Elementary Number Theory and Cryptography, Epiphany 2015

Problem Sheet 2, (Due: Monday Feb 9, at 12:00 in CM116)

Please hand in Problems 1(a,b), 2, 5(b), 8
The Problems 1(c), 3, 5(a), 7, 10 will be discussed in the tutorials, (depending on time).

1. Find all primitive Pythagorean triples x, y, z (if any) in which: a) $x = 30$; b) $x = 40$; c) $x = 60$.

2. If x, y, z is a primitive Pythagorean triple show that $x+y$ and $x-y$ are congruent modulo 8 to either 1 or 7.

3. For any natural number n show the existence of at least n Pythagorean triples having the same first member.

4. A Pythagorean triangle is defined to be a right triangle whose sides are of integral length. Moreover given a triangle we define the inscribed circle of the triangle as the largest circle which fit inside the triangle, which is equivalent to each of the triangle's three sides being a tangent to the circle. Show that the radius of the inscribed circle of a Pythagorean triangle is always an integer.

5. By using Fermat's descent method prove that

   (a) $\sqrt{6}$ is irrational; (b) $\sqrt[3]{2}$ is irrational.

6. Show that the area of a Pythagorean triangle (see 4 for a definition) can never be equal to a square (i.e. n^2 for some $n \in \mathbb{N}$).

7. Prove that the Diophantine equation $x^4 - 4y^4 = z^2$ has no solutions in positive integers x, y, z.

8. Prove that there exists no Pythagorean triangle (see Problem 4 for a definition) whose area is twice a perfect square, that is $2n^2$ for $n \in \mathbb{N}$. (Hint: You can use Problem 7 without a proof).

9. Find all Pythagorean triangles whose areas are equal to their perimeter.

10. Prove that the equation $x^4 - y^4 = 2z^2$ has no solutions in positive integers x, y, z. (Hint: Show that $x \equiv y \pmod 2$, and hence $x^2 + y^2 = 2a^2$, $x + y = 2b^2$, and $x - y = 2c^2$ for some a, b, c.)