Exercises

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Table of contents

Introductory questions

These are questions that you might already be able to do! They cover some of the main ideas we'll need in the course: some of them can be answered systematically, and others emphasise the importance of clearly defining the space we're working in.

Warm-up

Exercise 0.1. In a film dramatizing the early history of probability, three actors are required to portray three mathematicians, each with a distinctive (false) beard. Due to a mix-up in the make-up department, the false beards are randomly distributed to the actors (one beard each). What is the probability that at least one beard goes to the right actor?

Exercise 0.2.

a. On the last day of a basketball season, only teams A and B can possibly win the league. Team A will win the league if either they win their last game, or if Team B lose their last game (A and B are not playing each other). Games do not end in draws, and we naively assume that all outcomes are equally likely. What is the probability of A winning the league?

b. Now suppose it is a football season, and draws are possible. Again we assume all outcomes are equally likely, and that A needs to do at least as well as B to in the league. What is the probability of A winning the league?

Exercise 0.3. Three dice are rolled. Find the probability that the total score is (a) 3; (b) 4; (c) 5; (d) 6.

Exercise 0.4. Three shots are fired at a target that is divided into three regions. Each shot hits the target and is equally likely to hit any of the three regions. Find (a) the probability that all three regions are hit; (b) the probability that precisely two regions are hit.

Exercise 0.5. Two chess players want to decide fairly who should play white in a particular game by flipping a coin. Unfortunately, the only coin that they have is biased to heads. How can they still decide fairly by flipping?

Workout

Exercise 0.6. The good, the bad, and the uncertain: A, B and C are to fight in a three-cornered pistol duel (a 'truel'). All know that A's chance of hitting his target is 0.3, C's is 0.5, and B never misses. They are to fire at their choice of target one after the other, in the order A, B, C, cyclically (with gunmen who are hit taking no further part), until only one man remains. What is A's best strategy?

Exercise 0.7. Mr and Mrs Smith have two children.

a. They call one of them, Derek (a boy), down from his room and introduce you. What is the probability that both of the Smiths's children are boys?
b. They tell you their eldest child is Derek (a boy). Does this change the probability that both of the Smiths's children are boys? If so, how?

Assume that boys and girls are equally likely.

Exercise 0.8. Jimmy the gambler has three dice, each with an unusual configuration of spots on their six faces:

$$\begin{aligned} &\text{Die A has faces:} \quad 1,1,4,4,4,4; \\ &\text{Die B has faces:} \quad 3,3,3,3,3,3; \\ &\text{Die C has faces:} \quad 2,2,2,2,5,5. \end{aligned}$$

Jimmy picks a die, I pick a die, and we roll for high stakes, the larger number winning. Jimmy lets me pick the first die. Is he being nice? Which die should I pick?

Stretch

Exercise 0.9.

a. A recent study showed that most great male mathematicians were eldest sons (great female mathematicians were not studied). Must this mean that first-born sons are more likely to have mathematical ability than sons born later?
b. Another study showed that a far greater proportion of the population died of tuberculosis (TB) in Arizona than in any other state in America. Must this mean that Arizona's climate favours getting TB?
c. Yet another study was carried out on alleged sex-discrimination in graduate school admissions at Berkeley, California. About 44% of men applying for graduate work were accepted, whereas only 35% of women applying for graduate work were accepted. The qualifications of men and women were roughly the same. Surprisingly, when the data were examined to identify the departments where discrimination occurred, it turned out that in each department women had a greater chance of being accepted than men. How can this apparent paradox be explained?

1 Questions for Chapter 1

1.1 Warm-up

Exercise 1.1. Consider the sample space $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let

$$A = \{2, 4, 6, 8\}, \qquad B = \{3, 6, 9\}, \qquad C = \{2, 3, 5, 7\}.$$

Write down the corresponding sets for the following events:

1. A and B;
2. C or A;
3. B minus C;
4. not A.

Exercise 1.2. Prove **(C2)**, using just **(A2)** and **(C1)**.

Exercise 1.3. Prove **(C3)**, using just **(A2)** and **(C2)**.

Exercise 1.4. Prove **(C4)**, using just **(A1)** and **(C2)**.

Exercise 1.5. On each day of a certain week, you are to observe whether it rains. Arrange the following events in order of increasing probability, with an explanation of your answer. Do not make any assumptions of independence.

E_1 is the event that it rains on Monday.
E_2 is the event that it rains on Monday and Tuesday.
E_3 is the event that it rains on Monday, Tuesday and Wednesday, but not on Thursday.
E_4 is the event that it rains on Monday and Tuesday, but not on Thursday.

Exercise 1.6. Prove **(C10)**, using just **(A2)** and **(C7)**.

Exercise 1.7. There is a probability of 3/4 that Eric goes to the pub on Sunday evening (event A), and there is a probability of 1/3 that Eric arrives late to work on Monday morning (event B). Without making any assumptions on the dependence of these two events, show that

$$\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}.$$

Exercise 1.8. Anne and Bob meet on 'Blind Date'. If the probability that Anne falls in love with Bob is 0.4, the probability that Bob falls in love with Anne is 0.2 and the probability that both Anne and Bob fall in love with each other is 0.1, then find the probability that

a. at least one person falls in love;

   hint:use **(C6)**

b. neither person falls in love;

   hint:use **(C2)** plus part a.

c. exactly one of the two people fall in love.

   hint:use **(C1)** plus part a.

Exercise 1.9. Consider the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. Which of the following are σ-algebras, and which are not? Justify your answers.

i. $\mathcal{F}_1 = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, \Omega\}$;
ii. $\mathcal{F}_2 = \{\emptyset, \{1, 2, 3\}, \{4, 5\}, \{6\}, \Omega\}$;
iii. $\mathcal{F}_3 = \{\emptyset, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{5\}, \{6\}, \{5, 6\}, \Omega\}$.

1.2 Workout

Exercise 1.10. Prove **(C1)**, using just **(A3)**.

Exercise 1.11. Prove **(C5)**, using just **(A1)** and **(C1)**.

Exercise 1.12. Prove **(C6)**, using just **(A3)** and **(C1)**.

Exercise 1.13. Using **(C2)** and **(C8)**, prove *Boole's other inequality*: for events $A_1, A_2, \ldots$,

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{\infty} \mathbb{P}(A_i^c).$$

Exercise 1.14. A special case of **(C8)** (Boole's inequality) is that for any finite collections of sets $A_1, A_2, \ldots, A_k$,

$$\mathbb{P}(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} \mathbb{P}(A_i).$$

Prove this result directly, using just **(A1)** and **(C6)**, by induction.

Exercise 1.15. Using **(C6)**, prove that for events A, B, and C we have

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B \cap C).$$

Exercise 1.16. Suppose that Ω is finite and $A_1, \ldots, A_n$ is a *partition* of Ω (so the A_i are non-empty, pairwise disjoint, and $\bigcup_i A_i = \Omega$). Let $\mathcal{F}$ be the set of all possible unions of some of the A_is (i.e., the sets $\bigcup_{i \in I} A_i$ for all $I \subseteq \{1, 2, \ldots, n\}$, a total of 2^n sets in all). Verify that $\mathcal{F}$ is a σ-algebra.

Exercise 1.17. Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras over Ω. Show that their intersection $\mathcal{F} \cap \mathcal{G}$ is also a σ-algebra over Ω.

Exercise 1.18. Show that a σ-algebra $\mathcal{F}$ is closed under countable intersections: i.e., if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

Let $\mathcal{F}$ be a σ-algebra over Ω and suppose that $B \in \mathcal{F}$. Consider $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$, and show that $\mathcal{G}$ is a σ-algebra over B.

1.3 Stretch

Exercise 1.19. Consider the scenario of tossing a coin repeatedly and indefinitely. Consider the sample space Ω being the set of all sequences $\omega = (\omega_1, \omega_2, ...)$ where ω_i is the outcome of the ith roll, H or T. The notation for this is $\Omega = \{\mathrm{H}, \mathrm{T}\}^{\mathbb{N}}$. Consider the events $E_i = \{\omega_i = \mathrm{H}\}$ and let

$$A = \{\text{roll H infinitely often}\} = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} E_m.$$

Express the event A^{c} (i) in symbols, and (ii) in words.

Exercise 1.20. Prove **(C7)**, using just **(A3)**, by induction.

Exercise 1.21. Prove **(C8)**, using just **(A4)** and **(C5)**.

Exercise 1.22. The purpose of this exercise is to prove that **(A4)** (countable additivity) is equivalent to **(C7)** (finite additivity) plus **(C9)** (continuity along monotone limits).

a. Show that **(A4)** implies **(C9)**.

   hint: first show that it suffices to establish the first half of C9 (for increasing events); then express $\cup_{m=1}^{\infty} A_m$ as a countable union of pairwise disjoint events.

b. Show that **(C7)** and **(C9)** together imply **(A4)**.

Exercise 1.23. Suppose A, B, C are events with

$$\begin{aligned} &\mathbb{P}(A) = 0.5, && \mathbb{P}(B) = 0.7, && \mathbb{P}(C) = 0.6, \\ &\mathbb{P}(A \cap B) = 0.3, && \mathbb{P}(B \cap C) = 0.4, && \mathbb{P}(C \cap A) = 0.2, \\ &\mathbb{P}(A \cap B \cap C) = 0.1. \end{aligned}$$

a. Find the probability that exactly two of A, B, C occur.
b. Find the probability that exactly one of A, B, C occurs.

Exercise 1.24. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be an increasing sequence of σ-algebras over a sample space Ω. Consider $\mathcal{F} = \bigcup_i \mathcal{F}_i$.

a. Show that $\mathcal{F}$ satisfies **(S1)** and **(S2)**, and that for any $A, B \in \mathcal{F}$, we have $A \cup B \in \mathcal{F}$ (this is weaker than **(S3)**.
b. (Hard!) Show that $\mathcal{F}$ need not be a σ-algebra by considering the example where $\Omega = \mathbb{N}$ and where $\mathcal{F}_n$ consists of all subsets of $\{1, 2, \dots, n\}$ and their complements in $\mathbb{N}$.

Assignment 1

Question 1 (Exercise 1.14)

A special case of **(C8)** (Boole's inequality) is that for any finite collections of sets $A_1, A_2, \dots, A_k$,

$$\mathbb{P}(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} \mathbb{P}(A_i).$$

Prove this result directly, using just **(A1)** and **(C6)**, by induction.

Question 2 (Exercise 1.23)

Suppose A, B, C are events with

$$\begin{aligned} \mathbb{P}(A) &= 0.5, & \mathbb{P}(B) &= 0.7, & \mathbb{P}(C) &= 0.6, \\ \mathbb{P}(A \cap B) &= 0.3, & \mathbb{P}(B \cap C) &= 0.4, & \mathbb{P}(C \cap A) &= 0.2, \\ \mathbb{P}(A \cap B \cap C) &= 0.1. \end{aligned}$$

a. Find the probability that exactly two of A, B, C occur.
b. Find the probability that exactly one of A, B, C occurs.

Question 3

Write a summary, of around forty words, explaining your current best understanding of what a partition is.

Next week, you will be asked to look at pairs of these summaries, submitted by your peers, and select the summary which you think best encompasses this idea.

2 Questions for Chapter 2

2.1 Warm-up

Exercise 2.1. Find the probability that in a family of six children, the second child is a girl but they are not all girls (assuming all outcomes equally likely).

Exercise 2.2. A bag contains three red and two white marbles. These are randomly laid out in a row.

a. Write down the list of all outcomes, each of which is equally likely. Count by hand the total number of outcomes.
b. Count by hand the number of outcomes in your list for which the white marbles appear side by side.
c. From these two counts, derive the probability that the two white marbles appear side by side.

Exercise 2.3. In the following exercises, a *deck* of cards consists of 52 cards in total—there are 4 *suits* (hearts, diamonds, clubs, and spades), and 13 *denominations*: 2, 3, ..., 10, Jack, Queen, King, and Ace. Every combination of suit and value occurs exactly once, resulting in $4 \times 13 = 52$ cards in total.

You draw three cards, without replacement, from a well shuffled deck. What is the probability that the first is a King, the second is a Queen, and the last is a Jack, precisely in that order?

Exercise 2.4. An n card *hand* is simply a selection of n out of 52 cards from a deck, selected without replacement, whose ordering is irrelevant.

a. How many three card hands are there?
b. How many three card hands are there, where each card is a face card (Jack, Queen, or King)?
c. What is the probability of a three card hand where each card is a face card?

Exercise 2.5. You will need a calculator for these two problems.

a. In a hand of 13 cards, what is the chance that all cards have different values?
b. In a hand of 5 cards, what is the chance that all cards have different values?

2.2 Workout

Exercise 2.6. Probability as a branch of mathematics was born in the correspondence between Pascal and Fermat concerning some famous problems which, story has it, were raised by the Chevalier de Méré in 1654.

De Méré knew that it was advantageous to bet on the occurrence of at least one 6 in a series of 4 tosses of a die—maybe this was a common gambler's experience. He argued that it must be at least as advantageous to bet on the occurrence of at least one pair of 6's in a series of 24 tosses with a pair of dice. Fortune however disappointed him. He complained to Pascal about "preposterous mathematics which had deceived him".

Find the probabilities of

a. at least one 6 in 4 tosses of a die; and
b. at least one pair of 6's in 24 tosses of a pair of dice.

Comment on the values you obtain. You will need a calculator.

Exercise 2.7. From a well-shuffled standard pack of 52 cards, you pick two cards, one after the other without replacement. Find the probability that at least one of them is a heart. *hint*:use either C2 or C6.

Exercise 2.8. A five card hand is called a *full house* when it contains 3 of one value plus 2 of another value (for instance, 3 sixes and 2 queens).

a. In how many ways can you pick the suits of the cards that appear in a full house consisting of, say, 3 sixes and 2 queens?

   hint: In how many ways can you pick the suits of the 3 sixes? In how many ways can you pick the suits of the 2 queens?

b. In how many ways can you pick the values that appear in a full house?

c. How many five card hands are a full house? You may need a calculator.

   hint: Use parts a and b.

d. Find the probability of being dealt a full house in a five card hand. You will need a calculator.

   hint: Use part c.

Exercise 2.9. In the national lottery, you identify 6 balls from a set of 49 balls. Then 6 balls are selected at random, without replacement, from the 49 balls and you get a prize if at least 3 of these match the balls you identified, regardless of the order. (We ignore the bonus ball for now.)

What is the probability that you get a prize of some sort? You will need a calculator.

hint: How many outcomes match exactly 3 of the balls you selected? How many outcomes match exactly 4 of the balls you selected?

Exercise 2.10. In the national lottery (see Exercise 2.9), in fact, after the 6 balls are selected at random, a seventh bonus ball is selected at random. You get a very special prize if, of the six balls you identified, exactly 5 were in your selected group of 6, and your remaining selected ball matches the bonus ball.

What is the probability that you get a very special prize? You will need a calculator.

Exercise 2.11. A small village has population 30, of whom 20 support the Red party and 10 support the Blue party. An opinion pollster samples precisely 10 villagers at random. Find the probability that precisely 7 of the 10 villagers in the sample support the Red party. You will need a calculator.

Exercise 2.12. A small village has population 30. 15 villagers are from socio-economic group X (whatever that means), 10 from socio-economic group Y and 5 from socio-economic group Z. There is no overlap between socio-economic groups. Again, 10 villagers are sampled at random. What is the probability that precisely 5 villagers in the sample are from group X, precisely 3 from group Y and precisely 2 from group Z? You will need a calculator.

2.3 Stretch

Exercise 2.13. Following on from Exercise 2.6, De Méré's other query was the *probleme des partis*. A and B had contracted to play a set of games each of which has equal chances for each player. The player who first reaches five points would take the stakes. Owing to circumstances beyond their control, they are compelled to stop playing at a point where A has won 3 games and B has won 2 games. How should the stakes be divided?

hint: In the present case with A leading by 3 to 2, at most 4 more games are required before somebody reaches 5 points.

Exercise 2.14. Recall the birthday problem discussed in lectures.

a. Among 5 people, what is the probability that exactly 2 share a birthday?
b. Among n people, what is the probability that exactly r share a single birthday ($2 \leq r \leq n$)?
c. If your answer to part (b) is $q(n, r)$, explain whether or not $\sum_{r=2}^{n} q(n, r)$ is equal to the answer to the birthday problem for n, i.e., the probability that among n people, at least two share a birthday.

Exercise 2.15. A bag contains 5 red, 3 blue and 3 green beads. You draw out four beads at random i.e. every set of beads is equally likely. What are the chances that you draw:

a. four beads of one colour;
b. at least one of each colour;
c. beads with exactly two of the colours?

hint: For this exercise, counting is easiest if you assume that all beads are distinguishable.

hint: For part c, use parts a and b.

Exercise 2.16. Seven standard cubic dice are thrown.

a. What is the probability that all six numbers show up on the dice? [You will need a calculator.]
b. What is the probability that exactly five numbers show up on the dice? [You will need a calculator.]

3 Questions for Chapter 3

3.1 Warm-up

Exercise 3.1. Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and assume that all outcomes are equally likely. Let

$$A = \{2, 4, 6, 8\}, \quad B = \{3, 6, 9\}, \quad C = \{2, 3, 5, 7\}.$$

Calculate:

- $\mathbb{P}(A \mid B)$;
- $\mathbb{P}(B \mid C)$;
- $\mathbb{P}(C \cup B \mid A)$.

Exercise 3.2. In a survey it is found that 40 percent of the population like dogs, 60 per cent like cats, and 70 per cent of those who like dogs also like cats.

(a) Find the probability that a randomly selected member of the population likes both dogs and cats.

(b) Find the probability that a randomly selected member of the population likes dogs or cats but not both.

(c) What proportion of those who like cats also like dogs?

Exercise 3.3. A multiple-choice question in an exam has 4 possible answers. A student's knowledge is reflected by the number N of given answers he is able to eliminate as not being the correct answer ($N = 3$ means he knows the answer; $N = 0$ means he has not got a clue and cannot eliminate any possibilities). Let p_k be the probability that $N = k$ (so $p_0 + p_1 + p_2 + p_3 = 1$).

- Find a formula for the probability that the student gets the answer right, in terms of p_0, p_1, p_2, p_3.
- The student gets the answer right. Find a formula for the (conditional) probability that he was certain of the answer beforehand.

Exercise 3.4. Show that if events A and B are independent then A is also independent of B^{c}.

hint: Use **C1** and **C2**.

Exercise 3.5. Show that if events A and B are independent then the complementary events A^{c}, B^{c} are also independent.

hint: Use Exercise 3.4.

Exercise 3.6. Consider a family of two children, whose sexes are represented by the sample space $\Omega = \{\mathrm{BB}, \mathrm{BG}, \mathrm{GB}, \mathrm{GG}\}$, with each outcome being equally likely. Consider the events

$$B = \{\mathrm{BB}, \mathrm{BG}, \mathrm{GB}\},\ S = \{\mathrm{BB}, \mathrm{GG}\},\ G_1 = \{\mathrm{GG}, \mathrm{GB}\},\ \text{and } G_2 = \{\mathrm{GG}, \mathrm{BG}\}.$$

- Are G_1 and G_2 independent?
- Are G_1 and B independent?
- Show that G_1, S are independent and that G_2, S are independent. Are G_1, G_2, S mutually independent?

Exercise 3.7. A set Ω contains n items. A random subset S is selected by flipping a fair coin for each item and including it in S if the coin shows heads.

(i) Use a counting argument to find the number of distinct subsets of Ω.

(ii) Assume that the coins are flipped independently. Use the definition of "independence of multiple events" to show that every subset has an equal chance to be selected.

3.2 Workout

Exercise 3.8. Let C and D be any events for which $\mathbb{P}(C) > 0$ and $\mathbb{P}(D) > 0$. Prove that, if $\mathbb{P}(C \mid D) > \mathbb{P}(C)$, then $\mathbb{P}(D \mid C) > \mathbb{P}(D)$.

Exercise 3.9. Let $B \in \mathcal{F}$ be an event with $\mathbb{P}(B) > 0$. Use the definition of conditional probability to show that $\mathbb{P}(\,\cdot \mid B)$ satisfies the axioms **A1–A4**; for example, to verify **A3** you must check that for any disjoint events C and D,

$$\mathbb{P}(C \cup D \mid B) = \mathbb{P}(C \mid B) + \mathbb{P}(D \mid B).$$

Exercise 3.10. Suppose that two by-elections are to be held on the same day. Let E_1 denote the event that Labour win seat 1, and E_2 denote the event that Labour win seat 2. Party strategists believe that $\mathbb{P}(E_1) = 0.6$ and that $\mathbb{P}(E_2) = 0.4$. Which of the following numbers is the most reasonable estimate for $\mathbb{P}(E_1 \mid E_2)$, and why?

(a) 0.3; (b) 0.4; (c) 0.5; (d) 0.6; (e) 0.7.

Exercise 3.11. Consider a family with three children; suppose that boys and girls are equally likely.

- What is the probability that there is at least one boy?
- What is the (conditional) probability that there is at least one boy, given that there is at least one girl?
- What is the (conditional) probability that there is at least one boy, given that the first child is a girl?

Exercise 3.12. You roll two fair dice. What is the conditional probability of rolling at least one six, given that the total score is at least 8?

Exercise 3.13. From a survey of mixed-doubles tennis pairings, it is found that 30% of the men are left-handed, 20% of the women are left-handed, and 50% of the partners of left-handed men are left-handed. A pairing is selected at random from those surveyed.

- What is the probability that both of the pair are left-handed?
- What is the probability that exactly one of the pair is left-handed?
- What is the (conditional) probability that the man is left-handed, given that the woman is left-handed?

Exercise 3.14. Paul is to take a driving test, with probability 0.6 of passing. If he fails he will take a re-test, with a (conditional) probability 0.8 of passing. If he fails the re-test he will take a third test, with a (conditional) probability 0.5 of passing.

- What is the probability that Paul fails the first test and passes the second?
- What is the probability that he fails all three tests?

Exercise 3.15. Prove property **P3** of conditional probability for every k, using **P2** and induction.

Exercise 3.16. Re-do the birthday problem (i.e. find the probability that n people in a room have all different birthdays) using property **P3** of conditional probability.

Exercise 3.17. Suppose a proportion 0.001 of the population (i.e., 0.1%) have a certain disease. A diagnostic test is carried out for the disease: the outcome of the test is either *positive* or *negative*. It is known from past experience that the test is 90 per cent reliable, i.e. a person with the disease will test (correctly) positive with probability 0.9, while a person without the disease will test (incorrectly) positive with probability 0.1.

(a) A person is tested for the disease. What is the probability that they test positive?

(b) A person's test result is positive. What is the probability that they have the disease? Comment on your answer.

Exercise 3.18. Consider the following experiment. We have two fair six-sided dice, one red and one blue. A fair coin is tossed; if the coin comes up 'heads', the two dice are each rolled twice, while if the coin comes up 'tails', the two dice are each rolled just once. All dice rolls are independent, given the number of dice rolls.

Let A be the event that the total score on the red die is at least 6. Let B be the event that the total score on the blue die is at least 6.

- Are A and B independent? Or conditionally independent? Justify your answer.
- Given that A and B both occur, find the (conditional) probability that the coin came up 'heads'.

Exercise 3.19. Prove the "equivalent forms for independence" theorem , using the definition of conditional probability.

Exercise 3.20. Show that if A, B, C are independent then A, B and C^c are independent.

hint: Use **C1**, **C2**, and Exercise 3.4.

Exercise 3.21. Three players, A, B, and C, take it in turns (in that order) to roll a standard fair six-sided die. The first player who rolls a 6 wins the game. What are the probabilities of victory for each of the three players?

3.3 Stretch

Exercise 3.22. A traveller arrives at dusk at a small village. She knows that 5% of the villagers are vampires, 10% are werewolves and the remainder are, for want of a better word, normal. She meets a villager, and asks him if he is normal. She knows that normal people always answer this question truthfully. She also knows that vampires are basically honest, so that any particular vampire has only a 10% chance of lying to this question. Werewolves are less honest and have a 30% chance of lying.

(a) What is the probability that the villager will claim to be normal?

(b) The villager claims to be normal. What is the probability that he is lying?

Exercise 3.23. A plane is missing, and it is presumed that it was equally likely to have gone down in any of three possible regions. The probability that the plane will be found upon a search of region 1, when the plane actually is in that region, is judged to be 0.9. The corresponding figures for regions 2 and 3 are 0.95 and 0.85 respectively. What is the conditional probability that the plane is in region 3 given that searches in regions 1 and 2 have been unsuccessful?

Exercise 3.24. Alan, Bernard and Charles are prisoners of the mad king, Max. Max has decided to kill two of the prisoners and release the third, but the prisoners have no information as to who is to be set free. Alan says to the jailer (who knows Max's decision) "As I know that at least one of my comrades is to be executed, you will give me no information if you tell me the name of one prisoner, other than myself, who is to be executed."

The jailer accepts this argument, and tells Alan that Bernard is to be executed. Alan now reasons that either he or Charles is to be released, so that his probability of release has increased from 1/3 to 1/2. Which argument is correct, the argument that convinced the jailer or the one that Alan used or neither?

Hint: Let A denote the event that Alan is to be executed, B the event that Bernard is to be executed and J_B the event that the jailer names Bernard. Show that B is not the same as J_B by finding $P(B \mid A^{\mathrm{c}})$ and $P(J_B \mid A^{\mathrm{c}})$.

Exercise 3.25. Suppose a murder has been committed on an island inhabited by $n+2$ people. You can assume that one of them did it.

A suspect is identified. Genetic material left at the scene of the crime matches that of the suspect. It is not known whether the genetic material of the other inhabitants of the island matches that found at the scene of the crime. The police estimate the chance that the suspect is guilty to be $\alpha/(n+2)$. All others on the island are *a priori* assumed equally likely to be guilty.

The proportion of people in the general population of the world, whose genetic characteristics match those of the material at the scene of the crime is p. Matters are complicated by the fact that the suspect has one relative living on the island, and because of this relationship this relative has probability q of having genetic characteristics that match those of the material at the scene of the crime.

Find a formula for the probability that the suspect is guilty, given the evidence.

Exercise 3.26. At a party you hear that Leo's birthday is in an earlier month of the year than Matt's (denote this event by '$L < M$'). Then you meet Keanu and wonder "what is the chance that Keanu's birthday is in an earlier month than Leo's?". Suppose that each of the three is equally likely to be born in any month of the year and calculate:

(a) $P(L < M)$; (*hint*: Use **P4**, with the partition of events L_j = Leo's birthday is in month j, with $j = 1, 2, \dots, 12$.)

(b) $P\big(L_j \mid L < M\big)$, for $j = 1, 2, \dots, 12$;

(c) $P(K < L \mid L < M)$.

hint: For the final part of the question, use the conditional version of the partition theorem i.e.,

$$P(A \mid C) = \sum_j P\big(A \mid B_j \cap C\big)\, P\big(B_j \mid C\big)$$

where the B_j form a partition; also use that conditionally on L_j, $K < L$ is independent of $L < M$.

Exercise 3.27. Continuing from Exercise 3.7, generate two subsets S_1 and S_2 independently, and find

(i) $P(|S_1| = m)$ for $0 \le m \le n$, (where $|S_1|$ is the number of elements of S_1);

(ii) $P(S_1 \subseteq S_2)$; (*hint*: use **P4**)

(iii) $P(S_1 \cup S_2 = \Omega)$. (*hint*: use part b))

You may use that any specific subset S is generated with probability 2^{-n} (as proven in Exercise 3.7).

Assignment 2

Question 1 (Exercise 3.17)

Suppose a proportion 0.001 of the population (i.e., 0.1%) have a certain disease. A diagnostic test is carried out for the disease: the outcome of the test is either *positive* or *negative*. It is known from past experience that the test is 90 per cent reliable, i.e. a person with the disease will test (correctly) positive with probability 0.9, while a person without the disease will test (incorrectly) positive with probability 0.1.

(a) A person is tested for the disease. What is the probability that they test positive?

(b) A person's test result is positive. What is the probability that they have the disease? Comment on your answer.

Question 2 (Exercise 3.26)

At a party you hear that Leo's birthday is in an earlier month of the year than Matt's (denote this event by '$L < M$'). Then you meet Keanu and wonder "what is the chance that Keanu's birthday is in an earlier month than Leo's?". Suppose that each of the three is equally likely to be born in any month of the year and calculate:

(a) $P(L < M)$; (*hint*: Use **P4**, with the partition of events L_j = Leo's birthday is in month j, with $j = 1, 2, \dots, 12$.)

(b) $P\big(L_j \mid L < M\big)$, for $j = 1, 2, \dots, 12$;

(c) $P(K < L \mid L < M)$.

hint: For the final part of the question, use the conditional version of the partition theorem i.e.,

$$P(A \mid C) = \sum_j P\big(A \mid B_j \cap C\big)\, P\big(B_j \mid C\big)$$

where the B_j form a partition; also use that conditionally on L_j, $K < L$ is independent of $L < M$.

Question 3 (comparative judgement question)

Write a summary, of around forty words, explaining your current best understanding of what it means for two events to be independent of each other. The best summaries are ones that use key ideas, rather than repeating the original definition.

Next week, you will be asked to look at pairs of these summaries, submitted by your peers, and select the summary which you think best encompasses this idea.

4 Questions for Chapter 4

4.1 Stretch

Exercise 4.1. Get some practice in thinking about subjective probability by specifying your own subjective probabilities for your final grade in 1H, and another event of concern to you (your choice, but try to make it interesting!). Explain how you decided your probability values.

Remember, these are your own subjective values, so as long as they obey the probability axioms, they cannot be wrong; however, your reasoning can, of course, be faulty. For instance, you cannot get first class marks in all courses and then have a second class overall mark.

Discuss your solution with your tutor.

Exercise 4.2. Provide an intuitive justification for **A3**, namely

$$P(A \cup B) = P(A) + P(B) \text{ whenever } A \cap B = \emptyset$$

under

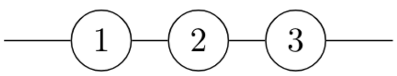
i. the relative frequency interpretation, and
ii. the betting interpretation.

In other words, write down the definitions of the three probabilities $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(A \cup B)$ under these interpretations, and show, for each interpretation, why A3 should hold. For the betting interpretation, you may assume that a combination of fair bets is also fair.

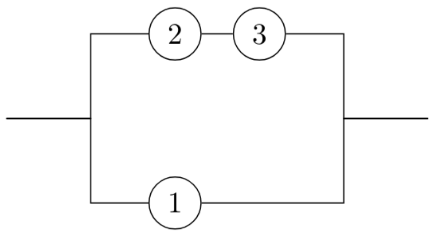
5 Questions for Chapter 5

5.1 Warm-up

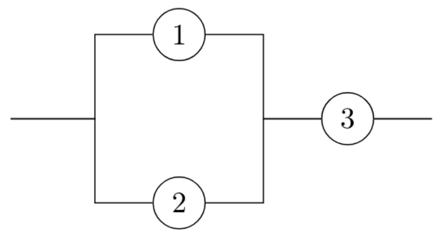
Exercise 5.1. The diagrams below represent systems with components that fail independently: the probability that component i works is $p_i \in [0, 1]$. For each system: i. find the probability that the system works, as a function of p_1, p_2, and p_3; ii. find the probability that the system works in case $p_1 = p_2 = p_3 = 0.7$; iii. find the conditional probability that component 1 works, given that the system works, in case $p_1 = p_2 = p_3 = 0.7$.



(a) Two components in series



(b) Three components in parallel



(c) A system with four components

5.2 Workout

Exercise 5.2. The colour of a certain kind of mouse is determined by the inheritance of allele pairs of a gene, which are denoted BB, Bb and bb. B is dominant (BB or Bb give a black mouse, while bb gives a brown mouse). When mating, each parent passes one of its allele pair (each choice being equally likely) independently for each offspring.

In a certain experiment, two mice of type Bb are mated to produce an offspring, Mickey.

a. Find the probability that Mickey is black.

For parts (b)–(d) suppose that Mickey is indeed black.

b. What is the probability that Mickey is type BB?
c. Mickey is mated with a brown mouse. They have three offspring. Find the probability that all the offspring are black.
d. Suppose that all three offspring of part (c) are indeed black. What is the probability now that Mickey is of type BB?
e. Finally, suppose that in the original experiment the genotypes of Mickey's parents are unknown. However, each of the two mice, independently, has probability 1/3 of genotype BB, and probability 2/3 of genotype Bb.

Repeat parts (a), (b), (c) and (d) using this information.

Exercise 5.3. Suppose a (homologous) gene has two alleles A and a, of which a is recessive and also harmful, so aa individuals have some genetic disorder. Suppose the proportion of individuals of genotype Aa in the healthy population (i.e. those not of type aa) is λ. Suppose a married couple, assumed to have come together by random mate selection, are both healthy. Suppose they have one healthy child. If they then decide to have a second child, what is the probability that it will also be healthy:

(a) as a function of λ, and

(b) in the special case $\lambda = 0.1$.

5.3 Stretch

Exercise 5.4. The diagram below represents a system with components that fail independently. Suppose that component i works with probability $\mathbb{P}(W_i) = p_i$. Find the probability that the system works as a function of $p_1, \dots, p_5$.

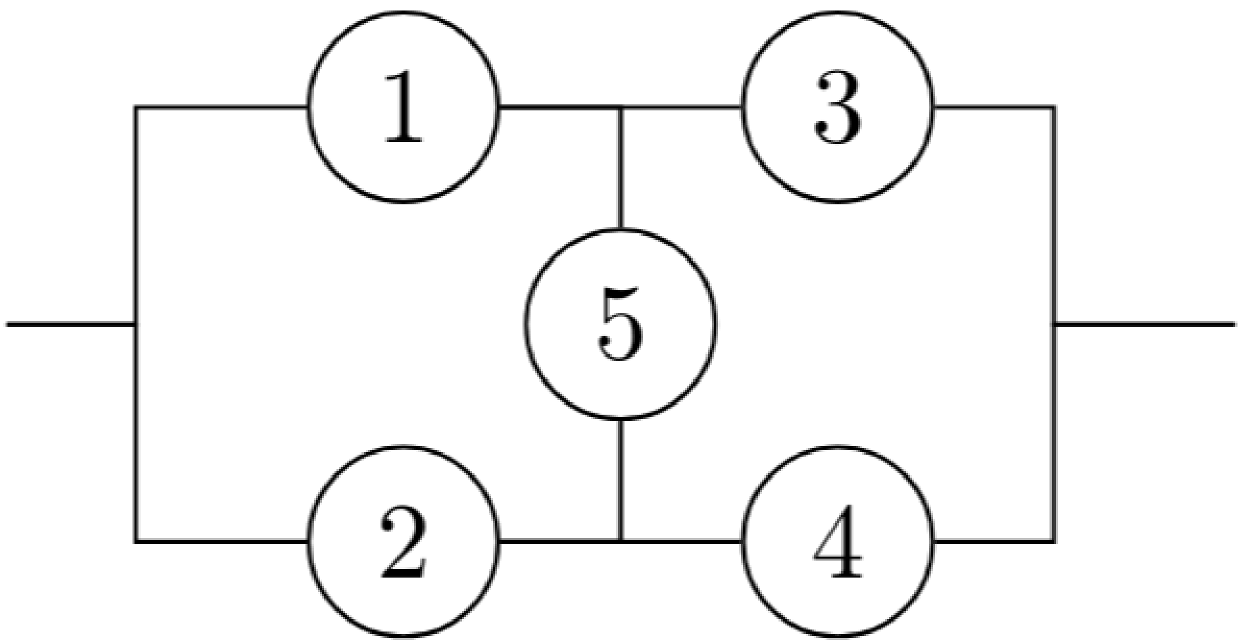


Figure 5.2: A system with five components

hint: Use the partition theorem with $\{W_5, W_5^c\}$

Exercise 5.5. Suppose that a certain sex-linked gene has genotypes AA, Aa, aa in females, and genotypes A, a in males. Suppose that, for a particular generation, the genotypes AA, Aa, aa occur with frequencies $u, 2v, w$. Let

$$p = u + v, \qquad q = v + w$$

(so that p, q with $p + q = 1$ are the proportions of A and a in the female population). Suppose that the frequencies of the two genotypes A, a in the male population are p', q' (where $p' + q' = 1$).

Assume random mating between males and females in a large population. Remember that, for a sex-linked gene, the son receives one gene from the mother and none from the father. Let p_1, q_1 be the proportions of A, a in females of the next generation, and let p_1', q_1' be the proportion of A, a in males of the next generation.

Show that

$$p_1' - p_1 = \frac{p - p'}{2} \quad \text{and} \quad \frac{2}{3}\,p_1 + \frac{1}{3}\,p_1' = \frac{2}{3}\,p + \frac{1}{3}\,p'.$$

Interpret these results.

6 Questions for Chapter 6

6.1 Warm-up

Exercise 6.1. Let $X\colon \Omega \to X(\Omega)$ be a random variable. For any $B_1, B_2 \subseteq X(\Omega)$, show that:

(i) $\{X \in B_1\}^{\mathrm{c}} = \{X \in B_1^{\mathrm{c}}\}$;

(ii) $\{X \in B_1\} \cup \{X \in B_2\} = \{X \in B_1 \cup B_2\}$; and

(iii) $\{X \in B_1\} \cap \{X \in B_2\} = \{X \in B_1 \cap B_2\}$.

Exercise 6.2. A bag contains 10 counters, numbered 1 to 10. The first 6 of them are worth £1 each, the next 2 are worth £5 each, the next one is worth £10 and the last one is worth £50. You draw a counter from the bag at random, its value being X pounds. Let $\Omega = \{1, 2, \dots, 10\}$.

(a) Write down the function corresponding to random variable X, i.e. for each $\omega \in \Omega$, write down $X(\omega)$.

(b) What subset of Ω does the event $\{X \leq 5\}$ correspond to? Find $P(X \leq 5)$.

(c) Write down the probability mass function $p()$ of X, i.e. write down in a table all values X can take and their corresponding probabilities.

(d) Use the probability mass function to find $P(X \geq 10)$.

Exercise 6.3. Suppose X is a discrete random variable with possible values 1, 2, and 3, and with probability mass function $p(x) = cx^2$ for $x \in \{1, 2, 3\}$. Calculate:

(a) the value of the constant c,

(b) the value of $P(X \geq 2)$, and

(c) the value of $P(X \in \{1, 3\})$.

Exercise 6.4. Use the equality

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

to show that $\sum_{x=0}^{n} p(x) = 1$ when $X \sim \mathrm{Bin}(n, p)$.

Exercise 6.5. Suppose $X \sim \mathrm{Bin}(n, p)$. Calculate $p(x)$ when

(i) $n = 7$, $p = 0.3$ for $x = 0, 1, 2$;

(ii) $n = 10$, $p = 0.95$ for $x = 9, 10$;

(iii) $n = 10$, $p = 0.05$ for $x = 1, 0$.

Use these to find $P(X > 2)$ in part (i), $P(X \leq 8)$ in part (ii), and $P(X \geq 2)$ in part (iii).

Exercise 6.6. Let $X \sim \mathrm{Bin}(n, p)$, and let $Y = n - X$ (i.e. if X counts the number of successes in a binomial scenario, then Y counts the number of failures in that same scenario). Show that $Y \sim \mathrm{Bin}(n, 1-p)$. *hint*: Remember that $\binom{n}{x} = \binom{n}{n-x}$.

Exercise 6.7. Use the equality

$$\exp(\lambda) = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

to show that $\sum_{x=0}^{\infty} p(x) = 1$ when $X \sim \mathrm{Po}(\lambda)$.

Exercise 6.8. Let $X \sim \mathrm{Po}(\lambda)$. Consider $f(\lambda) := e^{-\lambda}\lambda^x/x!$ as a function of $\lambda > 0$ (so x is fixed). Find the λ which maximises f. (This value is of interest when trying to estimate λ from an observation $X = x$.)

Exercise 6.9. Let $X \sim \mathrm{U}(3, 7)$.

(a) Sketch the probability density function $f(\cdot)$ of X.

(b) Find $P(X \in [4, 6])$, and mark the corresponding area on your graph.

(c) Find $P(X \in [1, 5])$, and mark the corresponding area on your graph.

Exercise 6.10. Let $X \sim \mathcal{E}(1)$.

(a) Sketch $f(x)$.

(b) Find $P(X \in [1, 2])$, and mark the corresponding area on your graph.

Exercise 6.11. In calculus, you showed that $\int_{-\infty}^{\infty} e^{-z^2}\,\mathrm{d}z = \sqrt{\pi}$ via double integration and change to polar coordinates. Use this equality to evaluate

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,\mathrm{d}x.$$

(Thereby, you have shown that $f(x)$ integrates to one when $X \sim \mathcal{N}(\mu, \sigma^2)$.)

Exercise 6.12. Let $a < b$, and $X \sim \mathrm{U}(a, b)$.

(a) Calculate, and sketch, the cumulative distribution function F of X.

(b) With $[a, b] = [3, 7]$, use F to calculate $P(X \in [4, 6])$ and $P(X \in [1, 5])$. (You may wish to check your answer against your solution for Exercise 6.9)

Exercise 6.13. Let $\beta > 0$, and $X \sim \mathcal{E}(\beta)$.

(a) Calculate, and sketch, the cumulative distribution function F of X.

(b) With $\beta = 1$, use F to calculate $P(X \in [1, 2])$. (You may wish to check your answer against your solution for #exr-expsketch.)

Exercise 6.14. Let $\beta > 0$ and $a > 0$. Show that if $X \sim \mathcal{E}(\beta)$ then $aX \sim \mathcal{E}(\beta/a)$.

Exercise 6.15. Prove the "standardizing the Normal distribution" theorem, by showing that $F_{(X-\mu)/\sigma}(z) = F_Z(z)$, and $F_{\sigma Z+\mu}(x) = f_X(x)$, when $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Z \sim \mathcal{N}(0, 1)$.

Exercise 6.16. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Use the tables below (you may pick for z the nearest number) to evaluate $P(X \in [-1, 2])$ when:

$\mu = 0$, $\sigma^2 = 1$;

$\mu = 2$, $\sigma^2 = 4$;

$\mu = -1.2$, $\sigma^2 = 1.62$.

z	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\Phi(z)$	0.540	0.580	0.618	0.655	0.691	0.726	0.758	0.788	0.816	0.841
z	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\Phi(z)$	0.864	0.885	0.903	0.919	0.933	0.945	0.955	0.964	0.971	0.977
z	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
$\Phi(z)$	0.982	0.986	0.989	0.992	0.994	0.995	0.996	0.997	0.998	0.999

6.2 Workout

Exercise 6.17. Let Ω be a sample space, let $X \colon \Omega \to X(\Omega)$ be a random variable on Ω, and let $P()$ be a probability on Ω, so $P()$ satisfies A1–A4. Show that the function $\mathbb{P}_X()$ defined by

$$\mathbb{P}_X(B) := P(X \in B)$$

for all $B \subseteq X(\Omega)$, also satisfies A1–A4, with $X(\Omega)$ instead of Ω. (This establishes that $\mathbb{P}_X()$ is indeed a probability, as mentioned in Section 6.1 of the notes.) *hint*: To prove **A3/A4**, use parts iii. and iv. of Exercise 6.1.

Exercise 6.18. A bag contains m red marbles and n blue marbles. You randomly take r marbles from the bag, without replacement, with $r \leq m + n$. Let X denote the number of red marbles you end up with. Find the probability mass function $p(x)$ for all $x \in \mathbb{N}$.

Exercise 6.19. Let $X \sim \text{Bin}(n, p)$. By considering the ratios $p(x+1)/p(x)$ for successive x, or otherwise, find a formula in terms of n and p for the value(s) of x where $p(x)$ is maximal over $x \in \{0, 1, \dots, n\}$. In other words, find the most probable observed value for X.

Exercise 6.20. Let $X \sim \text{Bin}(n, p)$. Consider $g(p) := \binom{n}{x} p^x (1-p)^{n-x}$ as a function of $p \in (0, 1)$ (so n and x are fixed). Find the p which maximizes g. (This value is of interest when trying to estimate p from an observation $X = x$.)

Exercise 6.21. Let $X \sim \text{Bin}(n, p)$ and let A_n denote the event that X is even: $A_n = \{\omega : X(\omega) \text{ is even}\}$. By considering the partition formed by {first trial a success}, {first trial a failure}, show that $P(A_n) = p + (1-2p)P(A_{n-1})$ for $n \in \mathbb{N}$. Hence prove by induction that

$$P(A_n) = \frac{1 + (1-2p)^n}{2} \quad \text{for all } n \in \mathbb{Z}_+.$$

Exercise 6.22. A boxer, at the start of his career, decides that he will retire after his first defeat. In each fight he has probability q of being defeated, with successive fights being assumed independent. Let X be the total number of fights in the boxer's career.

(a) What is the distribution of X?

(b) Show that $P(X > x) = (1 - q)^x$ for $x \in \{0, 1, 2, 3, ...\}$. Interpret this result.

(c) Find a formula for the probability that X is even (simplify your formula as much as possible). Evaluate this probability when $q = 0.4$.

Exercise 6.23. Let $X \sim \mathrm{Po}(\lambda)$. Calculate $p(x)$ when:

(i) $\lambda = 2.1$ for $x = 0, 1, 2$;

(ii) $\lambda = 9.5$ for $x = 9, 10$;

(iii) $\lambda = 0.5$ for $x = 1, 0$.

Use these to find $P(X > 2)$ in item (i), and $P(X \geq 2)$ in item (iii).

(Compare these values with those from #exr-binomialcalc.)

Exercise 6.24. Let $X \sim \mathrm{Po}(\lambda)$ where $\lambda > 1$. Show that $p(x)$, as a function of x, increases monotonically, and then decreases. For what $x \in \mathbb{Z}_+$ is $p(x)$ maximal?

Exercise 6.25. Let $X \sim \mathrm{Po}(\lambda)$ and $Y \sim \mathrm{Po}(\mu)$ be independent. By computing the probability mass function of $X + Y$, show that $X + Y \sim \mathrm{Po}(\lambda + \mu)$. Here independence means that $P(X = x, Y = y) = P(X = x)\, P(Y = y)$ for all x, y: you may want to look ahead to Chapter 7.

Exercise 6.26. Let $\beta > 0$, and $X \sim \mathcal{E}(\beta)$. For any real numbers $s > 0$ and $t > 0$, show that

$$P(X > s + t \mid X > s) = P(X > t)\,.$$

[This is called the *memoryless property*, and says that, given that X is bigger than s, the chance that it is at least t bigger than s is the same no matter how big s is.]

Exercise 6.27. Let $\beta > 0$ be some constant parameter, and let X be a continuous random variable with probability density function

$$f(x) = \frac{\beta}{2} e^{-\beta|x|} \text{ for } x \in \mathbb{R}.$$

(This is called the *two-sided exponential* or *Laplace* distribution.) Calculate, and sketch, the cumulative distribution function F of X.

6.3 Stretch

Exercise 6.28. A sequence of flips of a fair coin produces n heads and m tails. A *run* is a consecutive sequence of flips with the same outcome. Let R denote the number of runs (of both H and T). Find $p_R(2k)$ and $p_R(2k+1)$ for $k \in \{1, \ldots, \min(m, n)\}$. *hint*: This is a “sheep and fences” problem.

Exercise 6.29. A distributor of bean seeds determines from extensive tests that 1% of a large batch of seeds will not germinate. She sells the seeds in packets of 200 and guarantees that at least 98% of the seeds will germinate.

(i) Find the probability that any particular packet violates the guarantee

(a) exactly, using a binomial distribution;

(b) using the Poisson approximation.

(ii) A gardener buys 13 packets. What is the probability that at least one packet violates the guarantee? In your calculation, you should use the exact probability from the previous part, rather than the approximation.

Exercise 6.30. Suppose that the probability distribution of the number of eggs, N, laid by an insect is Poisson with parameter λ. Suppose also that the probability that an egg develops into an insect is q (independently for each egg).

(a) Show that the probability distribution of the number of insects produced, H, is Poisson with parameter λq. In other words, show that

$$P(H = h) = e^{-\lambda q} \frac{(\lambda q)^h}{h!} \qquad \text{for } h \in \{0, 1, 2, \dots\}.$$

hint: Use the partition theorem **(P4)** with the events $\{N = n\}$, for $n = 0, 1, 2, \dots$.

(b) What is the probability distribution of $N - H$?

(c) Show that H and $N - H$ are independent, i.e. for all j, h,

$$P(\{H = h\} \cap \{N - H = j\}) = P(H = h)\, P(N - H = j)\,.$$

Note: you will need the exponential series several times.

Exercise 6.31. Let $X \sim \mathrm{Bin}(1, p)$ be a Bernoulli random variable and let $Y \sim \mathrm{U}(0, 1)$ be a uniform random variable, and suppose that X and Y are independent. Let $S = X + Y$ and $M = \max\{X, Y\}$.

(a) What is the cumulative distribution function of S? Is S discrete, continuous, or neither?

(b) What is the cumulative distribution function of M? Is M discrete, continuous, or neither?

hint: Compute the cumulative distribution function. Use the partition theorem $P(A) = P(X = 0)\, P(A|X = 0) + P(X = 1)\, P(A|X = 1)$ for suitable events A; independence (which we look at in Chapter 7) means that conditioning on the value of X does not change the distribution of Y.

Assignment 3

Question 1 (Exercise 6.29)

A distributor of bean seeds determines from extensive tests that 1% of a large batch of seeds will not germinate. She sells the seeds in packets of 200 and guarantees that at least 98% of the seeds will germinate.

(i) Find the probability that any particular packet violates the guarantee

  (a) exactly, using a binomial distribution;

  (b) using the Poisson approximation.

(ii) A gardener buys 13 packets. What is the probability that at least one packet violates the guarantee? In your calculation, you should use the exact probability from the previous part, rather than the approximation.

Question 2 (Exercise 6.7)

Use the equality

$$\exp(\lambda) = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

to show that $\sum_{x=0}^{\infty} p(x) = 1$ when $X \sim \mathrm{Po}(\lambda)$.

Question 3 (comparative judgement question)

Write a summary, of around forty words, explaining your current best understanding of what a random variable is. Feel free to use any exampples or related functions to help you illustrate your explanation. The best summaries are ones that use key ideas, rather than repeating the original definition.

Next week, you will be asked to look at pairs of these summaries, submitted by your peers, and select the summary which you think best encompasses this idea.

7 Questions for Chapter 7

7.1 Warm-up

Exercise 7.1. The discrete random variables X and Y have the following joint probability mass function:

$f(x, y)$	$y = -1$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	$1/12$	$1/4$	0	0
$x = 1$	$1/12$	$1/12$	$1/12$	$1/12$
$x = 3$	0	0	$1/4$	$1/12$

Find the marginal probability mass functions $f_X(x)$ and $f_Y(y)$.

Exercise 7.2. A fair coin is tossed three times. Suppose X denotes the number of heads in the first two tosses and Y denotes the number of heads in the last two tosses.

(a) Make a table of the joint probability distribution of X and Y.

(b) Use your table for the joint probability mass function to confirm that the marginal probability mass functions of X and Y are both $\text{Bin}(2, 1/2)$.

(c) Compute $P(X = Y)$.

Exercise 7.3. Continuing from Exercise 7.2, confirm that X given $Y = 2$ is not binomial.

Exercise 7.4. Continuing from Exercise 7.1, write in a table the conditional probability mass function $f_{Y|X}(y|x)$.

Exercise 7.5. Let Y be a random variable with $P(Y = +1) = P(Y = -1) = 1/2$. Let X be another random variable, taking values in $\mathbb{Z}$, independent of Y.

Show that the random variables Y and XY are independent *if and only if* the distribution of X is symmetric, i.e., $P(X = k) = P(X = -k)$ for all $k \in \mathbb{Z}$.

Exercise 7.6. Suppose that X and Y have joint probability density function

$$f(x, y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

By integrating the joint probability density function, calculate:

(i) $P(X < 1/2, Y > 1/2)$, and

(ii) $P(X > Y)$.

Exercise 7.7. Continuing from Exercise 7.6:

(a) Show that X and Y are independent, with $X \sim \text{Exp}(2)$, and $Y \sim \text{Exp}(3)$.

(b) Confirm that $P(X < 1/2, Y > 1/2)$, which you calculated in , is equal to $P(X < 1/2)\, P(Y > 1/2)$.

Exercise 7.8. Continuing from Exercise 7.1 and Exercise 7.4, find the probability mass functions of $S = X + Y$, $D = X - Y$, and $M = \max(X, Y)$.

7.2 Work-out

Exercise 7.9. Let $U_1, U_2 \sim \text{U}(0, 1)$ be independent. By calculating the relevant cumulative distribution functions, show that $\max(U_1, U_2)$ has the same distribution as $\sqrt{U_1}$.

Exercise 7.10. A fair die is thrown. The score is divided by two and rounded up giving score X. A fair coin is then thrown X times and the number of heads is Y (so if the die roll is 3, then $X = 2$ and the coin is tossed twice).

(a) Write down the marginal probability mass function $p_X()$ for X.

(b) What is the (conditional) distribution of Y given $X = x$? Using this conditional probability mass function, and the marginal from part (a), compute the joint probability mass function $p(x, y)$, and present it in a table.

(c) Calculate the marginal probability mass function $p_Y()$ of Y.

Exercise 7.11. Continuing from Exercise 7.10, write in a table the conditional probability mass function $p_{X|Y}(x|y)$.

Exercise 7.12. Consider the experiment described in Exercise 3.18. Let X be the total score on the red die and let Y be the total score on the blue die. The experimental set-up implies that X and Y are *conditionally independent*: give a definition of the concept of conditional independence given an event for discrete random variables. (We did not discuss this in lectures, but there is a natural definition that you should be able to find.) Are X and Y independent?

Exercise 7.13. Let $\beta_1 > 0$ and $\beta_2 > 0$. Suppose that $X_1 \sim \mathcal{E}(\beta_1)$, $X_2 \sim \mathcal{E}(\beta_2)$, and X_1 and X_2 are independent. Let $M = \min(X_1, X_2)$. Show that $M \sim \mathcal{E}(\beta_1 + \beta_2)$.