Elementary Number Theory and Cryptography, Epiphany 2015

Problem Sheet 3, (Due: Monday Feb 23, at 12:00 in CM116)

Please hand in Problems 1, 2, 10, 12(a), 13(a)
The Problems 3, 8, 11, 12(b), 13(b), 15(a) will be discussed in the tutorials, (depending on time).

1. Let $n \in \mathbb{N}$. Write 2^n, as the sum of two squares.

2. A natural number is called "triangular" if it equals $a(a+1)/2$ for some $a \in \mathbb{N}$. Prove that if n is a sum of two triangular numbers then $4n+1$ is a sum of two squares.

3. Prove that of any four consecutive natural numbers, at least one is not representable as a sum of 2 squares.

4. If a prime number is the sum of three squares of prime numbers then one of these primes must be equal to 3.

5. Let p be an odd prime. If p divides a^2+b^2 where a and b are relatively prime natural numbers then prove that $p \equiv 1 \pmod 4$. Deduce from this that any non-trivial divisor of a sum of two relatively prime squares is again the sum of two squares.

6. Prove that every prime number p of the form $8k+1$ or $8k+3$ can be written in the form $p = a^2 + 2b^2$.

7. For any $n > 0$ show that there exists a positive integer which can be expressed in n distinct ways as the difference of two squares.

8. If the positive integer n is not the sum of squares of two integers, show that n cannot be represented as the sum of two squares of rational numbers.

9. Prove that every prime $p \equiv 1 \pmod 4$ divides the sum of two relatively prime squares, where each square exceeds 3.

10. Establish that the equation
$$x^2 + y^2 + z^2 + x + y + z - 1 = 0,$$
has no solutions in the integers x, y, z.

11. Prove that the only prime which can be represented as a sum of two positive cubes is 2.

12. Write each of the rational numbers below as simple finite continued fraction:
    (a) 187/57
    (b) 71/55
    (c) 118/303

13. Determine the rational numbers represented by the following simple continued fractions:

    (a) $[-2; 2, 4, 6, 8]$,
    (b) $[4; 2, 1, 3, 1, 2, 4]$,
    (c) $[0; 1, 2, 3, 4, 3, 2, 1]$.

14. If $C_k = p_k/q_k$ is the k-th convergent of the simple continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ establish that $q_k \geqslant 2^{(k-1)/2}$.

15. Evaluate each of the following infinite simple continued fractions:

    (a) $\overline{[2; 3]}$
    (b) $[0; \overline{1, 2, 3}]$
    (c) $[2; \overline{1, 2, 1}]$.