Mathematical Finance Lecture Notes 2023-24

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Contents

1 Introduction

These lecture notes are based on the content we will cover in Mathematical Finance in Michaelmas term, 2023 - 24. They are based on previous sets of notes put together by Nic Georgiou, Andrew Wade, and others.

1.1 What is Mathematical Finance?

Mathematical Finance is the study of the mathematics used to model and analyse financial markets. These models are constructed to try to better understand how markets behave in reality, and to inform decisions about investments. In reality, these markets are incredibly complex, but under some simplifying assumptions, the mathematics becomes quite elegant, and allows us to develop methods for pricing and valuing portfolios based on a wide range of financial derivatives.

In Michaelmas term, we'll focus on *discrete-time* versions of these models, where we assume that trades can only happen at specified moments. We'll build up a theory using probabilistic concepts like filtrations, conditional expectation, and martingales. Because discrete-time models lead to countable probability spaces, we'll be able to do a lot of this work in very concrete settings, and calculate the prices of some quite complex financial products.

In Epiphany term, you'll see the *continuous-time* versions of the models. Here, the continuous (uncountable!) probability spaces mean that the theory becomes much more complex and subtle, and some of the generalisations to continuous time require some pretty sophisticated measure theory. The understanding of concrete fundamental concepts you build up in Michaelmas term will put you on a solid footing to start working with the more abstract theory to come in Epiphany.

1.1.1 Finance or probability?

I consider myself a probabilist, and that's the approach I'm bringing to this course. I see it as being about the mathematics of finance, rather than the economics, and we'll be coming at a lot of the material from a probability perspective. That said, we'll need to use some financial terminology to describe the concepts we're using, and many of the examples in the course will be from a financial context.

1.2 Financial background

The main focus of this course is on financial derivatives, and how they should be priced and can be hedged. In this chapter, we set up the framework under which we work, including a lot of the definitions we'll need. In Chapters 2 and 3 we'll build up our theory using an example of a discrete-time market; then in Chapters 4 and 5 we'll look at extensions to this theory.

1.2.1 Underlying and derivative assets

The assets for sale in financial markets fall into two broad categories: **underlying assets**, and **derivative** assets. Underlying assets have intrinsic value, such as currency, stocks, bonds, and commodities. In this course, we are particularly interested in two types of underlying asset: **bonds** and **shares**.

A **bond** is a risk-free asset with a predictable price and future value. For example, it can take the form of a loan between an investor and a borrower, with a fixed rate of return.

A **share** (or a single unit of a **stock**) is a risky asset, which future price and value are unpredictable. We will write S_t for the price of a share at time t. Throughout this course, we'll model the way the share price evolves using different probabilistic models, in discrete or continuous time.

Example 1.1

A cafe in Durham wants to order enough coffee beans to keep its cafes running for Epiphany term (hopefully, we already have enough coffee to keep us running for Michaelmas; otherwise, we're in trouble).

It orders 10,000kg of Arabica coffee beans, at about 30 Brazilian Real per kg, for a total of 300,000 BRL.

The exchange rate between BRL and GBP is currently 1:6.02, so three hundred thousand Real is worth about fifty thousand pounds, which the cafe will have to pay *on delivery* of the coffee, in January.

The issue here is the uncertainty associated with this transaction. Since we can't know what the exchange rate will be in January, it is impossible to know today what the price in GBP will be. If the exchange rate is still close to 1:6, then the cafe will have to pay £50k, but if the rate decreases to, say, 1:5, then the coffee will cost more like £60k. This introduces a *currency risk* which the cafe would presumably rather avoid.

They can avoid this currency risk in several different ways, including:

1. They could buy 300,000 BRL £1,000,000 today, at a price of 50,000 GBP, and keep this money in a bank account until January.
   - Pros: The currency risk is completely eliminated.
   - Cons: This is essentially paying for the coffee months before it's delivered. It's a lot of money to tie up, and the cafe might not have it on hand right now.
2. The cafe could "buy" a **forward contract** for 300,000 BRL with delivery three months from now. This is an agreement with (e.g.) a bank that the cafe will buy the Real from them, at an exchange rate which is agreed at time $t = 0$. The rate is called the *forward price*, and usually denoted K; here we might have $K = 5.5$. Typically forward contracts involve no upfront cost.
   - Pros: There is nothing to pay now, and the currency risk is completely eliminated; the cafe know exactly how much they will have to pay. If the exchange rate at $t = T$ is higher than K, then they will have saved some money in buying at the lower rate of 1 GBP : K BRL - but…
   - Cons: …if the exchange rate at $t = T$ is *lower* than K, then the cafe will still have committed to pay the higher rate, and will lose out compared to paying the market rate for their 300,000 BRL.
3. What the cafe really want is a contract allowing them the *option* to buy at an agreed price, but not the *obligation*. This is called a **European call option**: they agree on a *strike price* of K BRL : 1 GBP and an *expiry time* T with the broker, and pay a (hopefully fair!) upfront cost for the agreement. If the exchange rate at time T is higher than K, then they can exercise the option and pay the lower price; if the exchange rate at time T is lower than K, they can ignore the option and simply buy 300,000BRL at the market price.

- Pros: the currency risk is completely eliminated; the cafe will either pay K BRL : 1 GPB or the exchange rate at time T, whichever is lower.
- Cons: there is an upfront cost for the contract; how much should the cafe be willing to pay?

Both the forward contract and the European call option in this example are **derivative assets**. They do not have intrinsic value like the coffee beans or the currency; instead, their value derives from that of the underlying assets. We might also call these assets **contingent claims**, as their value is contingent on that of something else.

Derivative assets come in two types: **locks** and **options**. Lock products, such as the forward contract, are an agreement that the holder *will* buy (or sell) something from the writer at an agreed future date. Options, on the other hand, are an agreement that the holder *can* buy or sell something at the future date – but is not obliged to do so.

There are two main categories of options: **call** and **put**. Call options give the holder the right, but not the obligation, to *buy* the underlying asset at an agreed price; put options give the holder the right to *sell* it. The prefix **European** means that the option can only be exercised at exactly time T; **American** options can be exercised at any time $t \in [0, T]$.

1.2.2 Payoffs

The **payoff** of an asset is the amount of money it is worth to us at time T. We usually denote the payoff as Φ or, sometimes, Φ_T. For example, if we buy £P worth of bonds with interest rate r at time 0, the payoff at time T will be $\Phi_T = P(1+r)^T$. Similarly, the payoff of one stock at time T is $\Phi_T = S_T$: its value is the same as its price.

In entering a forward contract, we agree that at time T we will buy an asset at a forward price K. The payoff for this contract is $\Phi_T = S_T - K$: we are receiving assets worth S_T (the share) and giving away assets worth K (in cash). Depending on the value of S_T, this contract could have a negative payoff: if the share price is lower than K, buying one share for £K equates to losing money.

On the other hand, writing a forward contract has payoff of $\Phi_T = K - S_T$: we sell one share worth S_T and receive K in cash.

To calculate the payoff of a European call option, we need to consider two cases. If the asset price is higher than the strike price, $S_T > K$, then the holder can exercise the contract, buying the asset and handing over K in cash, for a payoff of $S_T - K$. On the other hand, if the asset price is lower than the strike price, $S_T < K$, then there is no reason to exercise the option and lose money; instead, the holder will do nothing.

As a result, the payoff of the option at time T is

$$\begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K. \end{cases}$$

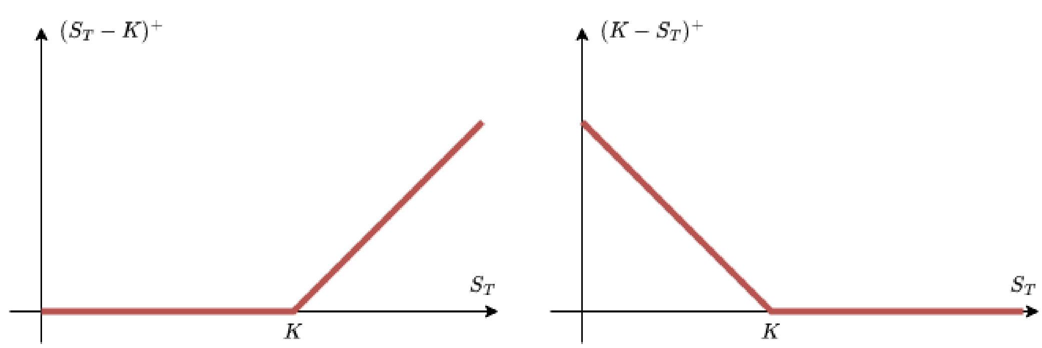


Figure 1: Payoff graphs for European call (left) and put (right) options with strike price K.

We can also view this as $\max(S_T - K, 0)$, which we write as $(S_T - K)^+$ (as it's the positive part of this value).

By the same argument, the return at time T of a put option is $\max(K - S_T, 0) = (K - S_T)^+$.

Remark. Notice that selling call options is not the same thing as buying put options! Mathematically, this is because

$$(K - S_T)^+ \neq -(S_T - K)^+$$

(one side is always positive or zero, and the other is always negative or zero). In terms of the transactions, the difference arises from the fact that the *choice* always lies with the buyer of the option; the seller is obligated to go along with the buyer's decision).

Exercise 1.1

Draw the graphs of the payoffs of the following combinations of options, as a function of the asset price:

- buying one call plus one put option with the same strike price K (this is known as a *straddle*)
- buying one call option and selling one put option with the same strike price K
- buying one call option with strike price K_1, and selling another call option with strike price K_2 (this is known as a *bull spread*; you'll want $K_2 > K_1$)

1.2.3 Risk

Investors in financial markets are hugely concerned with *risk*. Given two investments with the same expected return, investors will generally prefer the one with smaller variation. For instance, imagine you have £10,000 to invest, and you are choosing between the following strategies:

- Strategy A will return £0 with probability 0.45, or £20,000 with probability 0.55;
- Strategy B will return £10,000 with probability 0.5, or £12,000 with probability 0.5.

In both cases, the expected return is £11,000; but the variability (and hence the risk) is much higher with Strategy A. Nearly all investors would prefer Strategy B, as the potential losses are much smaller.

On the other hand, we could consider a third strategy: placing all of our money in a *risk-free* account, where it will sit and quietly earn interest.

- Strategy C will return £11,000 with probability 1.

Would you prefer Strategy C over Strategy B? For many investors, the low-risk (but not risk-free) option is most attractive.

In every market we study, we assume that there are opportunities for *risk-free investment* in the form of bonds. These bonds pay interest at a known, constant rate r.

Context: nominal and effective interest

If the interest is compounded once per year, then after one year an initial deposit of $B(0)$ will be worth $B(1) = B(0)(1 + r)$, and after t years, $B(t) = B(0)(1 + r)^t$.

If it is compounded n times per year (common choices for n include 2, 4, 12, 52, and 365) at a **nominal rate** r, then at each of n equally-spaced intervals, the value of the investment increases by a factor $(1 + r/n)$, so that $B(1/n) = B(0)(1 + r/n)$ and, more generally, $B(m/n) = B(0)(1 + r/n)^m$.

If we deposit £B in bonds at the start of the year, and interest is compounded n times at nominal rate r, then after one year our investment will be worth £$B(1 + r/n)^n$. The **effective interest rate** is the value r^* such that

$$1 + r^* = (1 + r/n)^n.$$

If you take out a loan or a mortgage, the APR you'll see advertised is the effective interest rate, expressed as a percentage.

Present-value analysis

Because we can always invest money into bonds for a guaranteed interest rate, a deposit of £1000 today is "worth more" than a deposit of £1000 in a year's time. To account for this, we calculate the **present value** of a cash flow. To do this, we multiply each deposit by the **discount factor**

$$\alpha = \frac{1}{1 + r}:$$

a deposit of £x at time t is worth $x\alpha^t$ "in today's money".

Example 1.2

Which of these cash flows has the highest (and the lowest) present value?

1. $x_i = 100$, $t_i \in \{1, 3, 5, 7, 9\}$
2. $x_i = 50$, $t_i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
3. $x_i = 200$, $t_i \in \{1, 3, 5, 7, 9\}$ and $x_i = -100, t_i \in \{2, 4, 6, 8, 10\}$
4. Wildcard: construct your own cash flow.

Example 1.3
You have a bank account in which the annual interest rate is 5%, compounded monthly You plan to pay in £D every month for thirty years (360 months), and then withdraw £1000 every month for the following twenty years (months 361 - 600). What is the minimum deposit D you should be making, to ensure you have enough in your account?

Here $r = 0.05/12 = 1/240$, so $\alpha = 241/240$. The present value of all the deposits is

$$D + D\alpha + D\alpha^2 + \cdots + D\alpha^{359} = D\sum_{i=0}^{359} \alpha^i = D\,\frac{1-\alpha^{360}}{1-\alpha}.$$

Next, the present value of all the withdrawals is

$$1000\alpha^{360}(1 + \alpha + \alpha^2 + \cdots + \alpha^{239}) = 1000\alpha^{360}\frac{1-\alpha^{240}}{1-\alpha}.$$

These are equal when

$$D = 1000\alpha^{360}\frac{1-\alpha^{240}}{1-\alpha}\,\frac{1-\alpha}{1-\alpha^{360}} = 1000\alpha^{360}\frac{1-\alpha^{240}}{1-\alpha^{360}} \approx 182.065.$$

1.2.4 Portfolios

In general, a portfolio is a description of the shares, options, and cash we hold at any given time. The values of any of these things can be negative if we engage in **short selling**: borrowing units of an asset at one time, to be returned later. In this course, it is always possible to short sell anything we need to: there is always someone willing to lend it to us. This is part of an assumption of **liquidity** which we will discuss later.

We can describe a portfolio in one of two different ways: in terms of the amount of each asset we *hold* at every time t, or in terms of the amount of each asset we *buy or sell*.

The trades defining a portfolio need not happen at deterministic (pre-determined) times. For example, we may wish to create a portfolio in which we buy shares of a stock at time 0, and then sell them as soon as the share price doubles. In that case, the time of our trade is $\inf\{t : S_t \geq 2S_0\}$. This is a permissible trade, because we will recognise the moment when it happens; on the other hand, we can't decide to sell "whenever the price is highest between time 0 and time T", because we can't know whether or not that moment has arrived. If you took Markov Chains last year, you might recognise that we're talking about stopping times.

We say that a portfolio is **self-financing** if each of the trades involved produces no **cash flow** – that is, if the total value of what we buy is equal to the total value of what we sell. In practice, this means that we use the principle that, aside from some **initial investment**, the trading strategy should take care of itself.

When it is possible to assemble a portfolio that is *certain* to produce a profit, we say that there is an **arbitrage opportunity** in the market. We can think of arbitrage as arising from two different possible sources. Firstly, if an underlying asset is guaranteed to out-perform the other assets in the market (including the risk-free investment in bonds), then there is arbitrage *inherent* in the market. In some sense, the whole market is rigged; there's a guaranteed win somewhere in there. Secondly, if an asset (usually a derivative asset) is *mis-priced*, then investors can create arbitrage by buying (or selling) the mispriced asset in large quantities, as we'll see in the next example.

Example 1.4
A company's shares are floated on the stock market. The value of one share is £100 today (at $t = 0$), and we know that tomorrow (at $t = 1$) the value will either increase to £200, with some unknown probability $0 < p < 1$, or decrease to £50 with probability $1 - p$.

We are a broker trading in stocks and options, and we're considering the following (European call) option: the holder has the right (but not the obligation) to buy one share for £150, at time $t = 1$. At time $t = 0$, what is the fair price C for this option? We have to be willing to both buy and sell options at this price, and we will assume that there's a wealthy trader "Agent A" who will take advantage and run us out of business if we misprice the option and create an arbitrage.

One approach is to think about the value of the option at time $t = 1$: if the share price is £50, then the option is worthless and we throw it away. If the share price is £200, then by exercising the option and immediately selling the share, the holder will make a profit of £50. So the expected value of the option is $50p$ – to decide what C should be, we'll need an estimate for p. Let's see what happens if we estimate that $p = 0.2$, so that $C = 10$.

Agent A comes along, and asks to sell us a share and buy 3 options at time $t = 0$. With this transaction, we initially hand over

$$100 - 3 \times 10 = 70.$$

An important note here is that Agent A can sell us as many shares as he likes, through *short selling*. He borrows the shares from some third party at time 0, and he will have to return them at time 1. In practice, this means that for every share he sells us now, he will have to buy one back later.

At time 1, if the share price has gone down, the option is worthless. In this case, he can buy one share of the stock for £50, return it to the third party, and leave with

$$70 - 50 = 20$$

in profit.

On the other hand, if the share price has gone up, each option represents a profit of £50. In this case, Agent A will spend £450 to exercise the options; return one share to the third party; and sell the two remaining shares for £400. This time, he leaves with

$$70 - 450 + 400 = 20$$

in profit.

In either case, Agent A has returned his borrowed share and made a £20 profit from us: there is no risk! This is an example of an *arbitrage opportunity*. Since Agent A has effectively limitless money, he can instead sell a million shares, buy three million options, and make £20 million profit - or even more - and ruin us.

In order to calculate the fair price for the option, we should think about Agent A's profit in general terms. Let's consider a portfolio consisting of x units of the stock and y units of the option; here, negative values for x and y represent short selling. The initial cost to Agent A is

$$100x + Cy.$$

At time 1, this portfolio is either worth $50x$ (if the share price goes down), or $200x + 50y$ (if the share price goes up). To eliminate the risk and make the two values equal to each other, we set $y = -3x$.

Overall, this portfolio has a risk-free profit of

$$50x - (100x - 3Cx) = (3C - 50)x.$$

If this is non-zero, Agent A can ruin us! If it's positive, he takes a hugely positive value for x (this is buying the stock and short selling the option). If it's negative, he can take a hugely negative value for x (this corresponds to short selling the stock and buying the option, as in the first part of the example). The only way to avoid arbitrage is to set $C = 50/3$.

A basic underlying principle of derivative pricing is:

> No market should inherently contain arbitrage opportunities, and no financial derivative should be priced in a way that creates an arbitrage opportunity.

In a liquid market, arbitrage opportunities are not stable and never exist for long: investors taking advantage of the opportunity (or *arbitrageurs*) will influence the supply and demand within the market and push the price back towards one that does not create arbitrage.

The principle that no derivative asset should be priced in a way that leads to arbitrage opportunities allows us to conclude that larger payoffs must mean larger prices.

Proposition 1.1. *Consider two self-financing portfolios, Portfolio 1 and Portfolio 2, with costs C_1 and C_2, and time-T payoffs V_1 and V_2. If $V_1 \geq V_2$ is true* whatever happens to the asset prices*, then we must have $C_1 \geq C_2$ or an arbitrage opportunity exists.*

(In other words: if Portfolio 1 is a better bet than Portfolio 2, it can't cost less.)

Proof. We use the golden principle of exploiting arbitrage opportunities: *buy cheap, and (short) sell expensive.*

Suppose $C_2 > C_1$. At time 0, by borrowing the second portfolio, selling it, and using the money to buy the first portfolio, our position is:

- holding Portfolio 1
- short Portfolio 2
- spare cash worth $C_2 - C_1$, which we can use to buy bonds.

Now at time T, we sell Portfolio 1 for V_1, pay V_2 to buy Portfolio 2 and return it to the lender, and have $V_1 - V_2 \geq 0$ left over from the transaction. In combination with the bonds which have been quietly earning interest, we are guaranteed to make a profit: this is an arbitrage opportunity. □

By applying 1.1 in both directions, we obtain the Law of One Price:

Theorem 1.1 (The Law of One Price (LOOP))**.** *If two self-financing portfolios have identical time-T payoffs, whatever happens to the asset prices, then at every time $t \leq T$, they must have the same price or an arbitrage opportunity will exist.*

An example of the Law of One Price is Put-Call Parity:

Theorem 1.2 (Put-Call Parity)**.** *Consider a European call option with cost C, and a European put option with cost P, on the same stock. If the initial share price is S_0, interest is compounded discretely at rate r, and both C and P have strike price K and expiry date T, then we must have*

$$P + S_0 = C + K(1+r)^{-T},$$

or else an arbitrage opportunity exists.

Proof. Consider two portfolios, X and Y. In X, we buy one call option and put $K(1+r)^{-T}$ into bonds, buying $K(1+r)^{-T}$ units, at time 0. The price at time 0 is $C + K(1+r)^{-T}$, and at time T the portfolio will be worth

$$V_T^X = (S_T - K)^+ + K(1+r)^{-T} \times B_T = (S_T - K)^+ + K = \max(S_T, K).$$

In Y, we buy one put option and one share of stock at time 0. The price at time 0 is $P + S_0$, and at time T portfolio Y will be worth

$$V_T^Y = (K - S_T)^+ + S_T = \max(K, S_T).$$

Since both portfolios have the same payoff, they must have the same initial cost by the Law of One Price to avoid an arbitrage opportunity; so

$$C + K(1+r)^{-T} = P + S_0.$$

□

We say that one portfolio **hedges** or **replicates** another if they always have the same payoff. A **complete** market is a market in which every contingent claim can be hedged with a portfolio consisting only of bonds and shares. Using the Law of One Price, we can calculate the fair price for any derivative asset at any time by first finding a hedging portfolio using bonds and shares, and then calculating its fair price – which should be a straightforward calculation.

1.3 Discrete-time model

In this part of the course, we are interested in discrete-time models: those in which trading can take place only at specific times.

A discrete-time model with one risky asset consists of:

1. A set of trading times $0 < t_1 < t_2 < t_3 < \cdots \leq T$. These do not have to be equally-spaced, but usually will be; we will almost always use $t_j = j$ so that trades happen at times $0, 1, 2, 3, \ldots, T$.
2. An outcome space Ω. Each element of Ω represents one possible way in which the share prices might evolve; this is how we will encode the risk or randomness in the model.
3. A risk-free asset, or bonds account, whose *price dynamics* are deterministic and given by

$$B(t) = (1 + r)^t, \qquad 0 \leq t \leq T.$$

4. A risky asset, whose price dynamics are defined for each $\omega \in \Omega$ by

$$S : \omega \mapsto \{S_1(\omega), \ldots, S_T(\omega)\}.$$

   The collection $\{S_1(\omega), \ldots, S_T(\omega)\}$ represents one possible *price path* for the share price.

Notes:

- The outcome space Ω should remind you of probability triples $(\Omega, \mathcal{F}, \mathbb{P})$ from Probability II. We're going to avoid defining a probability measure on Ω for now, but $\mathcal{F}$ will usually be the Borel σ-algebra $\mathcal{F} = 2^\Omega$.
- The interest rate r for the risk-free asset must be unique, to avoid arbitrage inherent in the market. If there were two different risk-free assets, with rates $r_1 < r_2$, then a portfolio in which we short-sell one bond at the lower rate, buying one bond at the higher rate, creates a guaranteed payoff of $(1 + r_2)^T - (1 + r_1)^T > 0$ with no initial investment.
- We view all price paths for the shares as left-continuous functions of time: $S_t = S_{t^-}$.

We make the following assumptions about the market:

a. The market is **liquid** and **divisible**: we can buy or sell any (real) quantity of bonds and of shares at each time t_i.
b. We do not have to pay transaction fees, and transactions happen instantaneously at each trading time.
c. There is no **bid-ask spread**: the prices at which we buy are the same as the prices at which we sell.
d. Shares produce no dividends.
e. Our actions have no impact on the pricing of any products in the market.

The second and third assumptions, together, are known as a **frictionless** market. They represent two of the major stumbling blocks that you will face if you set out to identify and exploit arbitrage opportunities in the real world.

In this course, we usually describe a portfolio as a sequence of *holdings vectors*, rather than in terms of the trades we need to make to achieve those holdings. For example, if we hold x bonds and y units of the stock at time t, we have $h_t = (x, y)$. A portfolio in which we short sell one share at time $t = 1$ and buy it back at time $t = 3$ would therefore look like:

$$h_t = \begin{cases} (0,0) & 0 \leq t < 1 \\ (0,-1) & 1 \leq t < 3 \\ (0,0) & 3 \leq t < T. \end{cases}$$

Note that this portfolio is defined at $t = 0$, but not at $t = T$: we assume that, at time T, we will exercise any contracts that are worth exercising, return anything we borrowed, and sell any remaining assets.

We can view a portfolio as a function $t \mapsto h_t$. It's a step function which is constant on the intervals $[0, 1)$, $[1, 2)$, and so on. In fact, any right-continuous piecewise step function can be seen as a portfolio, as long as the "jumps" happen only at trading times. We could also view a portfolio as a sequence of trades, rather than a sequence of holdings: to recreate the trades involved, we look for values of t for which $h_t - h_{t^-}$ is non-zero.

The **value** of a portfolio at time t is given by the total value of each of the assets held; for a portfolio $P = (h_t; t \in [0,T])$ we write

$$V_t = h_t \cdot (B_t, S_t) = x_t B_t + y_t S_t.$$

The amount of cash required to *initialise* a portfolio is given by V_0 - note that this can be negative, for example if we're short selling. The cash received at time $t = T$ when we *close out* the portfolio is V_T, which can also be negative - for example to buy back shares that were sold earlier.

Example 1.5
If our portfolio X is "sell 100 shares of the stock at time 0", then we can express X as a sequence of holdings by

$$X = (h_t^X, t \in [0,T]) \qquad \text{where} \qquad h_t^X = (0, -100)\ \forall t \in [0,T].$$

The value of X is always $V_t = -100 S_t$.

If our portfolio Y is "buy 100 shares at time 0, and sell them as soon as the share price doubles", then the holdings are given by

$$h_t^Y = \begin{cases} 100 & \sup_{s\in[0,t]}(S_s) < 2S_0 \\ 0 & \sup_{s\in[0,t]}(S_s) \geq 2S_0 \end{cases}.$$

We can write this as $h_t^Y = 100\ \mathbb{1}\left\{\sup_{s\in[0,t]}(S_s) < 2S_0\right\}$. Similarly, the value of the portfolio is given by $V_t = 100 S_t\ \mathbb{1}\left\{\sup_{s\in[0,t]}(S_s) < 2S_0\right\}$.

Question: what are the trades, holdings, and value of the portfolio $X + Y$?

The **cash flow** associated with a portfolio describes how its value changes as each trade occurs: if the portfolio has trades at times $t_1, \ldots, t_k$ and associated changes in value $c_1, \ldots, c_k$, then the cash flow is $\{(c_i, t_i)\}$.

We say that a portfolio is **self-financing** if the cash flow associated with it is trivial; in other words, if $V_t = V_{t^-}$ for every $t \in (0,T]$. Since B_t and S_t are left-continuous, we have

$$\begin{aligned} V_t - V_{t^-} &= x_t B_t + y_t S_t - (x_{t^-} B_{t^-} + y_{t^-} S_{t^-}) \\ &= (x_t - x_{t^-}) B_t + (y_t - y_{t^-}) S_t. \end{aligned}$$

The self-financing condition means that this value should always be 0.

Note that we can find a self-financing version of any portfolio by buying or selling bonds in the appropriate quantities to create a trivial cash flow. Although bonds are not the same as cash in practice, they play the same role in our mathematical model.

The **present value profit** of a self-financing portfolio P with value $V_t, t \in [0,T]$ is given by $\alpha^T V_T - V_0$. Remember that α is the discount factor per time period, so that $\alpha = (1+r)^{-1}$ if interest is compounded discretely at rate r per time period, or $\alpha = e^{-r}$ if interest is compounded continuously.

We say that a portfolio P is an **arbitrage portfolio** if $\alpha^T V_T - V_0 \geq 0$, *whatever happens to* S_t, and it is possible that $\alpha^T V_T - V_0 > 0$.

Alternatively, we can define an arbitrage portfolio as one which satisfies: (i) P is self-financing; (ii) $V_0 = 0$; and (iii) $V_T \geq 0$ whatever happens to the share prices and it is possible that $V_T > 0$.

Lemma 1.1. *These two definitions of an arbitrage portfolio are equivalent, in the sense that a market either allows both to exist, or neither.*

The proof of this Lemma is Exercise 1.14 on the problem sheet.

2 The one-period binomial model

In this chapter, we meet our first, and simplest, example of a discrete-time model: the one-period binomial model.

2.1 The model

The four elements of our model are:

1. A set of trading times. In this model, trades can happen at two times: $t = 0$ (today) and $t = T$ (next year; if you prefer, you could write $T = 1$).
2. An outcome space, $\Omega = \{0, 1\}$.
3. A bond, with price dynamics:
$$B(0) = 1, \qquad B(T) = 1 + r.$$
4. A share, whose price dynamics are:
$$S(0) = s, \qquad S(T)(\omega) = \begin{cases} su & \text{if } \omega = 1 \\ sd & \text{if } \omega = 0. \end{cases}$$

The probability space associated with this model is $(\Omega, \mathcal{F}, \mathbb{P})$, in which $\Omega = \{0, 1\}$, $\mathcal{F} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$, and $\mathbb{P}$ represents a selection from Ω: $\mathbb{P}(\omega = 0) = p = 1 - \mathbb{P}(\omega = 1)$. We will assume that $p \in (0, 1)$, and we will always assume that $u > d$.

To describe a one-period binomial model, we need to know the constants r, s, u, d, and (maybe) p.

2.2 Portfolios and arbitrage

In this market, a portfolio (or a **trading strategy**) is any vector $h = (x, y) \in \mathbb{R}^2$. We interpret the portfolio in the following way:

- buy x bonds in the risk-free asset, and y shares of the stock, at time 0. (If x and/or y is negative, this represents our short selling bonds and/or shares at time 0.)
- sell x bonds and y shares at time T. Remember that we always "close out" our position, that is, we are not holding onto shares for a later date; and if we have short sold any bonds or shares, we must buy them back at time T.

Our divisibility assumption means that any $h \in \mathbb{R}^2$ is a valid trading strategy.

Value and arbitrage

The **value process** of a portfolio h is the process V_t^h, given by

$$V_0^h = x + ys, \qquad V_T^h = \begin{cases} x(1+r) + ysu & \text{if } \omega = 1 \\ x(1+r) + ysd & \text{if } \omega = 0. \end{cases}$$

If h is an arbitrage portfolio, then we must have $x + ys = 0$, and $x(1 + r) + ysu$ and $x(1 + r) + ysd$ must be either both positive or both negative.

Example 2.1
Consider a one-period binomial market with $r = 0.1$, $s = 10$, $u = 1.2$, $d = 1.1$, $p = 0.2$.

We will look at the portfolio $h = (-10, 1)$. We have

$$V_0^h = -10 \times 1 + 1 \times 10 = 0;$$

and

$$V_T^h = -10 \times 1.1 + 1 \times S_T = \begin{cases} 1 & \text{if } \omega = 1 \\ 0 & \text{if } \omega = 0 \end{cases}.$$

We have found an arbitrage portfolio on this market: if we sell 10 bonds (or borrow 10 pounds, if you prefer) and buy one share of the stock at time 0, at worst we regain our money and we have a chance to make a profit.

Example 2.2
Let's change the parameters from the previous example, so that $d = 0.7$ instead of $d = 1.1$. Now S_T is either 1.2 or 0.7, so that the value process of the portfolio $h = (-10, 1)$ becomes

$$V_0^h = 0,$$
$$V_T^h = -10 \times 1.1 + 1 \times S_T = \begin{cases} 1 & \text{if } \omega = 1 \\ -4 & \text{if } \omega = 0 \end{cases}.$$

This is no longer an arbitrage portfolio, as we can now lose money through it.

Theorem 2.1. *There is no arbitrage inherent in the one-period binomial model, if and only if we have*

$$d < 1 + r < u. \tag{2.1}$$

Proof. First, we suppose that Equation (2.1) is true, and look for an arbitrage portfolio $h = (x, y)$. Since

$$V_0^h = x + sy = 0,$$

we must have $x = -sy$.

Now,

$$V_T^h = \begin{cases} sy(u - (1 + r)) & \text{if } \omega = 1 \\ sy(d - (1 + r)) & \text{if } \omega = 0 \end{cases}.$$

By Equation (2.1), $u-(1+r)$ must be positive and $d-(1+r)$ must be negative, so we can only have $\mathbb{P}(V_t^h \geq 0) = 1$ if $y = 0$. But then, $\mathbb{P}(V_t^h > 0) = 0$ and so no arbitrage portfolio exists.

Now, we suppose that one of the inequalities in Equation (2.1) does not hold; for instance, we have $(1 + r) \geq u$. Then we have $s(1 + r) \geq su > sd$ (Remember that $u > d$ is always true.)

Consider the portfolio $h = (s, -1)$: we sell one share short and invest all the money in the bond. We have $V_0^h = 0$, and

$$V_T^h s(1 + r) - sZ = \begin{cases} s(1 + r - u) & \text{if } \omega = 1 \\ s(1 + r - d) & \text{if } \omega = 0 \end{cases}.$$

So $V_T^h \geq 0$ when $\omega = 1$, and $V_T^h > 0$ when $\omega = 0$; we have $\mathbb{P}(V_T^h \geq 0) = 1$ and $\mathbb{P}(V_T^h > 0) > 0$, so we have found an arbitrage portfolio.

We can do a similar calculation if $1 + r \leq d$ (try it!) so that

- if Equation (2.1) holds, there can be no arbitrage portfolio
- if Equation (2.1) is broken in any way, an arbitrage portfolio must exist.

□

2.3 Contingent claims

Remember that a **contingent claim** is a contract between the buyer and the seller, in which the seller promises the random payoff Φ to the buyer at time T.

Mathematically, any random variable X can represent a contingent claim if we can find a **contract function** Φ such that $X = \Phi(\{S_t : t \in [0, T]\})$.

Example 2.3

Here are some examples of the contract functions for contingent claims we've already seen.

- In a European call option, we have $X = \Phi_{\text{call}}(S_T) = (S_T - K)^+$

- If the contingent claim is "one unit of every asset on the market", then $X = \Sigma_{j=1}^m S_T^j$.
- A **forward contract** on S is a contract in which the the asset is to be sold at a strike price K at expiry time T, and both buyer and seller are *obliged* to complete the transaction. In this case, the contract function is $\Phi_F(x) = x - K$, and $X = \Phi(S_T) = S_T - K$.
- In general, in European-style claims Φ only depends on the price at time T, S_T, but in general it can depend on the values of S_t at any (or all) times $t \in [0, T]$.

We say that a contingent claim is **reachable** if there exists a portfolio h consisting only of bonds and shares, such that

$$\mathbb{P}(V_T^h = X) = 1.$$

If every contingent claim X is reachable, we say that the market is **complete**.

Theorem 2.2. *If $u > d$, the one-period binomial model is complete.*

Proof. For any claim X with contract function Φ, we need to show that there exists a portfolio $h = (x, y)$ with

$$V_T^h = \begin{cases} \Phi(su) & \text{if } \omega = 1 \\ \Phi(sd) & \text{if } \omega = 0 \end{cases}.$$

In other words, we want to find a solution to the system

$$(1 + r)x + suy = \Phi(su)(1 + r)x + sdy = \Phi(sd).$$

If $u > d$, this system has a unique solution, namely,

$$x = \frac{1}{1+r}\frac{u\Phi(sd) - d\Phi(su)}{u - d} \tag{2.2}$$

$$y = \frac{\Phi(su) - \Phi(sd)}{s(u - d)}. \tag{2.3}$$

□

Question: what is a fair price for a contingent claim X?

At $t = T$, this is an easy problem to solve: we know how to calculate its value, by finding X using Φ. Writing $\Pi(X, t)$ for the price of X at time t, we must have $\Pi(X, T) = X$.

To calculate the fair price at time 0, we use the following *pricing principle*:

If a claim X is reachable with replication portfolio h, then the only reasonable price process for X is

$$\Pi(X, t) = V_t^h, \quad t = 0, T.$$

Theorem 2.3. *Suppose that a claim X is reachable with replication portfolio h, and that at time $t = 0$ the price of X is different from V_0^h. Then there is an opportunity for arbitrage.*

Proof. (This is an application of the Law of One Price, Theorem 1.1.)

Let $\Pi(X, 0)$ be the price of the claim, and consider the case $\Pi(X, 0) > V_0^h$. At time 0, we can short sell the claim, buy the portfolio h, and be left with $\Pi(X, 0) - V_0^h$, which we deposit in bonds.

Now at time T, the portfolio value V_T^h will exactly cover the claim we sold short, and we have a risk-free profit coming from our bonds, which are now worth $(\Pi(X, 0) - V_0^h)(1 + r)$.

In the case $\Pi(X, 0) < V_0^h$, we can follow a similar argument, this time by short selling the portfolio and buying the claim. □

Example 2.4
In the market from Example 2.2, let's find the fair price for a European call option with strike price $K = 9$ and maturity date T. The corresponding contingent claim is $X = (S_T - 9)^+$.

A hedging portfolio for X is a portfolio $h = (x, y)$ such that

$$(1+r)x + suy = (su - 9)^+, \qquad (1+r)x + sdy = (sd - 9)^+;$$

in other words,

$$1.1x + 12y = 3, \qquad 1.1x + 7y = 0.$$

The solution here is $y = \frac{3}{5}$, $x = -\frac{42}{11}$, so the value of this portfolio at time t is

$$V_0 = x + ys = \frac{-42}{11} + \frac{3}{5} \times 10 = \frac{24}{11}.$$

We conclude that the fair price for this call option is $\frac{24}{11}$.

2.4 The martingale measure

We say that a process (X_0, X_T) is a **martingale** under a measure $\mathbb{Q}$ if

$$\mathbb{E}_{\mathbb{Q}}[X_T] = X_0.$$

(We will see a more precise definition of martingales in more general contexts, later in the term.)

In this section, we'll look at the conditions we need to place on the model to guarantee the existence of such a measure for the process $(S_0, \alpha S_T)$ and determine what that measure must be.

Since $\alpha = \frac{1}{1+r}$ and $S_0 = s$, we are really looking for a measure under which

$$s = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[S_T].$$

If we write

$$\mathbb{Q}(Z = u) = q_u$$
$$\mathbb{Q}(Z = d) = q_d,$$

our martingale condition becomes

$$\frac{1}{1+r}(suq_u + sdq_d) = s,$$

and our question is really: under which conditions do there exist q_u and q_d which solve

$$uq_u + dq_d = 1 + r \tag{2.4}$$
$$q_u + q_d = 1 \tag{2.5}$$
$$0 < q_u, q_d < 1? \tag{2.6}$$

Surprisingly, we have already met such conditions in Equation (2.1).

Theorem 2.4. *The financial market $\mathcal{M} = (B_t, S_t)$ is arbitrage free if and only if there exists a martingale measure $\mathbb{Q}$.*

Proof. From Theorem 2.1, we know that the market is arbitrage free if and only if

$$d < 1 + r < u.$$

To see that Equations (2.5) have a solution if and only if $d < 1 + r < u$, we could use some convex analysis: this is exactly the condition under which $1 + r$ can be written as a convex combination of u and d.

Otherwise, we can look for the solutions directly. Solving the system of linear equations, we get

$$q_u = \frac{(1+r)-d}{u-d} \tag{2.7}$$

$$q_d = \frac{u-(1+r)}{u-d}. \tag{2.8}$$

Both q_u and q_d are positive if and only if the no-arbitrage condition holds. □

Theorem 2.4 is a version of the **first fundamental theorem of asset pricing**, which we will see in full detail later; we can also prove a simple version of the **second fundamental theorem of asset pricing**.

Theorem 2.5. *Suppose the financial market is arbitrage free. Then it is complete if and only if there is a* unique *martingale measure.*

Proof. If the market is arbitrage free, we must have $d < 1+r < u$, and in particular $d < u$. By Theorem 2.2, the market must be complete, and by Theorem 2.4 we know that a martingale measure exists.

To see that it is unique, suppose we have found two martingale measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$, such that

$$\frac{1}{1+r}\mathbb{E}_{\mathbb{Q}_i}[S_T] = S_0$$

holds for each of $i = 1, 2$. In other words, we have found q_1, q_2 which both satisfy

$$q_i u + (1-q_i)d = 1+r.$$

The only way q_1 and q_2 can be different is if $u = 1+r = d$, which contradicts our no-arbitrage assumption. □

Example 2.5

Using the market $\mathcal{M}$ from Example 2.2, with parameters $r = 0.1$, $s = 10$, $u = 1.2$, and $d = 0.7$, we can check that $d < 1+r < u$, so the market is arbitrage free; it is complete, so there exists a unique martingale measure. We have

$$q_u = \frac{1.1-0.7}{1.2-0.7} = \frac{4}{5}, \qquad q_d = \frac{1.2-1.1}{1.2-0.7} = \frac{1}{5},$$

and we can check that

$$\frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[S_T] = \frac{1}{1.1}\left[\frac{4}{5}\times 12 + \frac{1}{5}\times 7\right] = 10 = S_0.$$

2.5 Risk neutral valuation

Now that we know whether binomial market is complete, we can price any contingent claim. As we saw in Theorem 2.2, if $u > d$ any contingent claim X with contract function Φ is reachable, with hedging portfolio $h = (x, y)$ given by

$$x = \frac{1}{1+r}\frac{u\Phi(sd) - d\Phi(su)}{u-d} \tag{2.9}$$

$$y = \frac{\Phi(su) - \Phi(sd)}{s(u-d)}. \tag{2.10}$$

The price at time $t = 0$ of this portfolio is given by

$$\begin{aligned}
\Pi(X, 0) &= V_0^h \\
&= x + sy \\
&= \frac{1}{1+r}\left(\frac{u\Phi(sd) - d\Phi(su)}{u-d} + (1+r)\frac{\Phi(su) - \Phi(sd)}{u-d}\right) \\
&= \frac{1}{1+r}\left(\Phi(su)q_u + \Phi(sd)q_d\right),
\end{aligned}$$

where q_u and q_d are exactly the probabilities coming from the martingale measure $\mathbb{Q}$! We can interpret this as an expectation under $\mathbb{Q}$, to get the following pricing formula.

Theorem 2.6. *If the one-period binomial model is free of arbitrage, then the arbitrage-free price of a contingent claim X at time $t = 0$ is given by the* ***risk neutral valuation formula***

$$\Pi(X, 0) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[X],$$

where $\mathbb{Q}$ is the martingale measure (or ***risk-neutral measure****) uniquely determined by the relation*

$$\frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[S_T] = S_0$$

or, equivalently, given explicitly in Equations (2.8)*. Furthermore, the claim can be replicated using the portfolio set out in Equations* (2.10)*.*

Example 2.6
Using our same market from the other examples in this chapter, let's price the call option with strike price $K = 9$. We have $q_u = \frac{4}{5}$ and $q_d = \frac{1}{5}$, so by the risk-neutral valuation formula,

$$\begin{aligned}\Pi(X, 0) &= \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[(S_T - 9)^+] \\ &= \frac{1}{1.1}(3 \times q_u + 0 \times q_d) \\ &= \frac{1}{1.1} \times 3 \times \frac{4}{5} = \frac{24}{11}.\end{aligned}$$

2.6 Some generalisations

We can extend the one-period binomial model slightly, and the first and second fundamental theorems of asset pricing will still hold. Here are two (non-examinable) examples.

What happens when $u = d$

If $u = d$, both assets offer guaranteed *rates of payoff* per unit of investment: we have

$$\frac{B_T}{B_0} = 1 + r \qquad \text{and} \qquad \frac{S_T}{S_0} = u = d.$$

If these rates are not the same, there will be arbitrage in the market, with an arbitrage portfolio formed by buying the asset with the higher rate of payoff, and selling the asset with the lower rate. Meanwhile, we can slightly modify the argument from the proof of Theorem 2.1 to show that if $u = d = 1 + r$, there is no arbitrage portfolio on the market. Hence the no-arbitrage condition becomes $u = d = 1 + r$.

For any portfolio h, $V_T^h = xB_T + yS_T$ has a fixed value; so the only contingent claims X which are reachable are those which are also constant. Any contingent claim whose final value is genuinely random cannot be reached in this market: the market is **not** complete. On the other hand, any claim of the form $X = \Phi(S_T)$ *is* still reachable.

For *any* probability measure $\mathbb{Q} = (q_u, q_d)$, we have

$$uq_u + dq_d = u = d,$$

so martingale measures exist precisely when $u = d = 1 + r$. This means that the first fundamental theorem still holds: we have no arbitrage, if and only if $u = d = 1 + r$, if and only if there exists a martingale measure.

Finally, if any martingale measure exists, we must have **multiple** martingale measures. However, as we have observed, the market is not complete, so the second fundamental theorem also holds in this setting.

Many assets, and/or many outcomes, at time T

Finally, while keeping a single time period (that is, trades at time $t = 0$ and $t = T$ only), we can consider markets with more than 2 assets and/or more than 2 possible states of the market at time T.

Suppose the market has m assets, whose prices at time 0 are given by the deterministic vector $S_0 = (S_0^1, \dots, S_0^m)$, and that at time T the price vector S_T takes one of n possible values. In other words, there are n vectors $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$ and n positive probabilities $p_1, \dots, p_n$ such that

$$\mathbb{P}(S_T = \mathbf{a}_j) = p_j, \qquad j = 1, \dots, n.$$

These vectors also define an $m \times n$ matrix A, in which the ith row lists the possible values for the price of the ith asset at time T, and the jth column is the vector $\mathbf{a}_j$.

Arbitrage on this market is defined in the same way as on the simple market: any portfolio $h = (x_1, \dots, x_m) \in \mathbb{R}^m$ is an arbitrage portfolio if it satisfies (i) $V_0^h = 0$; (ii) $\mathbb{P}(V_T^h \geq 0) = 1$; and (iii) $\mathbb{P}(V_T^h > 0) > 0$. The value of h at time T, V_T^h, is random, and takes one of the n values in the row vector hA. So finding an arbitrage portfolio is equivalent to finding a vector $h \in \mathbb{R}^m$ such that $h \cdot S_0 = 0$, and hA has all non-negative entries, with at least one strictly positive.

To define a martingale measure on this market, we look for a measure $\mathbb{Q} = (q_1, \dots, q_n)$ such that, for each $i = 1, \dots, m$, $S_0^i = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[S_T^i]$. In terms of the matrix A, a martingale measure is equivalent to a vector $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ such that $S_0 = \frac{1}{1+r} qA^T$, and such that each $q_j \in (0, 1)$.

The first and second fundamental theorems of asset pricing both hold here. The first is a consequence of a result known as Farkas' Lemma, concerning the solvability of systems of linear equalities: it tell us, roughly speaking, that given the system of equations defining h, and the system of equations defining q, exactly one of these has a solution. For the second, we can relate the completeness of the market to the (row) rank of A being equal to n, while the uniqueness of a solution to $S_0 = \frac{1}{1+r} qA^T$ corresponds to the nullity of A being 0. By the rank-nullity theorem, the first situation holds if and only if the second does too.

3 The multi-period binomial model: Part 1

3.1 Introduction

In this chapter, we extend the one-period binomial model to allow for more trading times. Our model becomes:

1. A set of trading times, $\{0, 1, 2, \dots, T\}$.
2. An outcome space, $\Omega = \{0, 1\}^T$.
3. A bond, with price dynamics:
$$B(0) = 1, \qquad\qquad B(t) = (1 + r)^t.$$
4. A share, whose price dynamics are:
$$S_0(\omega) = s, \quad \text{for all } \omega \in \Omega,$$
and for each $t = 1, 2, \dots, T$,
$$S_t(\omega) = Z_t(\omega) S_{t-1}(\omega),$$
where $Z_1, Z_2, \dots, Z_T$ are random variables defined on Ω, with
$$Z_t(\omega) = \begin{cases} u & \text{if } \omega_t = 1, \\ d & \text{if } \omega_t = 0. \end{cases}$$

Notes:

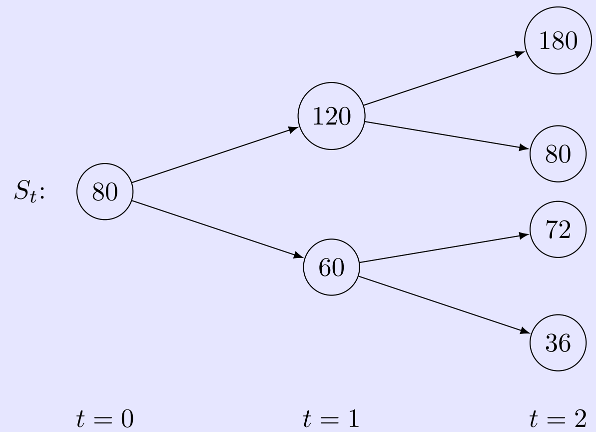
- In this model, Ω is the set of all sequences of length T, in which each element is a 0 or a 1. The sequence 00…0 corresponds to the share price moving "down" T times, while the sequence 11…1 corresponds to the share price moving "up" T times. When we discuss an element $\omega \in \Omega$, we will sometimes write $\omega = \omega_1 \omega_2 \dots \omega_T$.
- The Borel σ-algebra on Ω is $\mathcal{F}_T = 2^\Omega$ and consists of all possible subsets of the 2^T elements of Ω. For instance $\{\omega \in \Omega : \omega_1 = 0\} \in \mathcal{F}_T$. More on this later.

- As before, the price dynamics of the risk-free asset evolve deterministically via the risk-free interest rate.
- We assume that $S_0 > 0$ is fixed, and that if we know the share price is S_t at time t, there are two possible values for the share price at time $t+1$: $S_t \times u$ and $S_t \times d$. Although we will sometimes work with models in which u and d take the same values for all t and s, this is not necessarily the case, as you will see in Example 3.1.

Example 3.1

Consider a two-period model with the possible share prices $S_t, t = 0, 1, 2$, given by the tree:

A tree diagram, with prices 120 and 60 at time 1, and 180, 80, 72, and 36 at time 2.



- At $t = 0$, we have

$$u = \frac{120}{80} = \frac{3}{2} \qquad \text{and} \qquad d = \frac{60}{80} = \frac{3}{4}.$$

- At $t = 1$, when $S_1 = 120$, we have

$$u = \frac{180}{120} = \frac{3}{2} \qquad \text{and} \qquad d = \frac{80}{120} = \frac{2}{3}.$$

- At $t = 1$, when $S_1 = 60$, we have

$$u = \frac{72}{60} = \frac{6}{5} \qquad \text{and} \qquad d = \frac{36}{60} = \frac{3}{5}.$$

As in the one-period version of the model, the values of p_u and p_d do not matter to us, as long as $0 < p_u, p_d < 1$. For this reason, we do not ask whether p_u always takes the same value regardless of t and s; both possibilities are entirely reasonable.

The multi-period binomial model is free from inherent arbitrage if and only if there is no inherent arbitrage to be found by considering any of its constituent parts. In other words, it is free from inherent arbitrage as long as

$$d < 1 + r < u$$

holds everywhere in the model.

A portfolio on a multi-period market is now more complicated to describe, since we need to specify the amounts we hold of each asset at each time t, and this is allowed to depend on how the share price changes over time. However, we can still price contingent claims in this model, by using the risk-neutral valuation formula to find the no-arbitrage price at each node of the tree.

3.2 Portfolios and the self-financing condition

In this market, a portfolio is a sequence of pairs of random variables $h_t = (x_t, y_t)$, $t = 1, \dots, T$. Each pair is interpreted as follows:

- x_t is the number of units of the risk-free asset in your portfolio at time $t-1$ and kept until time t.
- y_t is the number of units of the risky asset in your portfolio at time $t-1$ and kept until time t.

The amounts x_t and y_t can depend on the present and past history $S_0, S_1, \dots, S_{t-1}$ of the stock prices; it is reasonable to assume that we can find functions f_x and f_y such that

$$x_t = f_x(S_0, \dots, S_{t-1}) \qquad y_t = f_y(S_0, \dots, S_{t-1}).$$

Example 3.2
Consider a market with $T = 2$, in which the time-0 share price is 10, at time 1 the share price will either be 15 or 8, and at time 2 the share price will be 18, 12, or 6. For this example, we use $r = 0$.

Our portfolio is "buy 20 shares at time 0; at time 1, sell half (and buy bonds) if the price goes up, or sell all of them if the price goes down".

In either case, we short sell $20 \times 10 = 200$ bonds to buy the initial shares; at time 1, we use the cash from our sale to buy either $10 \times 15 = 150$ or $20 \times 8 = 160$ bonds.

To express this portfolio as a sequence of vectors, we write:

$$\text{at time 0,} \qquad (x_1, y_1) = (-200, 20)$$
$$\text{at time 1,} \qquad (x_2, y_2) = \begin{cases} (-50, 10) & \text{if } \omega_1 = 1 \\ (-40, 0) & \text{if } \omega_1 = 0. \end{cases}$$

The value of a portfolio at time $t = 0, \dots, T$ is given by

$$V_t = x_{t+1} B_t + y_{t+1} S_t,$$

with the convention that $x_{T+1} = x_T$ and $y_{T+1} = y_T$. In other words, this is the value of the portfolio *after* the trade at time t, and *before* the next price change.

Example 3.3
In the market in the previous example, if $\omega_1 = 1$ (so the share price increases to 15 at time 1), then the value of the portfolio at time 1 is

$$V_1^h = x_2 B_1 + y_2 S_1 = -50 + 10 \times 15 = 100.$$

Between times $t-1$ and t, the values of the bond and the stock change from B_{t-1} and S_{t-1} to B_t and S_t, so that the portfolio is now worth $x_t B_t + y_t S_t$; we adjust our holdings according to the price changes, so that we now hold (x_{t+1}, y_{t+1}). This adjustment is **self-financing** if

$$x_t B_t + y_t S_t = x_{t+1} B_t + y_{t+1} S_t. \tag{3.1}$$

If (3.1) holds for all $t = 1, \dots, T$, we say the portfolio is **self-financing**. This condition basically means that you cannot take out money from the portfolio or put in money to the portfolio: it should finance itself.

Remark. Using the expression for the value V_t of the portfolio at time t, the self-financing constraint (3.1) can be rewritten as the condition

$$V_t - V_{t-1} = x_t(B_t - B_{t-1}) + y_t(S_t - S_{t-1}). \tag{3.2}$$

The financial meaning of this is that the change in the value $V_t - V_{t-1}$ of a self-financing trading strategy is due only to the gains/losses

$$x_t(B_t - B_{t-1}) + y_t(S_t - S_{t-1})$$

due to price changes of the assets. Summing both sides of (3.2) over t we obtain

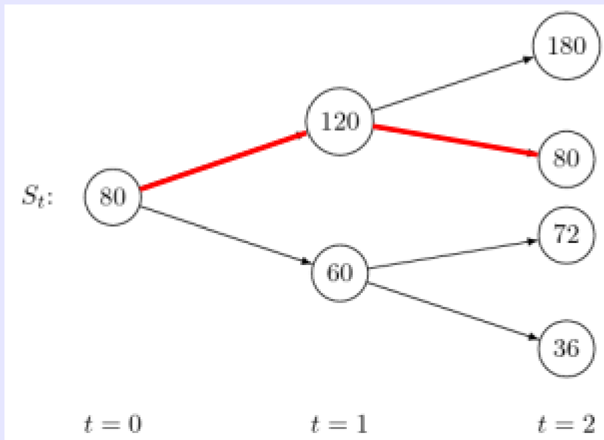
$$V_T - V_0 = \sum_{t=1}^{T} (V_t - V_{t-1}) = \sum_{t=1}^{T} x_t [B_t - B_{t-1}] + \sum_{t=1}^{T} y_t [S_t - S_{t-1}].$$

You will see this form when we start to define stochastic integration and self-financing conditions in continuous time.

3.3 The martingale measure

To construct the martingale measure in this model, we take the *product measure* based on the martingale measure found in Chapter 2. To find $\mathbb{Q}(\omega)$, we trace out the path represented by ω in the pricing tree. We calculate the one-step martingale measures at each node, and multiply them together.

Example 3.4
In the market from Example 3.1, the choice $\omega = 10$ represents the share price increasing in the first instance, and decreasing in the second:



At $t = 0$, we have

$$q_u = \frac{1 - \frac{3}{4}}{\frac{3}{2} - \frac{3}{4}} = \frac{1}{3} \qquad \text{and} \qquad q_d = \frac{\frac{3}{2} - 1}{\frac{3}{2} - \frac{3}{4}} = \frac{2}{3}.$$

At $t = 1$ and $S_1 = 120$, we have

$$q_u = \frac{1 - \frac{2}{3}}{\frac{3}{2} - \frac{2}{3}} = \frac{2}{5} \qquad \text{and} \qquad q_d = \frac{\frac{3}{2} - 1}{\frac{3}{2} - \frac{2}{3}} = \frac{3}{5}.$$

So

$$\mathbb{Q}(\omega = 10) = \frac{1}{3} \times \frac{3}{5} = \frac{1}{5}.$$

Next, for $A \in \mathcal{F}$,we define $\mathbb{Q}(A)$ using the elements $\omega \in A$:

$$\mathbb{Q}(A) = \sum_{\omega \in A} \mathbb{Q}(\omega).$$

Exercise 3.1

"Prove" (convince yourself) that

Suggestion: in the model with $T = 2$, calculate $\mathbb{Q}(11)$ and $\mathbb{Q}(10)$ and check that they add up to q_u. Do the same calculation for $T = 3$, and then think about the general case.

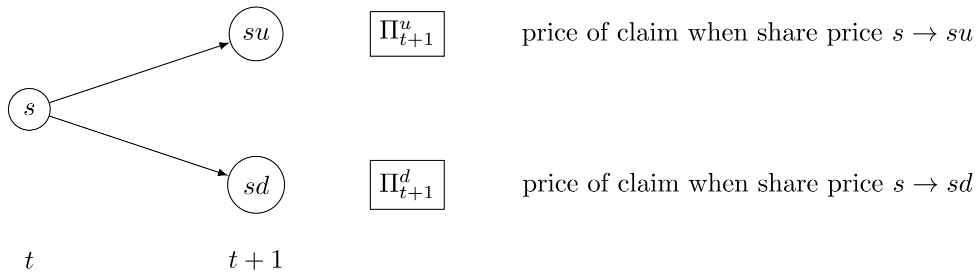
3.4 Pricing European contingent claims

3.4.1 Pricing contingent claims via the risk-neutral valuation formula

We work from $t = T - 1$ to $t = 0$, using backwards induction.

We denote the price of a contingent claim $X = \Phi(S_T)$, evaluated at time t, by $\Pi(X,t)$. We will also sometimes use the notation $\Pi_t(\Phi)$.

Let's focus on a single node of the tree. Suppose that the share price at time t is $S_t = s$, and the two possible values for S_{t+1} are su and sd. We also suppose that we know the two possible values for the fair price for the contingent claim at time $t+1$. We denote these Π^u_{t+1} and Π^d_{t+1} .



Using the same approach as in Chapter 2, we can find the martingale probabilities:

$$q_u = \frac{1+r-d}{u-d} \qquad q_d = \frac{u-(1+r)}{u-d},$$

and then the price Π_t at time t is given by

$$\Pi_t = \frac{1}{1+r}\left(q_u \Pi^u_{t+1} + q_d \Pi^d_{t+1}\right).$$

Working through all the nodes in this way, we can eventually find Π_0 as a linear combination of Π^u_1 and Π^d_1.

Example 3.5

For the market in Example **??**, with $r = 0$, let's calculate the fair price of a European call option with expiry date $T = 2$ and strike price $K = 70$.

First, we find the martingale measure:

- at $t = 0$, we have
$$q_u = \frac{1-\frac{3}{4}}{\frac{3}{2}-\frac{3}{4}} = \frac{1}{3} \qquad \text{and} \qquad q_d = \frac{\frac{3}{2}-1}{\frac{3}{2}-\frac{3}{4}} = \frac{2}{3}.$$
- at $t = 1$ and $S_1 = 120$, we have
$$q_u = \frac{1-\frac{2}{3}}{\frac{3}{2}-\frac{2}{3}} = \frac{2}{5} \qquad \text{and} \qquad q_d = \frac{\frac{3}{2}-1}{\frac{3}{2}-\frac{2}{3}} = \frac{3}{5}.$$
- at $t = 1$ and $S_1 = 60$, we have
$$q_u = \frac{1-\frac{3}{5}}{\frac{6}{5}-\frac{3}{5}} = \frac{2}{3} \qquad \text{and} \qquad q_d = \frac{\frac{6}{5}-1}{\frac{6}{5}-\frac{3}{5}} = \frac{1}{3}.$$

Now, we work backwards to find the price of the option at each node. When $t = 2$, we know that $\Pi_2 = (S_2 - 70)^+$, so that the option is worth $110, 10, 2$, or 0, when S_2 equals $180, 80, 72$, and 36, respectively.

Now when $t = 1$ and $S_1 = 120$, we have

$$\begin{aligned}\Pi_1 &= \frac{1}{1+r}\left(q_u \Pi^u_2 + q_d \Pi^d_2\right) \\ &= \frac{1}{1+0}\left(\frac{2}{5} \times 110 + \frac{3}{5} \times 10\right) \\ &= 50.\end{aligned}$$

When $t = 1$ and $S_1 = 60$, we have

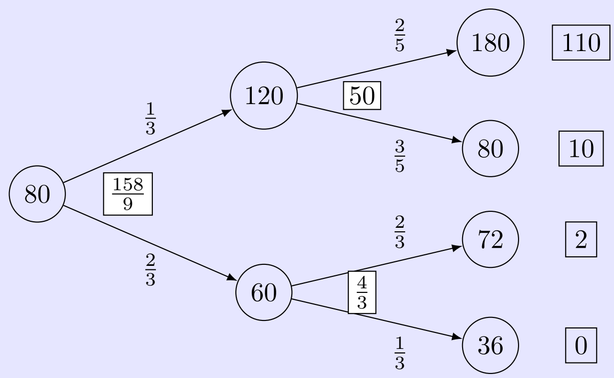
$$\begin{aligned}\Pi_1 &= \frac{1}{1+r}\left(q_u \Pi_2^u + q_d \Pi_2^d\right)\\ &= \frac{1}{1+0}\left(\frac{2}{3}\times 2 + \frac{1}{3}\times 0\right)\\ &= \frac{4}{3}.\end{aligned}$$

Finally, we $t = 0$, we have

$$\begin{aligned}\Pi_0 &= \frac{1}{1+r}\left(q_u \Pi_1^u + q_d \Pi_1^d\right)\\ &= \frac{1}{1+0}\left(\frac{1}{3}\times 50 + \frac{2}{3}\times \frac{4}{3}\right)\\ &= \frac{158}{9};\end{aligned}$$

so the no-arbitrage price of the call option is $\frac{158}{9}$.

These calculations are often recorded in a single tree: we write the share prices inside the nodes, the martingal probabilities along the arrows, and the prices of the claim next to each node in rectangles.



Once we have calculated the martingale probabilities at each node, the martingale measure for S_T is the product measure $\mathbb{Q}$, which we obtain by multiplying the probabilities along each arrow. For instance, in Example **??**, $\mathbb{Q}(S_T = 180) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$. Under this measure, we have

$$S_0 = \frac{1}{(1+r)^T}\mathbb{E}_{\mathbb{Q}}[S_T],$$

and we can view the initial price $\Pi_0(\Phi)$ as the present value of the expectation of $\Phi(S_T)$:

$$\Pi_0(\Phi) = \frac{1}{(1+r)^T}\mathbb{E}_{\mathbb{Q}}[\Phi(S_T)].$$

Warning: of course you already know this, but $\mathbb{E}_{\mathbb{Q}}[\Phi(S_T)]$ is not the same thing as $\Phi(\mathbb{E}_{\mathbb{Q}}[S_T])$.

Example 3.6
Consider the market as described in Examples **??** and **??**. Using the martingale probabilities on each arrow, we can calculate the martingale measure $\mathbb{Q}$ for S_2 (here $T = 2$). For example, the stock price S_2 equals 180 exactly when the price S_t follows the path $80 \to 120 \to 180$ through the tree, so $\mathbb{Q}(S_2 = 180) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$. There are a total of 4 possible values that S_2 can take: 180, 80, 72 or 36, and these have probabilities $\frac{2}{15}$, $\frac{1}{5}$, $\frac{4}{9}$ and $\frac{2}{9}$.

The expectation of S_2 with respect to this measure is

$$\mathbb{E}_{\mathbb{Q}}[S_2] = \frac{2}{15} \times 180 + \frac{1}{5} \times 80 + \frac{4}{9} \times 72 + \frac{2}{9} \times 36 = 80.$$

As expected this equals S_0, since in this example $r = 0$.

Now, to find the price Π_0 at time 0 of the call option with strike price 70 and expiry date $T = 2$, we calculate the present value of the expectation under $\mathbb{Q}$ of the value at time T of the call option, $(S_2 - 70)^+$. We find that

$$\mathbb{E}_{\mathbb{Q}}[(S_2 - 70)^+] = \frac{2}{15} \times 110 + \frac{1}{5} \times 10 + \frac{4}{9} \times 2 + \frac{2}{9} \times 0 = \frac{158}{9},$$

which agrees with our previous calculation.

3.4.2 Pricing contingent claims by hedging

For now, we will focus on European claims – that is, claims whose contract functions are of the form $X = \Phi(S_T)$. In Chapter 5, we will extend our theory to include claims in which the payoff can depend on the past history of the stock price.

Theorem 3.1. *For a contingent claim X, suppose that there exists a* ***self-financing*** *trading strategy $h_t = (x_t, y_t)$, $t = 1, 2, \dots, T+1$ whose value process V_t^h, $t = 0, \dots, T$ satisfies*

$$\mathbb{P}(V_T^h = X) = 1.$$

Then for each $t = 0, 1, \dots, T$, we must have

$$\Pi_t(\Phi) = V_t^h,$$

or else the market will contain arbitrage.

Proof. We first suppose that $V_t^h < \Pi_t(\Phi)$ holds for some t. Then, at this time, we can (short) sell the contingent claim and use the money to buy the portfolio, using the remaining $\Pi_t(\Phi) - V_t^h$ to buy bonds.

Now at time T, the value of our short-sold contingent claim is exactly covered by our portfolio, and we have a risk-free profit of $[\Pi_t(\Phi) - V_t](1+r)^{T-t}$.

Similarly, if $V_t^h > \Pi_t(\Phi)$, we can buy the contingent claim and sell the portfolio, and obtain a risk-free profit of $[V_t - \Pi_t(\Phi)](1+r)^{T-t}$. □

We can now construct the hedging portfolio for any claim, by backwards induction from $t = T$. We know the price of the claim at all nodes at time T; to move backwards in time by one step, we need to find x_t and y_t such that

$$x_{t-1}B_{t-1} + y_{t-1}S_{t-1} = V_{t-1} = x_t B_{t-1} + y_t S_{t-1}$$

holds, whichever node we reach at time t. The two possiblities give us two equations in two unknowns:

$$\begin{aligned} x_t(1+r)B_{t-1} + y_t S_{t-1} u &= \Pi_t^u, \\ x_t(1+r)B_{t-1} + y_t S_{t-1} d &= \Pi_t^d, \end{aligned}$$

which we can solve to get

$$x_t = \frac{1}{(1+r)B_{t-1}} \frac{u\Pi_t^d - d\Pi_t^u}{u-d}, \quad y_t = \frac{\Pi_t^u - \Pi_t^d}{S_{t-1}u - S_{t-1}d}. \tag{3.3}$$

We can check that

$$\begin{aligned} \Pi_{t-1} = V_{t-1} = x_t B_{t-1} + y_t S_{t-1} &= \frac{1}{1+r} \frac{u\Pi_t^d - d\Pi_t^u}{u-d} + \frac{\Pi_t^u - \Pi_t^d}{u-d} \\ &= \frac{1}{1+r}[q_u \Pi_t^u + q_d \Pi_t^d], \end{aligned} \tag{3.4}$$

recalling that the martingale probabilities are $q_u = \dfrac{1+r-d}{u-d}, q_d = \dfrac{u-(1+r)}{u-d}$.

We can repeat these calculations at every node at time $t-1$ (remembering that the values of u and d might be different at different nodes), to find the price $V_{t-1} = \Pi_{t-1}(\Phi)$ and the portfolio (x_t, y_t) at every node at time $t-1$.

So, starting at time T, we use Equations (3.3) to find (x_T, y_T) at every node at time $T-1$ so that the value of the portfolio matches the prices $\Pi_T(\Phi)$ at time T. We calculate the value of this portfolio V_{T-1}, which then must be equal to the price of the claim $\Pi_{T-1}(\Phi)$ at time $T-1$. Then, using our inductive step for $t = T-1, T-2, \ldots, 1$, we can find the price of the claim and the portfolio at every node of the tree.

Remark. Just as in the one-period case, we can use the martingale measure to calculate the fair price for the claim **without** finding the hedging portfolio, through Equation (3.4).

Example 3.7
Let's calculate the hedging portfolio for the call option from Example **??**. We will use the fair prices of the call option at each time, that we calculated earlier.

We start at time $t = 1$, with the case $S_1 = 120$. We need to find (x_2, y_2) such that the value of the portfolio matches the fair price of the option (either 110 or 10) at time 2. Since $r = 0$, we have $B_t = 1$ for all t and Equations (3.3) become

$$x_2 = \frac{\frac{3}{2} \times 10 - \frac{2}{3} \times 110}{\frac{3}{2} - \frac{2}{3}} = -70, \quad y_2 = \frac{110 - 10}{180 - 80} = 1,$$

and we have

$$V_1 = x_2 + 120y_2 = -70 + 120 \times 1 = 50,$$

which matches the fair price for the claim we found earlier.

If instead, $S_1 = 60$, then our (x_2, y_2) will need to give values of either 2 or 0 at time 2. Now Equations (3.3) give

$$x_2 = \frac{\frac{6}{5} \times 0 - \frac{3}{5} \times 2}{\frac{6}{5} - \frac{3}{5}} = -2, \quad y_2 = \frac{2 - 0}{72 - 36} = \frac{1}{18},$$

and

$$V_1 = x_2 + 60y_2 = -2 + 60 \times \frac{1}{18} = \frac{4}{3},$$

so again we have matched the price of the claim.

Now at time $t = 0$, we need to find (x_1, y_1) giving values of either 50, if the share price goes up, or 4/3, if the share price goes down. We find

$$x_1 = \frac{\frac{3}{2} \times \frac{4}{3} - \frac{3}{4} \times 50}{\frac{3}{2} - \frac{3}{4}} = -\frac{142}{3}, \quad y_1 = \frac{50 - \frac{4}{3}}{120 - 60} = \frac{73}{90},$$

and we have

$$V_0 = x_1 + 80y_1 = -\frac{142}{3} + 80 \times \frac{73}{90} = \frac{158}{9}.$$

Our hedging portfolio is:

$$\text{at time } 0, \qquad (x_1, y_1) = \left(-\frac{142}{3}, \frac{73}{90}\right)$$

$$\text{at time } 1, \qquad (x_2, y_2) = \begin{cases} (-70, 1) & \text{if } \omega_1 = 1 \\ (-2, \frac{1}{18}) & \text{if } \omega_1 = 0. \end{cases}$$

We can check the self-financing condition: if the share price goes up, then

$$x_1 + 120y_1 = -\frac{142}{3} + 120 \times \frac{73}{90} = 50$$

and

$$x_2 + 120y_2 = -70 + 120 \times 1 = 50.$$

Similarly, if the share price goes down, then

$$x_1 + 60y_1 = -\frac{142}{3} + 60 \times \frac{73}{90} = \frac{4}{3},$$

and

$$x_2 + 60y_2 = -2 + 60 \times \frac{1}{18} = \frac{4}{3}.$$

Our portfolio, consisting of (x_1, y_1) and two possible outcomes for (x_2, y_2) is self-financing, and replicates the returns of the random variable $(S_T - 70)^+$, whatever happens to the share prices.

3.5 Recombinant trees

In this section, we consider the special case where u and d have the same values at every node. This means that, for example, if the stock moves up and then down, it will have the same value as if it moved down and then up. We can represent the model using a diagram which has $t+1$ nodes at each time t, rather than 2^t, as in Figure 2. In this situation, we say that the diagram is **recombinant** (and in fact, it is no longer a tree).

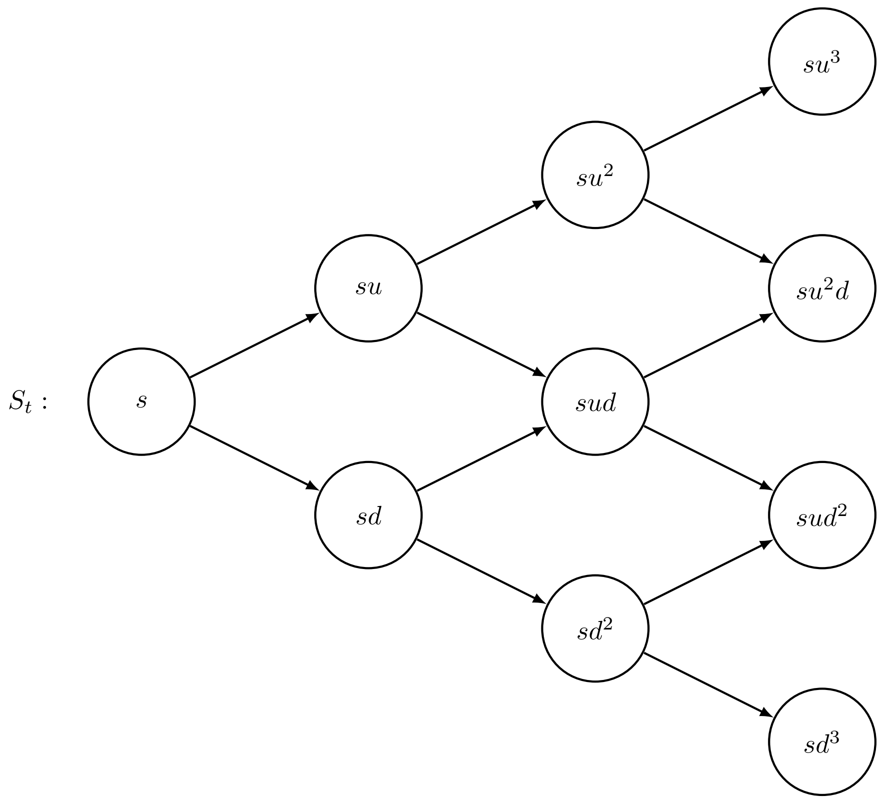


Figure 2: A recombinant 'tree'.

Now, the possible values of the share price at time t are

$$S_t \in \{su^k d^{t-k}, \quad k = 0, 1, \dots, t\},$$

where k denotes the total number of up-moves that have occurred (up to time t). This means that we can specify each node in the tree by a pair (t, k) with $k = 0, 1, \dots, t$.

The contingent claim $\Phi(S_T)$ can be replicated using a self-financing portfolio. If $V_t(k)$ denotes the value of the portfolio at node (t,k), then $V_t(k)$ can be computed recursively by

$$\begin{aligned} V_T(k) &= \Phi(su^k d^{T-k}), \\ V_t(k) &= \frac{1}{1+r}[q_u V_{t+1}(k+1) + q_d V_{t+1}(k)]. \end{aligned} \tag{3.5}$$

where just as before q_u and q_d are the martingale probabilities at a node, given by

$$q_u = \frac{1+r-d}{u-d}, \quad q_d = \frac{u-(1+r)}{u-d}.$$

(In fact, these are the martingal probabilities at every node, since u and d are fixed.)

We can also calculate the hedging portfolio: writing $x_t(k)$ and $y_t(k)$ for the amounts held at node $(t-1,k)$, we have

$$x_t(k) = \frac{1}{(1+r)^t}\frac{uV_t(k) - dV_t(k+1)}{u-d}, \quad y_t(k) = \frac{V_t(k+1) - V_t(k)}{su^k d^{t-1-k}(u-d)}.$$

The recursive equations (3.5) are sometimes referred to as the **binomial algorithm**. We can use this algorithm to work out the fair price of a contingent claim at time 0, by starting with the nodes at time $t = T$ and working backwards to $t = 0$. This gives us the following result.

Theorem 3.2. *The arbitrage-free price at $t = 0$ of a contingent claim $\Phi(S_T)$ is given by*

$$\Pi_0(\Phi) = \frac{1}{(1+r)^T}\mathbb{E}_{\mathbb{Q}}(\Phi(S_T)),$$

where $\mathbb{Q}$ is the martingale measure for S_T. More explicitly, we can write

$$\Pi_0(\Phi) = \frac{1}{(1+r)^T}\sum_{k=0}^{T}\binom{T}{k} q_u^k q_d^{T-k}\Phi(su^k d^{T-k}).$$

Proof. It is straightforward to check that the two expressions for $\Pi_0(\Phi)$ are equivalent, by calculating $\mathbb{Q}(S_T = su^k d^{T-k})$. This is the probability that there are exactly k up steps, and $T-k$ down steps, among the T total steps, so is $\binom{T}{k} q_u^k q_d^{T-k}$. In other words, under the measure $\mathbb{Q}$, the price S_T equals $S_0 u^Y d^{T-Y}$, where $Y \sim \mathrm{Bin}(T, q_u)$, and therefore

$$\mathbb{E}_{\mathbb{Q}}[\Phi(S_T)] = \mathbb{E}[\Phi(S_0 u^Y d^{T-Y})] = \sum_{k=0}^{T}\Phi(S_0 u^k d^{T-k})\mathbb{P}[Y = k].$$

It is also easy to see that the explicit expression for $\Pi_0(\Phi)$ follows by applying the equations (3.5). Indeed, it is possible to show by induction on t, that the value $V_{T-t}(i)$ of the portfolio at a node $(T-t, i)$, for $T-t \in [0,T]$ and $i = 0, \ldots, T-t$, satisfies

$$V_{T-t}(i) = \frac{1}{(1+r)^t}\sum_{k=0}^{t}\binom{t}{k} q_u^k q_d^{t-k}\Phi(su^{i+k} d^{T-i-k}),$$

and then by the Law of One Price, we have that $\Pi_0(\Phi) = V_0(0)$, which gives the required expression. $\square$

Exercise: work out the above proof by induction. Hint: in the inductive step use the identity $\binom{t}{k-1} + \binom{t}{k} = \binom{t+1}{k}$.

3.6 The Cox–Ross–Rubinstein formula

We use the pricing formula we found in the previous section to give a formula for the price of a European call option, known as the Cox–Ross–Rubinstein formula.

As usual, let K be the strike price of the call option, and T the expiry date. The Cox–Ross–Rubinstein formula for the price of such a call option assumes that the underlying stock can be modelled by the recombinant tree model we studied in the previous section, for appropriate choices of u, d and r. As it stands, we haven't yet considered how suitable the binomial model is as a model of stock price fluctuations. We'll look at this question in Chapter 5, but for now we suppose that we have some appropriately chosen values for u, d and r, and we price the option in terms of these parameters.

We can use the formula from Theorem 3.2 to price the call option, by setting $\Phi(S_T) = (S_T - K)^+$. In fact, we give a more general form, for the price at time $T - t \in [0, T]$.

Theorem 3.3 (Cox-Ross-Rubinstein formula)**.** *The price at time $T-t$ of a European call option with strike price K and expiry date T is given by*

$$\Pi_{T-t} = S_{T-t} \sum_{k=k^\star}^{t} \binom{t}{k} m_u^k m_d^{t-k} - \frac{K}{(1+r)^t} \sum_{k=k^\star}^{t} \binom{t}{k} q_u^k q_d^{t-k}, \tag{3.6}$$

where q_u, q_d are the risk-neutral probabilities, $m_u = \dfrac{q_u u}{1+r}, m_d = \dfrac{q_d d}{1+r}$ and $k^\star$ is the smallest integer k that satisfies

$$k \log\left(\frac{u}{d}\right) > \log\left(\frac{K}{S_{T-t} d^t}\right). \tag{3.7}$$

Proof. The price $\Pi_{T-t}(\Phi)$ at time $T-t$ of the claim $\Phi(S_T)$ is a function of the share price S_{T-t} given by

$$\frac{1}{(1+r)^t} \sum_{k=0}^{t} \binom{t}{k} q_u^k q_d^{t-k} \Phi(S_{T-t} u^k d^{t-k}).$$

Since $\Phi(x) = (x-K)^+$, the terms appearing in this sum will be zero unless $S_{T-t} u^k d^{t-k} > K$. We can rewrite this condition as $(u/d)^k > K/(S_{T-t} d^t)$, and since $u/d > 1$, there is a smallest integer $k^\star$ given by (3.7), such that if $k \geq k^\star$ then the condition is satisfied. (Note that if $k^\star > t$ then the sum becomes zero, and $\Pi_{T-t} = 0$.)

In other words, we can write the price of a European call option as

$$\Pi_{T-t} = \frac{1}{(1+r)^t} \sum_{k=k^\star}^{t} \binom{t}{k} q_u^k q_d^{t-k} (S_{T-t} u^k d^{t-k} - K)$$

and multiplying out the bracket and using the expressions for m_u and m_d yields the formula (3.6). □

Example 3.8
Consider a 3-period model with $u = 1.2$, $d = 0.8$ and $r = 0.1$. Suppose the share price at time 0 is $S_0 = 100$. We calculate the CRR price at time 0 of a European call option with strike price 70 and expiry date $T = 3$.

First, we calculate the martingale probabilities:

$$q_u = \frac{1.1 - 0.8}{1.2 - 0.8} = \frac{3}{4}, \quad q_d = \frac{1.2 - 1.1}{1.2 - 0.8} = \frac{1}{4}.$$

Then, we calculate m_u and m_d:

$$m_u = \frac{3}{4} \times \frac{1.2}{1.1} = \frac{9}{11}, \quad m_d = \frac{1}{4} \times \frac{0.8}{1.1} = \frac{2}{11}.$$

To find the correct value of $k^\star$, we look at the possible values for the share price at time T. We see that S_3 can be one of 51.2, 76.8, 115.2 or 172.8, and so the call option only has positive value at time T if at least one up-move occurs, so $k^\star = 1$.

(Alternatively, we can calculate $\log(K/S_0 d^3)/\log(u/d) = \log(70/51.2)/\log(3/2) \approx 0.771$, and find that the smallest integer greater than this is $k^\star = 1$.)

Then the CRR price is given by

$$V_0 = S_0 \sum_{k=1}^{3} \binom{3}{k} m_u^k m_d^{3-k} - \frac{K}{(1+r)^3} \sum_{k=1}^{3} \binom{3}{k} q_u^k q_d^{3-k}$$
$$= 100 \times \frac{1323}{1331} - \frac{70}{1.331} \times \frac{63}{64} = \frac{253575}{5324} \approx 47.63.$$

Remark. Note that $m_u, m_d > 0$ and $m_u + m_d = \frac{q_u u}{1+r} + \frac{q_d d}{1+r} = 1$, so that m_u and m_d can also be interpreted as probabilities. Hence we can write (3.6) as

$$\Pi_{T-t} = S_{T-t}\mathbb{P}[X_t \geq k^\star] - \frac{K}{(1+r)^t}\mathbb{P}[Y_t \geq k^\star],$$

where X_t and Y_t are Binomial random variables: $X_t \sim \text{Bin}(t, m_u), Y_t \sim \text{Bin}(t, q_u)$.

4 Filtrations, Conditional Expectation, and Martingales

In this chapter, we extend the probability theory behind our pricing of contingent claims in the binomial model. This will allow us to put the important financial concepts on a proper mathematical foundation, and to prepare for Epiphany term. In continuous time, our intuition is less helpful, and we will need to approach the material using a rigorous mathematical theory.

4.1 The probability space

Remember that in the multi-period binomial model, we are working in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in which:

- $\Omega = \{0,1\}^T$ is the set containing all strings of length T in which every character is a 0 or a 1
- $\mathcal{F} = \mathcal{F}_T$ is the Borel σ-algebra on Ω, i.e., $\mathcal{F}_T = 2^\Omega$.
- we have fundamentally not been interested in learning more about $\mathbb{P}$, as long as we know that $\mathbb{P}(\omega) > 0$ for every $\omega \in \Omega$.

We can view this probability space as equivalent to a coin-toss space, in which we toss a sequence of coins and record a 1 each time a coin lands heads, and a 0 each time a coin lands tails. In case you've forgotten how equivalent measures work, the next subsection might be helpful:

4.1.1 Side note: Equivalent measures

Recall that two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent* to each other if

$$\mathbb{P}(A) > 0 \text{ if and only if } \mathbb{Q}(A) > 0.$$

The measures do not need to assign the same probabilities to each event, but they should agree that either A is a possible event (both $\mathbb{P}(A)$ and $\mathbb{Q}(A)$ are strictly positive), or that A is an impossible event (both are zero).

Example 4.1
In the context of the multi-period binomial mode, we have two equivalent measures: $\mathbb{P}$, the ''objective measure", representing our estimate of how likely a head is to occur in reality, and $\mathbb{Q}$, the martingale measure that we use for pricing.

Example 4.2
For any $0 < p < 1$, we can define a probability measure $\mathbb{P}$ on Ω_T as follows.

Let #heads(ω) be the number of times that 1 appears in $\omega = \omega_1 \dots \omega_T$, and #tails($\omega$) $= T -$ #heads(ω) be the number of 0s Then

$$\mathbb{P}(\omega) = p^{\#\text{heads}(\omega)}(1-p)^{\#\text{tails}(\omega)},$$

and for $A \in \mathcal{F}$,

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega).$$

For a different $0 < q < 1$, we can define a different measure $\mathbb{Q}$ on Ω_T with

$$\mathbb{Q}(\omega) = q^{\#\text{heads}(\omega)}(1-q)^{\#\text{tails}(\omega)}, \qquad \mathbb{Q}(A) = \sum_{\omega \in A} \mathbb{Q}(\omega).$$

Here $\mathbb{P}$ and $\mathbb{Q}$ are equivalent to each other as long as both $0 < p < 1$ and $0 < q < 1$.

Remark. In the *recombinant* multi-period binomial model, the martingale measure $\mathbb{Q}$ is defined exactly as in Example 4.2, and we set $q = q_u$. On the other hand if u and d are not fixed throughout the model, then for each $\omega \in \Omega_T$, $\mathbb{Q}(\omega)$ becomes a product of q_us and q_ds, which can be found by "multiplying along the branches of the tree".

4.1.2 Back to probability spaces

Remember that in Chapter 3, we defined the share prices in terms of the random variables Z_t:

$$S_t(\omega) = Z_t(\omega) S_{t-1}(\omega),$$

where $Z_1, Z_2, \dots, Z_T$ are random variables defined on Ω, with

$$Z_t(\omega) = \begin{cases} u & \text{if } \omega_t = 1, \\ d & \text{if } \omega_t = 0. \end{cases}$$

Example 4.3
In the recombinant market with $s = 4$, $T = 3$, $u = 2$, and $d = \frac{1}{2}$, we have

$$S_0(\omega_1\omega_2\omega_3) = 4, \qquad \text{for all } \omega \in \Omega_T;$$

$$S_1(\omega_1\omega_2\omega_3) = \begin{cases} 8 & \text{if } \omega_1 = 1, \\ 2 & \text{if } \omega_1 = 0; \end{cases}$$

$$S_2(\omega_1\omega_2\omega_3) = \begin{cases} 16 & \text{if } \omega_1 = \omega_2 = 1, \\ 4 & \text{if } \omega_1 \neq \omega_2, \\ 1 & \text{if } \omega_1 = \omega_2 = 0; \end{cases}$$

and

$$S_3(\omega_1\omega_2\omega_3) = \begin{cases} 32 & \text{if } \omega_1 = \omega_2 = \omega_3 = 1, \\ 8 & \text{if there are two 1s and one 0} \\ 2 & \text{if there are two 0s and one 1} \\ \frac{1}{2} & \text{if } \omega_1 = \omega_2 = \omega_3 = 0. \end{cases}$$

In the spirit of simplifying the notation in this chapter, we are going to **assume that u and d are fixed everywhere**. All of the theory works in exactly the same way when they're not, but we have to worry about which u, or which q_u, we're using all the time.

Under this nice assumption, the Zs are independent and identically distributed random variables. They encapsulate all the randomness in the probability space $(\Omega_T, \mathcal{F}, \mathbb{P})$: the 2^T elements of Ω_T are in one-to-one correspondence with the 2^T possible real-valued sequences $(z_1, z_2, \dots, z_T)$ describing the possible values of $(Z_1, Z_2, \dots, Z_T)$. In other words, if we know the values of all of $Z_1, Z_2, \dots, Z_T$, then we know which state $\omega \in \Omega_T$ we are in, and vice versa.

We say that $\mathcal{F} = 2^{\Omega_T}$ is equal to the σ-algebra **generated by the random variables** $Z_1, Z_2, \dots, Z_T$, and we write $\mathcal{F} = \sigma(Z_1, Z_2, \dots, Z_T)$.

4.2 Partial information

If we only know about the first t coin tosses (that is, we know what $Z_1, Z_2, \dots, Z_t$ are, but we do not know $Z_{t+1}, \dots, Z_T$), we can still say quite a lot about how the share prices evolve: we can already write down the values of $S_0, S_1, \dots, S_t$. Intuitively, this seems perfectly reasonable, but before we can say anything probabilistically, we'd better sort out which space we're working in.

We define a sequence of σ-algebras as follows:

- Take $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- Let $A_1 = \{\omega \in \Omega : \omega_1 = 1\}$, and let $A_0 = \{\omega \in \Omega : \omega_1 = 0\}$. Define

$$\mathcal{F}_1 = \{\emptyset, A_1, A_0, \Omega\}.$$

- Let $A_{11} = \{\omega \in \Omega : \omega_1 = \omega_2 = 1\}$, $A_{10} = \{\omega \in \Omega : \omega_1 = 1, \omega_2 = 0\}$, $A_{01} = \{\omega \in \Omega : \omega_1 = 0, \omega_2 = 1\}$, and $A_{00} = \{\omega \in \Omega : \omega_1 = \omega_2 = 0\}$. Define

$$\mathcal{F}_2 = \sigma(A_{11}, A_{10}, A_{01}, A_{00})$$

  as the smallest σ-algebra containing these four sets.
- Continue in this way, defining 2^t subsets which encapsulate the first t coin tosses, and building $\mathcal{F}_t$ as the smallest σ-algebra containing these subsets, until we reach $\mathcal{F}_T = 2^\Omega$.

Exercise 4.1

Construct $\mathcal{F}_2$; what does $\mathcal{F}_t$ look like?

Constructing these σ-algebras helps us to define what we mean by partial information. We say that we have *observed* a σ-algebra $\mathcal{G}$ if, for every event $A \in \mathcal{G}$, we know whether A has occurred, that is, if we know the value of the indicator function $\mathbb{1}(A)$. Now at time t, we know what the share price has been for *every* $k \leq t$; that is, we know *all* of the values $S_0, S_1, \dots, S_t$. So our σ-algebra $\mathcal{F}_t$ encodes *all* of the information about the share prices up to time t.

Remark. In this chapter, we move back and forth between thinking about having observed a *σ-algebra*, and having observed some *random variables*. Because $\mathcal{F}_t$ encodes all of the randomness in the first t share prices, and nothing else, we can write $\mathcal{F}_t = \sigma(S_0, S_1, \dots, S_t)$, or even $\mathcal{F}_t = \sigma(Z_1, \dots, Z_t)$.

Observing the σ-algebra $\mathcal{F}_t$ means knowing exactly which of the events in it have occurred, so when we say that we have observed $\mathcal{F}_t$ we can deduce the values of $S_1, \dots, S_t$. Similarly, we can go in the other direction: if we know *all* of the values $S_1, \dots, S_t$, then for every event $A \in \mathcal{F}_t$ we know the value of $\mathbb{1}(A)$.

Definition 4.1. A **filtration** is a non-descending sequence of σ-algebras

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n.$$

Filtrations help us to model partial information which is accumulating over time. We assume that we have a perfect memory and never forget anything that's happened; we just keep increasing the amount of information we have.

Exercise 4.2

Prove (convince yourself) that the sequence of σ-algebras we have just constructed is a filtration: for each t, $\mathcal{F}_t \subset \mathcal{F}_{t+1}$.

Definition 4.2. A random variable $X : \Omega \to \mathbb{R}$ is **measurable** with respect to a σ-algebra $\mathcal{G}$ if for every Borel set $B \in \mathcal{B}(\mathbb{R})$,

$$\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{G}.$$

In other words, the random variable X is measurable with respect to $\mathcal{G}$ if observing $\mathcal{G}$ allows us to determine the value of X.

Example 4.4
We have already met some random variables which are measurable with respect to filtrations:

- The share price S_t is always measurable with respect to $\mathcal{F}_t$, because it only depends on the first t coin tosses.
- The holdings in a portfolio at time $t-1$, x_t and y_t, are allowed to be random *as long as* they are measurable with respect to $\mathcal{F}_{t-1}$.
- A random variable T is a stopping time with respect to the filtration $(\mathcal{F}_t)$ if, for every t, $\{T \leq t\} \in \mathcal{F}_t$.

The maximal value that the share price will take over the period $[0, T]$ is *not* measurable with respect to $\mathcal{F}_t$ for any $t < T$, because we will only know what the maximum is when we reach time T.

Since the share price S_t is measurable with respect to $\mathcal{F}_t$, we sometimes write $S_t(\omega_1\omega_2 \ldots \omega_t)$, or even $S_t(\cdots\omega_t)$, rather than $S_t(\omega_1\omega_2 \ldots \omega_T)$.

Example 4.5
In Example 4.3, we can write $S_1(1) = 8$, or $S_2(01) = 4$.

4.3 Conditional expectation

If you have forgotten how conditional expectation with respect to a random variable works, go and check the Prerequisite notes then come back here.

Definition 4.3. Given a random variable X on Ω_T such that $\mathbb{E}_{\mathbb{Q}}[|X|] < \infty$, the **conditional expectation** of X with respect to $\mathcal{G}$ is the unique random variable ξ which satisfies:

- $\mathbb{E}_{\mathbb{Q}}[|\xi|] < \infty$.
- ξ is measurable with respect to $\mathcal{G}$
- For every event $A \in \mathcal{G}$, $\mathbb{E}_{\mathbb{Q}}[\xi \mathbb{1}(A)] = \mathbb{E}_{\mathbb{Q}}[X \mathbb{1}(A)]$.

We denote $\xi = \mathbb{E}_{\mathbb{Q}}[X|\mathcal{G}]$.

We can think of the conditional expectation with respect to a sub-σ-algebra as "the expectation of X, *given* the partial information contained in $\mathcal{G}$". If we use the sub-σ-algebra generated by a random variable Y, $\mathcal{G} = \sigma(Y)$, we will rediscover our original Probability-1 definition of conditional expectation ($\mathbb{E}[X|Y] = g(Y)$, and so on).

If $X_T = \Phi(S_1, \ldots, S_T)$ then the conditional expectation $\mathbb{E}_{\mathbb{Q}}[X_T|\mathcal{F}_t]$ can be thought of as the expected value of X_T if we fix the outcomes of $S_1, \ldots, S_t$ and average over the remaining randomness that determines $S_{t+1}, \ldots, S_T$, i.e., over $Z_{t+1}, \ldots, Z_T$.

This means that, for $\omega \in \Omega_T$, we have

$$\mathbb{E}_{\mathbb{Q}}[X_T|\mathcal{F}_t](\cdots\omega_t) = q_u \mathbb{E}_{\mathbb{Q}}[X_T|\mathcal{F}_{t+1}](\cdots\omega_t 1) + q_d \mathbb{E}_{\mathbb{Q}}[X_T|\mathcal{F}_{t+1}](\cdots\omega_t 0). \tag{4.1}$$

Example 4.6
In the recombinant market with $s = 4$, $T = 3$, $u = 2$, and $d = \frac{1}{2}$ (the same market as in Example 4.3), we have

$$q_u = \frac{1 - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{1}{3}, \quad q_d = \frac{2 - 1}{2 - \frac{1}{2}} = \frac{2}{3}.$$

Let's calculate the values of $\mathbb{E}_{\mathbb{Q}}[S_2|\mathcal{F}_1]$ and $\mathbb{E}_{\mathbb{Q}}[S_3|\mathcal{F}_1]$. The two events in $\mathcal{F}_1$ are $\omega_1 = 1$ and $\omega_1 = 0$, so we need to do two sets of calculations.

$$\mathbb{E}_{\mathbb{Q}}[S_2|\mathcal{F}_1](1) = 16 \times \frac{1}{3} + 4 \times \frac{2}{3} = 8, \tag{4.2}$$

$$\mathbb{E}_{\mathbb{Q}}[S_2|\mathcal{F}_1](0) = 4 \times \frac{1}{3} + 1 \times \frac{2}{3} = 2. \tag{4.3}$$

Here $\mathbb{E}_{\mathbb{Q}}[S_2|\mathcal{F}_1]$ is a random variable defined on Ω_3 which takes value 8 whenever $\omega_1 = 1$, and 2 whenever $\omega_1 = 0$. We see that, in both cases, $\mathbb{E}_{\mathbb{Q}}[S_2|\mathcal{F}_1] = S_1$.

Next,

$$\mathbb{E}_{\mathbb{Q}}[S_3|\mathcal{F}_1](1) = 32 \times \left(\frac{1}{3}\right)^2 + 8 \times 2 \times \frac{1}{3} \times \frac{2}{3} + 2 \times \left(\frac{2}{3}\right)^2 = 8, \tag{4.4}$$

$$\mathbb{E}_{\mathbb{Q}}[S_3|\mathcal{F}_1](0) = 8 \times \left(\frac{1}{3}\right)^2 + 2 \times 2 \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{2} \times \left(\frac{2}{3}\right)^2 = 2. \tag{4.5}$$

Once again, we have $\mathbb{E}_{\mathbb{Q}}[S_3|\mathcal{F}_1] = S_1$.

Important Properties of Conditional Expectations

- **Linearity.** For constants a_1, a_2, we have
$$\mathbb{E}[a_1 X + a_2 Y|\mathcal{F}_t] = a_1 \mathbb{E}[X|\mathcal{F}_t] + a_2 \mathbb{E}[Y|\mathcal{F}_t].$$
- **Taking out what is known.** If X depends only on the first t coin flips, then
$$\mathbb{E}[XY|\mathcal{F}_t] = X \cdot \mathbb{E}[Y|\mathcal{F}_t].$$
- **Iterated conditioning.** If $s \le t$ then
$$\mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[X|\mathcal{F}_s].$$
- **Independence.** If X depends only on coin tosses $t+1$ to T, then
$$\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[X].$$

4.4 Martingales

Definition 4.4. A sequence of integrable random variables $Y_0, Y_1, \cdots, Y_T$ is called a **martingale** under the measure $\mathbb{Q}$ if for each t, the value of Y_t depends on the outcome of the first t coin flips (*we say the sequence is adapted to the filtration*) and
$$\mathbb{E}_{\mathbb{Q}}[Y_{t+1}|\mathcal{F}_t] = Y_t, \quad t = 0, 1, \dots, T-1.$$

Martingales arose out of the study of gambling models. If $Y_0, Y_1, \dots, Y_t$ represent the fortune of a player, then $\mathbb{E}_{\mathbb{Q}}[Y_{t+1}|\mathcal{F}_t]$ represents the amount of money the player can expect to have after the next game, given their position and knowledge at the end of game t. The player will consider the game to be fair if $\mathbb{E}_{\mathbb{Q}}[Y_{t+1}|\mathcal{F}_t] = Y_t$; that is, if the sequence is a martingale.

Here are some simple properties of martingales:

- For any t, we have $\mathbb{E}_{\mathbb{Q}}[X_t] = \mathbb{E}_{\mathbb{Q}}[X_0]$, because
$$\mathbb{E}_{\mathbb{Q}}[X_t] = \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}[X_t|\mathcal{F}_{t-1}]\Big] = \mathbb{E}_{\mathbb{Q}}[X_{t-1}] \text{ for all } t \ge 1.$$
- Whenever $m > n$,
$$\mathbb{E}_{\mathbb{Q}}[X_m|\mathcal{F}_n] = X_n.$$
- Given a filtration $(\mathcal{F}_t)_{t \le T}$ and a random variable Z which is measurable with respect to $\mathcal{F}_T$, the sequence
$$X_t = \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_t]$$
is always a martingale with respect to $(\mathcal{F}_t)_{t \le T}$.

4.4.1 Martingales in the multi-period binomial model

Theorem 4.1. *The sequence of discounted stock prices*
$$\frac{S_t}{(1+r)^t}, \quad t = 0, 1, 2, \dots, T,$$
is a martingale under the risk-neutral measure $\mathbb{Q}$.

Remark. The converse of this statement is also true in the multi-period binomial model. That is to say, the martingale measure $\mathbb{Q}$ in an arbitrage-free and complete multi-period binomial model is determined by the property that $\frac{S_t}{(1+r)^t}$ forms a martingale sequence under $\mathbb{Q}$.

Proof. The discounted stock prices are definitely adapted to the filtration, because the sequence $S_1, \dots, S_t$ is adapted to the filtration.

For the conditional expectation, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\frac{S_{t+1}}{(1+r)^{t+1}}|\mathcal{F}_t\right](\omega_1\cdots\omega_t) &= q_u\frac{S_{t+1}(\omega_1\cdots\omega_t 1)}{(1+r)^{t+1}} + q_d\frac{S_{t+1}(\omega_1\cdots\omega_t 0)}{(1+r)^{t+1}} \\ &= \frac{S_t(\omega_1\cdots\omega_t)}{(1+r)^{t+1}}[q_u u + q_d d] \\ &= \frac{S_t(\omega_1\cdots\omega_t)}{(1+r)^t}.\end{aligned}$$

□

Theorem 4.2. *For any self-financing portfolio in which (x_{t+1}, y_{t+1}) is always measurable with respect to $\mathcal{F}_t$, the* ***discounted value process***

$$\frac{V_t}{(1+r)^t}, \quad t = 0, 1, \dots, T,$$

is a martingale under the risk-neutral measure.

Remark. For this proof, we need to rewrite the self-financing condition in a new form: the **wealth equation**. The definition of the value, $V_t = x_{t+1}B_t + y_{t+1}S_t$, and the self-financing condition $V_{t+1} = x_{t+1}B_{t+1} + y_{t+1}S_{t+1}$, together imply that

$$V_{t+1} = y_{t+1}S_{t+1} + (1+r)(V_t - y_{t+1}S_t).$$

Proof. First, we check that $\frac{V_t}{(1+r)^t}$ is adapted to the filtration; since $V_t = x_{t+1}B_t + y_{t+1}S_t$ and all parts of this are measurable with respect to $\mathcal{F}_t$, we can continue.

For the conditional expectation, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\frac{V_{t+1}}{(1+r)^{t+1}}|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{y_{t+1}S_{t+1} + (1+r)(V_t - y_{t+1}S_t)}{(1+r)^{t+1}}|\mathcal{F}_t\right] \\ \text{(linearity)} \rightarrow \quad &= \mathbb{E}_{\mathbb{Q}}\left[\frac{y_{t+1}S_{t+1}}{(1+r)^{t+1}}|\mathcal{F}_t\right] + \mathbb{E}_{\mathbb{Q}}\left[\frac{(1+r)(V_t - y_{t+1}S_t)}{(1+r)^{t+1}}|\mathcal{F}_t\right] \\ \text{(taking out what is known)} \rightarrow \quad &= y_{t+1}\mathbb{E}_{\mathbb{Q}}\left[\frac{S_{t+1}}{(1+r)^{t+1}}|\mathcal{F}_t\right] + \frac{V_t - y_{t+1}S_t}{(1+r)^t} \\ &= y_{t+1}\frac{S_t}{(1+r)^t} + \frac{V_t - y_{t+1}S_t}{(1+r)^t} \\ &= \frac{V_t}{(1+r)^t},\end{aligned}$$

showing that $V_t/(1+r)^t$ is a martingale. □

This proves the correctness of the **risk-neutral valuation formula** for pricing contingent claims:

$$V_t = \frac{1}{(1+r)^{T-t}}\mathbb{E}_{\mathbb{Q}}[V_T|\mathcal{F}_t].$$

We finish this section with a version of the First Fundamental Theorem for the multi-period binomial model. We require a definition of arbitrage on the multi-period binomial model.

Definition 4.5. A portfolio $h \equiv (h_t = (x_t, y_t), t = 0, 1, \dots, T+1)$ on the multi-period binomial model $\mathcal{M} = (B_t, S_t)$ is an **arbitrage** portfolio if it is self-financing and its value process $V_t^h = x_{t+1}B_t + y_{t+1}S_t$ satisfies:

$$V_0^h = 0, \quad \mathbb{P}(V_T^h \geq 0) = 1, \quad \mathbb{P}(V_T^h > 0) > 0.$$

Theorem 4.3. *The following conditions are equivalent for a multi-period binomial model $\mathcal{M} = (B_t, S_t)$, $t = 0, 1, \dots, T$, with interest rate r.*

(1) The model is arbitrage-free according to Definition 4.5.
(2) The condition $d < 1 + r < u$ holds, where $d < u$ are the two possible values of $Z_t = S_t/S_{t-1}$ at each time t. (Z_t equals u with probability p and d with probability $1 - p$ for some $0 < p < 1$).
(3) There is a measure $\mathbb{Q}$ defined by

$$q_u = \frac{1 + r - d}{u - d}, \quad q_d = \frac{u - (1 + r)}{u - d}$$

at each node of the tree, such that $\frac{S_t}{(1+r)^t}$ is a martingale under $\mathbb{Q}$.

Proof. We have done most of the work needed to prove this theorem. Let us show the implications (1) ⇒ (2) ⇒ (3) ⇒ (1).

- (1) ⇒ (2): Consider the number of periods T in the model. If $T = 1$ then the implication holds by Theorem 2.1. For $T > 1$, the 1-period model obtained by observing the market from $t = 0$ to $t = 1$ has no arbitrage, and so $d < 1 + r < u$ by Theorem 2.1.
- (2) ⇒ (3): This implication is Theorem 4.1 above.
- (3) ⇒ (1): Suppose $h_t = (x_t, y_t)$ is a self-financing portfolio that satisfies the conditions $\mathbb{P}(V_0^h = 0) = 1$ and $\mathbb{P}(V_T^h \geq 0) = 1$. Since the measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$, it follows that $\mathbb{Q}(V_0^h = 0) = 1$ and $\mathbb{Q}(V_T^h \geq 0) = 1$. Now, by Theorem 4.2, $\frac{V_t}{(1+r)^t}$ is a martingale under $\mathbb{Q}$ and so in particular,

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{V_T}{(1+r)^T}\right] = \mathbb{E}_{\mathbb{Q}}[V_0] = 0.$$

  This shows that $\mathbb{E}_{\mathbb{Q}}[V_T] = 0$ and thus V_T is a non-negative random variable with mean 0, which implies V_T is identically zero: $\mathbb{Q}(V_T > 0) = 0$. Consequently, $\mathbb{P}(V_T > 0) = 0$ as well so h is not an arbitrage portfolio.

□

5 Pricing options from around the world

5.1 An algorithm to price any derivative asset

This algorithm allows us to price any derivative asset via its hedging portfolio. Suppose we have an asset with $X_T(\omega)$, which is measurable with respect to $\mathcal{F}_T$.

First, we define T random variables, $X_{T-1}, \dots, X_1, X_0$, on our probability space, using the recursive formula

$$X_t(\cdots\omega_t) = \frac{1}{1+r}(q_u X_{t+1}(\cdots\omega_t 1) + q_d X_{t+1}(\cdots\omega_t 0)).$$

Here q_u and q_d are the martingale probabilities at each node, given by

$$q_u = \frac{1+r-d}{u-d}, \quad q_d = \frac{u-(1+r)}{u-d}.$$

Next, we let

$$y_t(\cdots\omega_{t-1}) = \frac{X_t(\cdots\omega_{t-1}1) - X_t(\cdots\omega_{t-1}0)}{S_t(\cdots\omega_{t-1}1) - S_t(\cdots\omega_{t-1}0)}, \quad t = 1\dots T.$$

Finally, set $V_0 = X_0$ for all $\omega \in \Omega_T$. Now, for $t = 0, \dots, T-1$, let

$$V_{t+1} = y_{t+1}S_{t+1} + (1+r)(V_t - y_{t+1}S_t). \qquad (\textbf{Wealth Equation})$$

Theorem 5.1. *The portfolio given by $y_t(\cdots\omega_{t-1})$ as defined above, with*

$$x_t(\cdots\omega_{t-1}) = \frac{V_{t-1}(\cdots\omega_{t-1}) - y_t(\cdots\omega_{t-1})S_{t-1}(\cdots\omega_{t-1})}{B_{t-1}},$$

is self-financing, has value process V_t as defined above, and replicates the contingent claim X. Moreover, for every t and ω, $V_t(\cdots\omega_t) = X_t(\cdots\omega_t)$.

Proof. First, it is clear from the definition of x_t that for each t, the relation

$$V_t = x_{t+1}B_t + y_{t+1}S_t$$

holds for every $\omega \in \Omega_T$.

Next, to see that the self-financing condition holds, we check that

$$V_{t+1} = x_{t+1}B_{t+1} + y_{t+1}S_{t+1}$$

for $t = 0, \dots, T$.

We have

$$\begin{aligned} x_{t+1}B_{t+1} + y_{t+1}S_{t+1} &= \frac{V_t - y_{t+1}S_t}{B_t}B_{t+1} + y_{t+1}S_{t+1} && (5.1)\\ &= (1+r)(V_t - y_{t+1}S_t) + y_{t+1}S_{t+1} && (5.2)\\ &= V_{t+1}, && (5.3) \end{aligned}$$

by definition.

Now, to see that V and X always coincide, we use induction. The base case $V_0 = X_0$ is true by definition. Next, we fix $\omega_1\cdots\omega_t$ and use $V_t(\omega_1\cdots\omega_t) = X_t(\omega_1\cdots\omega_t)$ to show that

$$\begin{aligned} &V_{t+1}(\cdots\omega_t 1) = X_{t+1}(\cdots\omega_t 1), \quad \text{and}\\ &V_{t+1}(\cdots\omega_t 0) = X_{t+1}(\cdots\omega_t 0). \end{aligned}$$

First, if $\omega_{t+1} = 1$, we use the fact that $S_{t+1}(\cdots\omega_t 1) = uS_t(\cdots\omega_t)$ in the wealth equation to write

$$V_{t+1}(\cdots\omega_t 1) = y_{t+1}uS_t + (1+r)(V_t - y_{t+1}S_t) \tag{5.4}$$
$$= y_{t+1}S_t(u-(1+r)) + (1+r)V_t. \tag{5.5}$$

(Here, we have stopped writing $\cdots\omega_t$ in the right hand side, to simplify the notation.)

Using the inductive hypothesis and the expression for y_{t+1}, we get

$$V_{t+1}(\cdots\omega_t 1) = \frac{X_{t+1}(1) - X_{t+1}(0)}{uS_t - dS_t} S_t(u-(1+r)) + (1+r)X_t \tag{5.6}$$
$$= (X_{t+1}(1) - X_{t+1}(0))\frac{u-(1+r)}{u-d} + (1+r)X_t \tag{5.7}$$
$$= (X_{t+1}(1) - X_{t+1}(0))q_d + [q_u X_{t+1}(1) + q_d X_{t+1}(0)] \tag{5.8}$$
$$= X_{t+1}(1), \tag{5.9}$$

as required. □

Exercise: Check that, when $\omega_{t+1} = 0$, we also have $V_{t+1} = X_{t+1}$.

5.2 Popular options

A **digital (or binary) option** is a contract whose payoff depends in a discontinuous way on the terminal price of the underlying asset. We can describe the payoff functions succinctly using indicator functions: for an event A we write 1_A for the random variable

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, 1_A takes the value 1 exactly when A occurs, and 0 otherwise. For example,

$$1_{\{S_T > K\}} = \begin{cases} 1 & \text{if } S_T > K, \\ 0 & \text{otherwise.} \end{cases}$$

The payoff of the **cash-or-nothing binary option** is given by

$$X = \eta 1_{\{S_T > K\}} \quad \text{for a call option,}$$
$$X = \eta 1_{\{S_T < K\}} \quad \text{for a put option,}$$

where η is a pre-specified amount; and the payoff of the **asset-or-nothing binary option** is given by

$$X = S_T 1_{\{S_T > K\}} \quad \text{(call),}$$
$$X = S_T 1_{\{S_T < K\}} \quad \text{(put).}$$

Gap options are contingent claims whose payoffs are given by

$$X = (S_T - \eta)1_{\{S_T > K\}} \quad \text{(call),}$$
$$X = (\eta - S_T)1_{\{S_T < K\}} \quad \text{(put).}$$

Here again η is a pre-specified amount.

Lookback options are contingent claims whose payoff depends not only on the terminal price of the underlying asset but also on asset price fluctuations during the option's life time. There are two **standard lookback options**. The payoff of the **standard lookback call option** is given by

$$LC_T = S_T - S_{\min},$$

where $S_{\min} = \min_{t\in[0,T]} S_t$ is the minimum value of the stock price during its lifetime. The payoff of the **standard lookback put option** is given by

$$LP_T = S_{\max} - S_T,$$

where $S_{\max} = \max_{t\in[0,T]} S_t$ is the maximum value of the stock price.

Example 5.1
Let $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}$. We find the prices of the lookback option

$$X_3 = \max\{S_0, S_1, S_2, S_3\} - S_3$$

at times $t = 0, 1, 2$.

First we find $q_u = \dfrac{\frac{5}{4} - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{1}{2}$ and $q_d = \dfrac{2 - \frac{5}{4}}{2 - \frac{1}{2}} = \frac{1}{2}$. Writing $S_{\max} = \max\{S_0, S_1, S_2, S_3\}$, we find the following values for $S_{\max}, S_3$ and X_3 as functions of $\omega \in \Omega_3$:

$\omega_1\omega_2\omega_3$	111	110	101	100	011	010	001	000
$S_{\max}(\omega_1\omega_2\omega_3)$	32	16	8	8	8	4	4	4
$S_3(\omega_1\omega_2\omega_3)$	32	8	8	2	8	2	2	$\frac{1}{2}$
$X_3(\omega_1\omega_2\omega_3)$	0	8	0	6	0	2	2	$\frac{7}{2}$

Then we use the recursive definitions for X_2, X_1, X_0 to find the prices at time $t = 2, 1, 0$:

$$\begin{aligned}
X_2(11) &= \frac{1}{1+r}[q_u X_3(111) + q_d X_3(110)] \\
&= \tfrac{4}{5} \times [\tfrac{1}{2}X_3(111) + \tfrac{1}{2}X_3(110)] = \tfrac{16}{5}, \\
X_2(10) &= \tfrac{4}{5} \times [\tfrac{1}{2}X_3(101) + \tfrac{1}{2}X_3(100)] = \tfrac{12}{5}, \\
X_2(01) &= \tfrac{4}{5} \times [\tfrac{1}{2}X_3(011) + \tfrac{1}{2}X_3(010)] = \tfrac{4}{5}, \\
X_2(00) &= \tfrac{4}{5} \times [\tfrac{1}{2}X_3(001) + \tfrac{1}{2}X_3(000)] = \tfrac{11}{5},
\end{aligned}$$

and

$$\begin{aligned}
X_1(1) &= \tfrac{4}{5} \times [\tfrac{1}{2}X_2(11) + \tfrac{1}{2}X_2(10)] = \tfrac{56}{25}, \\
X_1(0) &= \tfrac{4}{5} \times [\tfrac{1}{2}X_2(01) + \tfrac{1}{2}X_2(00)] = \tfrac{6}{5},
\end{aligned}$$

and finally $X_0 = \frac{4}{5} \times [\frac{1}{2}X_1(1) + \frac{1}{2}X_1(0)] = \frac{172}{125}$.

Barrier Call Options: Let $L, K > 0$ be two given parameters, and $T > 0$ a given maturity. There are four types of barrier call options on the underlying asset with strike price K, barrier L and maturity T:

1. When $L > S_0$:
   - an **up and out call option** is defined by the payoff at the maturity T:

     $$UOC_T = (S_T - K)^+ 1_{\{S_{\max} \le L\}}.$$

     The payoff is that of a European call option if the price process of the underlying asset never reaches the barrier L before maturity. Otherwise it is zero (the contract knocks out).
   - an **up and in call option** is defined by the payoff at the maturity T:

     $$UIC_T = (S_T - K)^+ 1_{\{S_{\max} > L\}}$$

     The payoff is that of a European call option if the price process of the underlying asset crosses the barrier L before maturity. Otherwise it is zero (the contract knocks out). We have

     $$UOC_T + UIC_T = C_T,$$

     where C_T is the payoff of the corresponding European call option.
2. When $L < S_0$:
   - an **down and out call option** is defined by the payoff at the maturity T:

     $$DOC_T = (S_T - K)^+ 1_{\{S_{\min} > L\}}.$$

     The payoff is that of a European call option if the price process of the underlying asset never reaches the barrier L before maturity. Otherwise it is zero (the contract knocks out).

- an **down and in call option** is defined by the payoff at the maturity T:

$$DIC_T = (S_T - K)^+ 1_{\{S_{\min} \leq L\}}$$

The payoff is that of a European call option if the price process of the underlying asset crosses the barrier L before maturity. Otherwise it is zero (the contract knocks out). Clearly,

$$DOC_T + DIC_T = C_T,$$

where C_T is the payoff of the corresponding European call option.

Barrier put options: Replace calls by puts in the previous definitions.

An **Asian option** is a generic name for the class of options whose terminal payoff is based on average asset values during some period within the options lifetime. Let T be the exercise date and T_0 be the beginning date of the averaging period, for some $0 \leq T_0 \leq T$. Then the payoff at expiry of an **Asian call option** is given by

$$C_T^A \equiv (A_S(T_0, T) - K)^+,$$

where $A_S(T_0, T)$ can be the continuous arithmetic average

$$A_S(T_0, T) = \frac{1}{T - T_0} \int_{T_0}^{T} S_t dt,$$

or a discrete average

$$A_S(T_0, T) = \frac{1}{n} \sum_{i=0}^{n-1} S_{T_i},$$

where n is a positive integer and $T_0 < T_1 < T_2 < \cdots < T_n \leq T$, or sometimes a geometric average

$$A_S(T_0, T) = \left(\prod_{i=0}^{n-1} S_{T_i} \right)^{1/n},$$

again for some $T_0 < T_1 < T_2 < \cdots < T_n \leq T$. An **Asian put option** can be defined similarly.

5.3 American Options

An **American call or put option** gives the right to buy or, respectively, to sell the underlying asset for the strike price K at any time between now and a specified future time T, called the **expiry time**. In other words, an American option can be exercised at any time up to and including expiry. The holder of an **American type contingent claim** with contract function $\Phi(x)$ will receive a payoff $\Phi(S_\tau)$ at time τ, where τ is a random variable chosen by the holder. The random variable τ must take values in $\{0, 1, \ldots, T\}$ and specifies the choice of the exercise time for the holder. This means that if the option will be exercised at time $\tau = t$ then the payoff will be $\Phi(S_t)$ at time t. Of course, it can be exercised only once. The holder does not have complete freedom to choose τ arbitrarily; it must be a **stopping time**, i.e., the decision to exercise the option at time t can only depend on what has happened upto time t and not on the future randomness. Some examples of stopping times are $\tau \equiv T$ (always exercise at time T), and $\tau = \inf\{t : S_t \geq L\} \wedge T$ (exercise at the first time that the share price is at least price L, or at time T if that never happens).

It it possible to show that the price of the American option at time 0 equals $\sup_\tau \{\mathbb{E}_\mathbb{Q}[(1+r)^{-\tau} \Phi(S_\tau)]\}$, where the supremum is taken over all stopping times τ. We give a rough argument, as follows. Suppose the holder exercises the American option according to the stopping time τ, so that the payoff to the holder is the amount $\Phi(S_\tau)$ at time τ. This is equivalent to a present value of $(1+r)^{-\tau} \Phi(S_\tau)$, so risk-neutral valuation tells us that the value at time 0 would be $\mathbb{E}_\mathbb{Q}[(1+r)^{-\tau} \Phi(S_\tau)]$. But since the holder is free to choose any stopping time τ, they will choose the τ that maximises this value at time 0, hence the value must be $\sup_\tau \{\mathbb{E}_\mathbb{Q}[(1+r)^{-\tau} \Phi(S_\tau)]\}$.

The following pricing algorithm allows us to compute the value of the American option at any time $t = 0, 1, \ldots, T$. Let V_t^A denote the price of the American option at time t (that has not been exercised yet). Using the risk-neutral valuation formula, we can price an American option inductively, as follows:

At $t = T$: $V_T^A = \Phi(S_T)$, because if we hold an American option at time $t = T$, the only choice is to exercise or not at the expiry time T, so it has the same value as the European version of the option.

At $t < T$: suppose we know the value of the American option at time $t+1$ is V^A_{t+1}, then $V^A_t = \max\left\{ \Phi(S_t)\,,\, \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[V^A_{t+1} \mid \mathcal{F}_t] \right\}$.

Why do we take a "max" here? It's because if we hold an American option at time t, we have the choice to either **exercise early** at time t, or **wait**. The value of exercising early is $\Phi(S_t)$, the contract function Φ evaluated at the current share price S_t; the **value of waiting** at time t is the risk-neutral price $\frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[V^A_{t+1} \mid \mathcal{F}_t]$, and we will choose whichever gives us more.

Summarising, we have the following **pricing algorithm** for American options:

$$V^A_t(\omega_1 \cdots \omega_t) = \begin{cases} \Phi(S_T(\omega_1 \cdots \omega_T)) & \text{if } t = T, \\ \max\left\{ \Phi(S_t(\omega_1 \cdots \omega_t)), \frac{1}{1+r}[q_u V^A_{t+1}(\omega_1 \cdots \omega_t 1) + q_d V^A_{t+1}(\omega_1 \cdots \omega_t 0)] \right\} & \text{if } t < T. \end{cases}$$

To see the algorithm in more detail, let's consider an American option expiring after 2 steps with the contract function $\Phi(x)$. The value of this option at time 2 (if it is not exercised before time 2) is clearly $\Phi(S_2(\omega_1\omega_2))$. At time 1 the option holder will have the choice to exercise immediately, with payoff $\Phi(S_1(\omega_1))$, or to wait until time 2, when the value of the option will become $\Phi(S_2(\omega_1\omega_2))$. **The value of waiting** at time 1 is therefore given by

$$\frac{1}{1+r}[q_u \Phi(S_2(\omega_1 1)) + q_d \Phi(S_2(\omega_1 0))].$$

In effect, the option holder has the choice between the "value of waiting" and the immediate payoff $\Phi(S_1(\omega_1))$. The American option at time 1 will, therefore, be worth the higher of these two:

$$V^A_1(\omega_1) = \max\{\Phi(S_1(\omega_1)), \frac{1}{1+r}[q_u \Phi(S_2(\omega_1 H)) + q_d \Phi(S_2(\omega_1 T))]\}.$$

The same reasoning applied at time 0 gives

$$V^A_0 = \max\{\Phi(S_0), \frac{1}{1+r}[q_u V^A_1(H) + q_d V^A_1(T)]\}.$$

Example 5.2
Consider an American put option with strike price $K = 80$ pounds expiring at time 2 on a stock with initial price $S_0 = 80$ pounds in a Binomial model with $u = 1.1, d = 0.95$ and $r = 0.05$. The stock values are:

t	0		1		2
					96.80
			88.00	<	
S_t	80.00	<			83.60
			76.00	<	
					72.20

The price of the American put will be denoted by P^A_t for $t = 0, 1, 2$ and its price at time 2 is $(80 - S_2)^+$ given in the following tree:

t	0		1		2
					0.00
			?	<	
P^A_t	?	<			0.00
			?	<	
					7.80

First observe that $q_u = \frac{1+r-d}{u-d} = \frac{2}{3}$ and $q_d = \frac{1}{3}$. At time 1 the option holder can choose between exercising the option immediately or waiting until time 2. In the up state at time 1 the immediate payoff is $(K - S_1)^+ = (80 - 88)^+ = 0$ and the value of waiting is $\frac{1}{1+r}[q_u \times 0 + q_d \times 0] = 0$. In the down state the immediate payoff is 4 pounds, while the value of waiting is $1.05^{-1} \times \frac{1}{3} \times 7.8 \approx 2.48$. The option holder will choose the higher value (i.e., to exercise the option in the down state at time 1). This gives the time 1 value of the American put

t	0		1		2
					0.00
			0.00	$<$	
P_t^A	?	$<$			0.00
			4.00	$<$	
					7.80

At time 0 the choice is, once again, between the payoff $(80 - S_0)^+$, which is zero, or the value of waiting, which is $1.05^{-1} \times \frac{1}{3} \times 4 \approx 1.27$ pounds. Taking the higher of the two completes the tree of the option prices:

t	0		1		2
					0.00
			0.00	$<$	
P_t^A	1.27	$<$			0.00
			4.00	$<$	
					7.80

Therefore the price of the American put is $P_0^A = 1.27$ pounds.

In comparison, the price of a European put is $P_0^E = 1.05^{-1} \times \frac{1}{3} \times 2.48 \approx 0.79$. Here we use 2.48 (not 4) in the calculation as European option is exercised at time 2.

What is the price of an American call in the above example? Although in general an American option is at least as valuable as the equivalent European option (because of the additional choice in when to exercise the option), for **call options (on a stock that does not pay dividends)** the American and European options have the same price.

Theorem 5.2. *The prices of American and European call options on a stock that pays no dividends are equal $C^A = C^E$, whenever the strike price K and expiry time T are the same for both options.*

Proof. The relation $C^A \geq C^E$ is clear as the American call option gives higher payoff (since you can exercise your right at any time) than the European call. (It's also possible to give an arbitrage argument to prove this.) Now if $C^A > C^E$, then

- write and sell an American call.
- buy a European call.
- invest the difference $C^A - C^E$ risk free with interest rate r.

If the American call is exercised at time $t \leq T$, then borrow a share and sell it for K to settle your obligation as a writer of the call option, investing K at the rate r. Then at time T you can use the European call to buy a share for K and close your short position in stock. Your arbitrage profit will be

$$(C^A - C^E)(1+r)^T + K(1+r)^{T-t} - K > 0.$$

If the American option is not exercised at all, you will end up with the European option and an arbitrage profit $(C^A - C^E)(1+r)^T$. This proves that $C^A = C^E$. $\square$

6 The Black–Scholes formula

6.1 Asset price behaviour

We have seen how to price contingent claims using the binomial model in terms of the given parameters u, d, r, etc. This is of course only of practical use if the binomial model gives a reasonable approximation to real stock price behaviours. In this chapter we look at the how well the binomial model matches observed data and look at some particular choices of u, d, p_u that give rise to the Black–Scholes formula for pricing a European call option.

We'd like for our binomial model to match the observed behaviour of stock prices, that the simple returns

$$\frac{S_{t+\delta t} - S_t}{S_t}$$

over short time periods of length δt are approximately Normally distributed, and have distribution $\mathcal{N}(m\delta t, \sigma^2 \delta t)$, for some values m and σ^2 that depend on the stock and other features of the market.

A common way to achieve this is to make the assumption that the price S_t is distributed as a **geometric Brownian motion** with **drift** μ and **volatility** $\sigma > 0$. Specifically,

$$\log \frac{S_{u+t}}{S_u} \sim \mathcal{N}((\mu - \sigma^2/2)t, \sigma^2 t) \tag{6.1}$$

for all $u \geq 0, t > 0$ and additionally S_{u+t}/S_u is independent of $S_v, v \leq u$.

Next term we'll study the theory of Brownian motion and this continuous-time model of stock prices in more detail; here, we just try to make our discrete-time binomial model approximate the geometric Brownian motion as well as it can.

6.2 Tuning the binomial model

To do this, we start by breaking our time interval $[0, T]$ into n steps, each of size $\Delta t = T/n$. We'll be letting $n \to \infty$, so we should think of n being large and Δt being small. This is a slight difference from how we set up the binomial model before, where the time steps between nodes of the tree were all of size 1, but it does not really complicate matters very much, as long as we use the correct value for the interest rate per time step. We want the interest rate *per unit time* to be $r > 0$, so we take the interest rate over the small time step of size Δt to be $r\Delta t$. This means the price B_t of the risk-free asset is given by

$$B_{k\Delta t} = (1 + r\Delta t)^k, \quad \text{for } k = 0, \dots, n.$$

Other than this difference, the binomial model is exactly the same; there are constants $u > d$ and objective probabilities p_u, p_d which specify how the price of the risky asset changes. Specifically, at each of the n steps the share price is multiplied by either u or d (with probability p_u or p_d), so that at time T (after n steps) the share price is

$$S_T = S_0 u^Y d^{n-Y},$$

where $Y \sim \mathrm{Bin}(n, p_u)$. Note that this is a description of S_T under the objective measure given by p_u and p_d. We will make use of the martingale measure later.

To match the volatility, we choose $u = \mathrm{e}^{\sigma\sqrt{\Delta t}}$ and $d = \mathrm{e}^{-\sigma\sqrt{\Delta t}}$ for some $\sigma > 0$.

Remark. Note that for $\mathrm{e}^{-\sigma\sqrt{\Delta t}} < 1 + r\Delta t < \mathrm{e}^{\sigma\sqrt{\Delta t}}$ to hold it is enough that $r\Delta t < \sigma\sqrt{\Delta t}$ and this holds if $n > T(r/\sigma)^2$. In other words, for large enough n the market with these parameters is arbitrage-free and we can sensibly price contingent claims on this market.

How well do these parameters approximate geometric Brownian motion? We see that

$$\log \frac{S_T}{S_0} = Y \log \frac{u}{d} + n \log d = \sigma\sqrt{\Delta t}(2Y - n) = \sigma\sqrt{T}\frac{2Y - n}{\sqrt{n}}.$$

Now, since $Y = \sum_{i=1}^n X_i$ is a sum of i.i.d. random variables, with $\mathbb{V}\mathrm{ar}\, X_1 < \infty$ we can apply the Central Limit Theorem, which says that

$$\frac{\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}(X_1)}{\sqrt{\frac{1}{n}\mathbb{V}\mathrm{ar}(X_1)}} = \frac{Y - \mathbb{E}(Y)}{\sqrt{\mathbb{V}\mathrm{ar}(Y)}} \to \mathcal{N}(0,1)$$

in distribution as $n \to \infty$.

At this point we specify p_u. What happens if we choose $p_u = 1/2$? Then $\mathbb{E}(Y) = np_u = n/2$ and $\mathbb{V}\mathrm{ar}(Y) = np_u(1-p_u) = n/4$, which means that

$$\frac{Y - \mathbb{E}(Y)}{\sqrt{\mathbb{V}\mathrm{ar}(Y)}} = \frac{1}{\sigma\sqrt{T}} \log\frac{S_T}{S_0} \to \mathcal{N}(0,1).$$

In other words, $\log(S_T/S_0)$ is approximately $\mathcal{N}(0, \sigma^2 T)$. This is a good first attempt, as we've managed to emulate the volatility σ using our binomial model, but this choice of p_u doesn't give us a general drift term. To get that we need to use a different value of p_u, which is a slight perturbation from $p_u = 1/2$.

If we choose $p_u = 1/2 + (\mu - \sigma^2/2)\sqrt{\Delta t}/2\sigma$, then we have

$$\mathbb{E}(Y) = np_u = n/2 + \sqrt{n}(\mu - \sigma^2/2)\sqrt{T}/2\sigma, \quad \mathbb{V}\mathrm{ar}(Y) = np_u(1-p_u) = n/4 + O(1),$$

so for this choice of p_u we find that

$$\frac{Y - \mathbb{E}(Y)}{\sqrt{\mathbb{V}\mathrm{ar}(Y)}} \approx \frac{1}{\sigma\sqrt{T}} \log\frac{S_T}{S_0} - (\mu - \sigma^2/2)\sqrt{T}/\sigma \to \mathcal{N}(0,1),$$

meaning that $\log(S_T/S_0)$ is distributed Normally as $\mathcal{N}((\mu - \sigma^2/2)T, \sigma^2 T)$, asymptotically as $n \to \infty$.

Aside. Actually, we haven't quite shown that the price S_t is distributed as a geometric Brownian motion; we need to show rather more, that $\log(S_{t+u}/S_u) \sim \mathcal{N}((\mu - \sigma^2/2)t, \sigma^2 t)$ for *all times* u and $t > 0$. This is in fact true but requires rather too much advanced probability for this course. The general scheme is (i) show convergence to a vector of suitable random variables $(S_{t_1}, S_{t_2}, \dots, S_{t_n})$ for any finite n; (ii) prove that Brownian motion paths are almost surely continuous (this is the hard bit); (iii) use path continuity and convergence at finite sets of times to show convergence of the trajectories to those of Brownian motion (this needs basic ideas about distances between functions).

6.3 Risk-neutral drift

We have chosen parameters

$$u = \mathrm{e}^{\sigma\sqrt{\Delta t}}, \quad d = \mathrm{e}^{-\sigma\sqrt{\Delta t}}, \quad p_u = \frac{1}{2} + \frac{(\mu - \sigma^2/2)\sqrt{\Delta t}}{2\sigma} \tag{6.2}$$

so that under the objective probabilities p_u, p_d, the random variable $\log(S_T/S_0)$ is approximately Normally distributed. However, to calculate the arbitrage-free price of a derivative, we need to use the martingale probabilities q_u, q_d, where

$$q_u = \frac{1 + r\Delta t - d}{u - d} = \frac{1 + r\Delta t - \mathrm{e}^{-\sigma\sqrt{\Delta t}}}{\mathrm{e}^{\sigma\sqrt{\Delta t}} - \mathrm{e}^{-\sigma\sqrt{\Delta t}}},$$

(and $q_d = 1 - q_u$).

Applying Taylor's theorem to $\mathrm{e}^x = 1 + x + x^2/2 + \cdots$, we get

$$q_u = \frac{\sigma\sqrt{\Delta t} + (r - \sigma^2/2)\Delta t + o(\Delta t)}{2\sigma\sqrt{\Delta t} + o(\Delta t)} = \frac{1}{2} + \frac{(r - \sigma^2/2)\sqrt{\Delta t}}{2\sigma} + o(\sqrt{\Delta t}).$$

We see that (ignoring the error term) the expression for q_u looks like the expression for p_u but with μ replaced by r. Consequently, this means that under the martingale measure $\log(S_T/S_0)$ is also asymptotically Normally distributed, but as $\mathcal{N}((r - \sigma^2/2)T, \sigma^2 T)$. In other words, under the martingale measure the price S_t is again a geometric Brownian motion with the same volatility σ, but with drift r, which is called the **risk-neutral drift**.

In summary, we have shown that it is possible to make the binomial model emulate observed stock price behaviour by choosing u, d and p_u according to (6.2). Furthermore if we use the risk-neutral probabilities q_u, q_d in the binomial model then there is a corresponding risk-neutral drift for the geometric Brownian motion model. This means that if we calculate the price $\Pi_0(\Phi)$ of a contingent claim $\Phi(S_T)$, as given by Theorem 3.2 (using interest rate $r\Delta t$ per step),

$$\Pi_0(\Phi) = \frac{1}{(1 + r\Delta t)^n}\mathbb{E}_{\mathbb{Q}}[\Phi(S_T)]$$

and let $n \to \infty$, we get the limiting expression

$$\mathrm{e}^{-rT}\mathbb{E}[\Phi(S_0\mathrm{e}^W)]$$

where $W \sim \mathcal{N}((r - \sigma^2/2)T, \sigma^2 T)$.

To price a European call option, we use this expression with $\Phi(S_T) = (S_T - K)^+$, and we find an expression in terms of an expectation under the risk-neutral measure for the cost $C(K, T, \sigma, S, r)$ of a European call option, which depends on the strike price K, expiry date T, the volatility σ of the stock, the current share price S and the interest rate r:

$$C(K, T, \sigma, S, r) = \mathrm{e}^{-rT}\mathbb{E}[(S\mathrm{e}^W - K)^+], \tag{6.3}$$

where $S = S_0$ and $W = \log(S_T/S) \sim \mathcal{N}((r - \sigma^2/2)T, \sigma^2 T)$.

Remark. In fact, there is a slight gap in our reasoning above. The formula (6.3) should contain a term of the form $\lim_{n\to\infty}\mathbb{E}[X_n]$ but we are claiming it equals $\mathbb{E}[\lim_{n\to\infty} X_n]$. We know that for arbitrary random variables these need not be the same (for example, the sequence of random variables X_n on state space $\Omega = (0, 1)$ with the uniform measure defined by

$$X_n(\omega) = \begin{cases} n & \text{if } \omega < 1/n, \\ 0 & \text{otherwise,} \end{cases}$$

has $\mathbb{E}[X_n] = 1$ for all n, but $\lim_{n\to\infty} X_n(\omega) = 0$ for all $\omega \in (0, 1)$ so $\mathbb{E}[\lim_{n\to\infty} X_n] = 0$), but in our case the random variables in question are well-behaved enough to mean that we can push this limit inside our expectation.

The expectation formula for the call option price can be evaluated with a straightforward computation involving the Normal distribution. Let's start with the answer and how to use it and then see how to find the answer.

6.4 The formula

The formula $C = \mathrm{e}^{-rT}\mathbb{E}[(S\mathrm{e}^W - K)^+]$ stated in (6.3) can be written the form

$$C = SN(d_1) - Ke^{-rT}N(d_1 - \sigma\sqrt{T}) \quad \text{where} \quad d_1 = \frac{(r + \sigma^2/2)T + \log(S/K)}{\sigma\sqrt{T}} \tag{6.4}$$

and N denotes the standard Normal cdf given by

$$N(x) = \int_{-\infty}^{x}\phi(z)\mathrm{d}z = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-z^2/2}\mathrm{d}z$$

(This is also commonly denoted by Φ but we've already used Φ as the notation for a general contract function, so we'll use N for the Normal cdf to avoid confusion.) Some text books define $d_2 = d_1 - \sigma\sqrt{T}$, for the input to N in the last term of the formula.

Example 6.1. The current price of a stock is $S = 20$, its volatility is $\sigma = 0.1$, the nominal interest rate is $r = 0.06$ (or 6% per year). The no-arbitrage price of a call option with strike price $K = 21$ which expires in 3 months is found as follows.

With $T = 0.25$, we have

$$d_1 = \frac{0.015 + 0.00125 + \log 20/21}{0.1 \times 0.5} \approx \frac{-0.032540}{0.05} = -0.65080$$

(remember that we use natural logs) and hence

$$C \approx 20N(-0.65080) - 21e^{-0.015}N(-0.70080) \approx 5.152 - 5.000 = 0.152$$

i.e. just about 15.2 pence (if the stock value is pounds).

Remark. There are a few important points to note.

1. If we wish to buy/sell a (K, T) call option at time $t \in (0, T)$ when the stock price is S_t then the appropriate price is $C(K, T-t, \sigma, S_t, r)$. This form is commonly encountered as option expiry dates are concentrated into just a few dates in the year so generally we look at dependence upon the variable t rather than T. Modify d_1 from (6.4) by replacing T with $T - t$.
2. Using put-call option parity the no-arbitrage price of a European (K, T) put option satisfies

$$P = C(K, T, \sigma, S, r) + Ke^{-rT} - S = Ke^{-rT}N(\sigma\sqrt{T} - d_1) - SN(-d_1)$$

3. The underlying mean drift μ of the stock price does not appear in the Black–Scholes formula as we use the risk-neutral drift but the *volatility* σ does appear and we have to find a value for this somehow! More on this later.
4. We have derived this formula via a limit using risk-neutral probabilities on an artificial tree model which we 'tuned'. In fact the original argument was rather different using self-financing portfolios and *delta hedging*.

6.5 Calculating the expectation

We start by introducing the indicator function that the option is worth something at time T, i.e.,

$$I_K = \begin{cases} 1 & \text{if } S_T > K, \\ 0 & \text{otherwise.} \end{cases}$$

Also suppose that $t = 0$ for the moment. In what follows, we write $\mathbb{E}_r$ to remind ourselves that we are taking the expected value using the risk-neutral drift. Then, recalling that $S_T = Se^W$ we have

$$\mathbb{E}_r[(S\mathrm{e}^W - K)^+] = \mathbb{E}_r[I_K(S\mathrm{e}^W - K)] = S\mathbb{E}_r[I_K\mathrm{e}^W] - K\mathbb{E}_r[I_K]$$

Lemma 6.1. *We have*

$$\mathbb{E}_r[(S\mathrm{e}^W - K)^+] = \mathrm{e}^{rT}SN(d_1) - KN(d_1 - \sigma\sqrt{T})$$

where $d_1 = [(r + \sigma^2/2)T + \log S/K]/\sigma\sqrt{T}$ *and* N *is the standard Normal cdf.*

Proof. Under the risk-neutral drift, $\log(S_T/S) = W \sim \mathcal{N}((r - \sigma^2/2)T, \sigma^2 T)$ so (key trick!) we can write it in the form

$$W = (r - \sigma^2/2)T + \sigma\sqrt{T} \cdot Z \tag{6.5}$$

where $Z \sim N(0, 1)$. Now

$$I_K = 1 \Leftrightarrow S\mathrm{e}^W > K \Leftrightarrow (r - \sigma^2/2)T + \sigma\sqrt{T}Z > \log(K/S) \Leftrightarrow Z > \sigma\sqrt{T} - d_1.$$

From this equivalence and the symmetry of the Normal density we get

$$\mathbb{E}_r[I_K] = \mathbb{P}_r(S\mathrm{e}^W > K) = \mathbb{P}(Z < d_1 - \sigma\sqrt{T}) = N(d_1 - \sigma\sqrt{T}). \tag{6.6}$$

It remains to evaluate $\mathbb{E}_r(I_K\mathrm{e}^W)$. Recalling that the standard Normal density is $\phi(z) = \mathrm{e}^{-z^2/2}/\sqrt{2\pi}$ we have

$$\begin{aligned} \mathbb{E}_r[I_K\mathrm{e}^W] &= \int_{\sigma\sqrt{T}-d_1}^{\infty} \exp[(r - \sigma^2/2)T + z\sigma\sqrt{T}]\phi(z)\,\mathrm{d}z \\ &= \mathrm{e}^{rT}\frac{1}{\sqrt{2\pi}}\int_{\sigma\sqrt{T}-d_1}^{\infty} \exp[-(z^2 - 2z\sigma\sqrt{T} + \sigma^2 T)/2]\mathrm{d}z \\ &= \mathrm{e}^{rT}\frac{1}{\sqrt{2\pi}}\int_{-d_1}^{\infty} \mathrm{e}^{-y^2/2}\,\mathrm{d}y \qquad (\text{set } y = z - \sigma\sqrt{T}) \\ &= \mathrm{e}^{rT}\mathbb{P}(Z > -d_1) = \mathrm{e}^{rT}N(d_1) \end{aligned} \tag{6.7}$$

Combining these two parts together completes the proof. □

Now, multiplying the statement of Lemma 6.1 by e^{-rT} gives us (6.4), the Black–Scholes formula.

Exercise: check that we can just replace T by $T - t$ throughout if the option is purchased at time t rather than 0.

6.6 Properties of C

We finish this chapter by stating some properties of C. We describe the behaviour of $C(K, T, \sigma, S, r)$ as the various inputs change, some of which we can see from the form of the Black–Scholes formula. We can show that C is

- decreasing and convex in K;
- increasing in T and hence $C(K, T-t, \sigma, S, r)$ is decreasing in t;
- increasing and convex in S;
- increasing in σ and increasing in r.

In fact, the behaviour of C as K or T changes does not depend upon the geometric Brownian motion model, and can be deduced by comparing the appropriate portfolios. The properties of P, the European put price follow directly from those of C and the put-call parity formula.

These properties can also be deduced by calculating various partial derivatives of $C(K, T-t, \sigma, S, r)$. Some of these derivatives have special names given by Greek letters, and they are collectively known as "the Greeks". We'll look more at these next term.

Exercise: calculate $\partial C/\partial x$ with respect to $x = S$, K, t, r, σ directly from (6.4). Their expressions involve the standard Normal cdf N and pdf ϕ, e.g.,

$$\frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_1 - \sigma\sqrt{T-t}) > 0.$$

It is interesting to observe that the price $C = C(K, T-t, \sigma, S, r)$ satisfies the following partial differential equation

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0.$$

We shall see this PDE again next term, when we study Itô calculus.

Exercise: show that $C(K, T-t, \sigma, S, r) = \alpha C(K/\alpha, T-t, \sigma, S/\alpha, r)$ for $\alpha > 0$ — this scaling property is entirely natural as we should find equivalent prices whether we choose to pay in units of pence, pounds or 100 pounds.