

It might be safer to play the game with heaps of counters instead of snakes. In this form, the general position has a number of heaps, and the move is to remove any positive number of counters from any one heap. In the normal play version, the winner is the person who takes the last counter. So this is the same as the snake game, with an n-edge snake replaced by a heap of n counters, and Fig. 17 becomes Fig. 18.

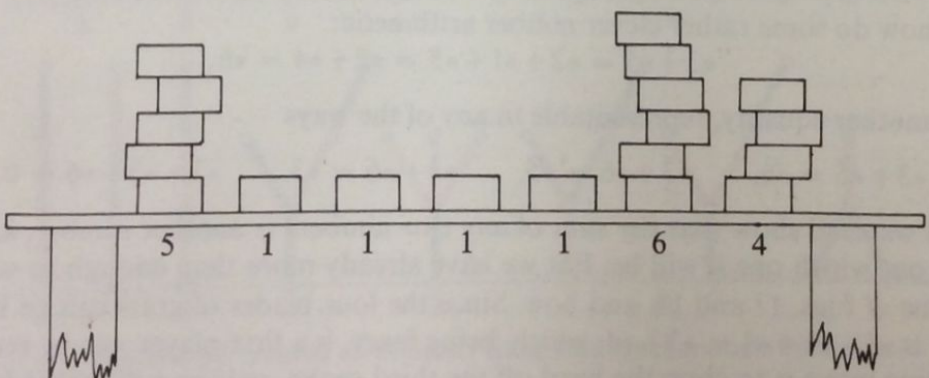


Figure 18. A Simple Nim Position.

The game is the celebrated game of Nim, analyzed by C.L. Bouton, and we shall meet it again and again, for R.P. Sprague and P.M. Grundy showed (independently) that it implicitly contains the additive theory of all impartial games. For the moment, we refrain from giving the theory in general (see the Extras to Chapter 3), and just describe a few simple positions and equalities.

GET NIMBLE WITH NIMBERS!

Firstly, note that a single non-empty heap is fuzzy, for the first player to move can take the whole heap. In the Hackenbush form, he chops the bottom edge of the snake. Next, two heaps of equal size add up to zero, for the impartiality ensures that a position is its own negative. So any pair of equal heaps in a position may be neglected—this allows us to neglect all four blades of grass in Fig. 17. On the other hand, the sum of two unequal heaps is a fuzzy game, for the first player can equalize them by reducing the larger one.

These remarks show that in a three-heap game, the player who first (fatally) equalizes two of the heaps or empties any heap is the loser, for in the first case his opponent can remove the third heap, and in the second, equalize the two non-empty heaps. But in the position $*1 + *2 + *3$, every move of the first player loses for one of these reasons, and so $*1+*2+*3 = 0$. Since nimbers are their own negatives this can also be written in any of the forms

$$*1+*2 = *3, \qquad *1+*3 = *2, \qquad *2+*3 = *1,$$

which are very useful in simplifying positions. For example, any situation in which there is one heap of size 2 and another of size 3 may be simplified by regarding these as a single heap of size 1.

From the position $*1 + *4 + *5$, if either player reduces one of the larger heaps to 2 or 3, the other player can reduce the other to 3 or 2 respectively. Since all other moves are fatal for one of our two reasons, this shows that $*1+*4+*5 = 0$, enabling us in general to replace two heaps of any two distinct sizes from 1,4,5 by one heap of the third size.

The equality $*2+*4+*6 = 0$ can be checked in a similar way. If either player reduces one of the larger heaps to 1 or 3, his opponent can reduce the other to the other, getting $*2 + *1 + *3$. The only other moves not obviously fatal are to reduce 2 to 1 or 6 to 5, and these counter each other, since $*1+*4+*5 = 0$.

We can now do some rather clever nimber arithmetic:

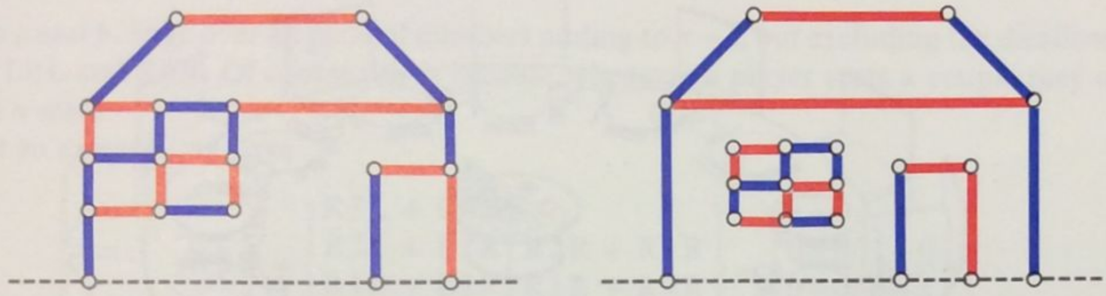
$$*3+*5 = *2+*1+*5 = *2+*4 = *6,$$

so we have another equality, representable in any of the ways

$$*3+*5 = *6, \qquad *3+*6 = *5, \qquad *5+*6 = *3, \qquad *3+*5+*6 = 0.$$

Later on we shall show that the sum of *any* two nimbers is another nimber, and give rules for working out which one it will be. But we have already more than enough to work out who wins the game of Figs. 17 and 18, and how. Since the four blades of grass can be neglected, the value of this is $*5+*6+*4 = *3+*4$, which, being fuzzy, is a first-player win by reducing 4 to 3. So one winning move is to chop the head off the third snake, reducing his value from $*4$ to $*3$. The diligent reader should check that the only other two winning first moves are to reduce $*5$ to $*2$ and $*6$ to $*1$. Our Most Assiduous Reader will prepare an extended nim-addition table using our examples as basis.

CHILDISH HACKENBUSH



(a) (b)

Figure 19. Childish and Grown-Up Pictures.

We call a Hackenbush picture *childish* because every edge is connected to the ground, perhaps via other edges. For example, the house of Fig. 19(a) is childish, but that of Fig. 19(b) is not, because the window will fall down and no longer be part of the position. The rule in ordinary Hackenbush is that edges which might make a picture non-childish are deleted as soon as they arise. However, in **Childish Blue-Red Hackenbush** (J. Schaer) you are only allowed to take edges which leave all the others connected to the ground; nothing may fall off. It might be thought that this is not a very interesting game. However Childish Blue-Red Hackenbush is far from trivial and the reader may like to verify the values of the positions in Fig. 20, and to compare them with the values of ordinary Blue-Red Hackenbush in Fig. 16 of Chapter 1.

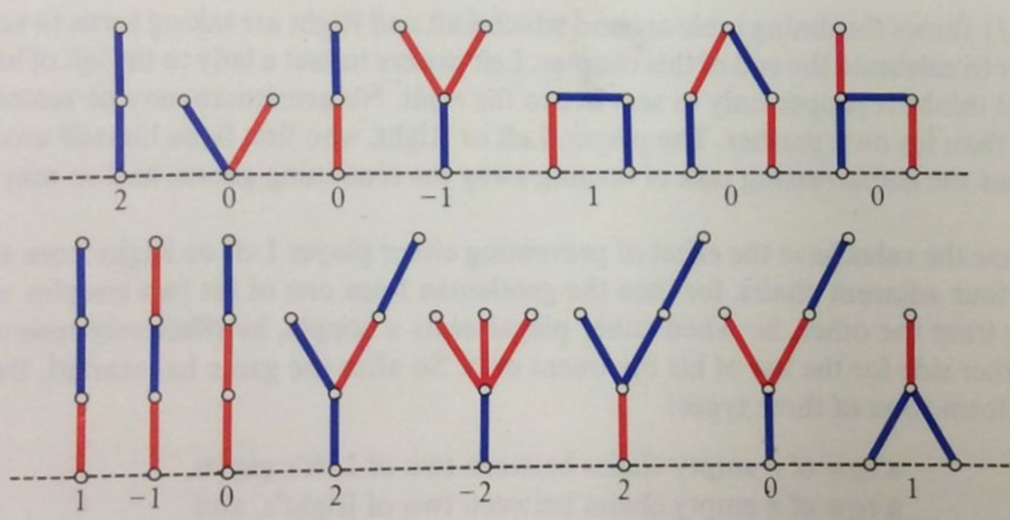


Figure 20. Values of Childish Blue-Red Hackenbush Positions.

Some Childish Hackenbush positions with non-integer values can be found in the Extras.

SEATING COUPLES

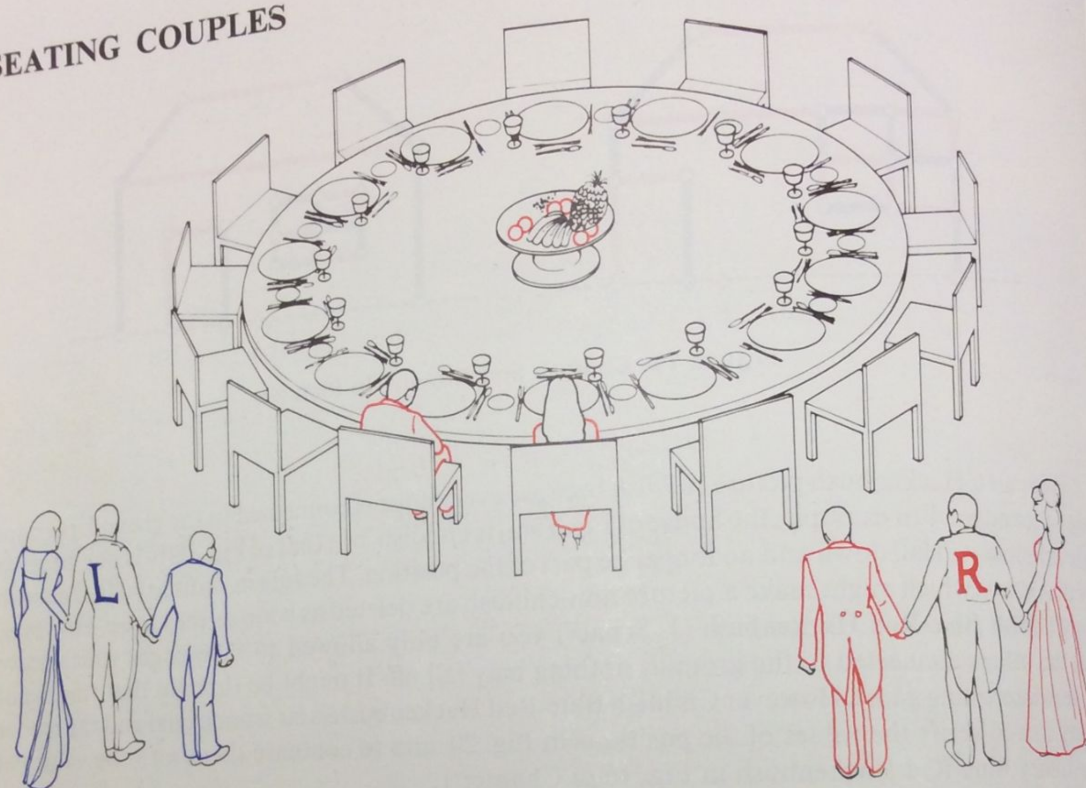


Figure 21. A Dinner to Celebrate the End of Chapter 2.

Figure 21 shows the dining table around which Left and Right are taking turns to seat couples for a dinner to celebrate the end of this chapter. Left prefers to seat a lady to the *left* of her partner, while Right thinks it proper only to seat her to the *right*. No gentleman may be seated next to a lady other than his own partner. The player, Left or Right, who first finds himself unable to seat a couple, has the embarrassing task of turning away the remaining guests, and so may be said to *lose*.

Of course the rules have the effect of preventing either player Left or Right from seating two couples in four adjacent chairs, for then the gentleman from one of his two couples will be next to the lady from the other. So when either player seats a couple, he effectively reserves the two seats on either side for the use of his opponent only. So after the game has started, the available chairs will form rows of three types:

LnL, a row of n empty chairs between two of Left's guests,
RnR, a row of n empty chairs between two of Right's, and
LnR or RnL, a row of n empty chairs between one of Left's guests and one of Right's.

Thus Fig. 21 is R 12 R. It is convenient to start the numbering from $n = 0$, but of course disallowing the positions L 0 L and R 0 R in which one player has illegally seated two adjacent couples. When we do this, we have

$$\begin{aligned} LnL &= \{LaL + LbL \mid LaR + RbL\} \\ RnR &= \{RaL + LbR \mid RaR + RbR\} \quad (= -LnL) \\ LnR &= \{LaL + LbR \mid LaR + RbR\} \quad (= RnL) \end{aligned}$$

where a and b range over all pairs of numbers adding to $n-2$, but excluding the disallowed positions L0L and R0R. Of course this is because whenever a player seats a couple they occupy 2 of the n seats.

As an example, we have

$$R5R = \left\{ \begin{array}{l|l} R3L + L0R & \\ R2L + L1R & R2R + R1R \\ R1L + L2R & R1R + R2R \\ R0L + L3R & \end{array} \right\} = \left\{ \begin{array}{l|l} *+0 & \\ 0+0 & 1+0 \\ 0+0 & 0+1 \\ 0+* & \end{array} \right\}$$

which simplifies to $\{0,*|1\}$. What value is this? To find out, we use the inequalities $-\frac{1}{4} \leqslant * \leqslant \frac{1}{4}$, which tells us that

$$\{0,-\tfrac{1}{4}|1\} \leqslant R5R \leqslant \{0,\tfrac{1}{4}|1\},$$

and so we must have $R5R = \frac{1}{2}$, since the Simplicity Rule tells us that this is the value of both $\{0,-\frac{1}{4}|1\}$ and $\{0,\frac{1}{4}|1\}$. Verify in like manner the first few entries of Table 6. Who wins Fig. 21?

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	…
LnL	–	0	-1	-1	$*$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$*$	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{16}$	…
LnR = RnL	0	0	0	$*$	$*$	$*$	0	0	0	$*$	$*$	$*$	0	0	0	…
RnR	–	0	1	1	$*$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$*$	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{16}$	…

Table 6. Values of Positions in Seating Couples.