MATH3101 Fluid Mechanics

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Contents

4 Compressible flow

Recall the continuity and momentum equations for an ideal (inviscid) fluid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{4.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho}\nabla p. \tag{4.2}$$

If the fluid is *compressible*, meaning that ρ varies in space, then this is not a closed system because there are 5 unknown functions (u_1, u_2, u_3, p, ρ) but only 4 equations. To close the system, one needs an additional physical equation.

▶ Gases are the typical example of a compressible fluid. Plasmas are another common example.

▶ Rather confusingly, it is possible to have an incompressible flow (meaning $\nabla \cdot \boldsymbol{u} = 0$) even if the fluid itself is compressible (varying ρ), but this is not generic.

4.1 Barotropic fluids

A fluid is *barotropic* if its pressure is a function of density only:

$$p = P(\rho). \tag{4.3}$$

If we have a particular relation $P(\rho)$, then this will close our system of equations. For such a fluid, we can tidy the momentum equation by writing $\frac{1}{\rho}\nabla P = \nabla h$, so that

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} - \nabla h. \tag{4.4}$$

The function $h(\rho)$ is called the *enthalpy*, and since $\frac{1}{\rho}P'\nabla\rho = h'\nabla\rho$ we have

$$h(\rho) = \int_0^\rho \frac{P'(\sigma)}{\sigma}\,\mathrm{d}\sigma. \tag{4.5}$$

Proposition 4.1. *The unforced Euler equations for a barotropic fluid satisfy*

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \quad \textit{where} \quad E = \int_V \rho\left(\frac{1}{2}|\boldsymbol{u}|^2 + \epsilon(\rho)\right)\mathrm{d}V \quad \textit{and} \quad \epsilon(\rho) = \int_0^\rho \frac{P(\sigma)}{\sigma^2}\,\mathrm{d}\sigma.$$

Here $\epsilon(\rho)$ is the *internal energy density*. Unlike in the incompressible case, kinetic energy is no longer conserved because it can be converted to internal energy (effectively heat) by compressing the same mass into a smaller volume.

▶ You may see ϵ called *specific* internal energy, meaning internal energy *per unit mass*.

Example → *Polytropic gas*: $P(\rho) = k\rho^\gamma$.
This is a very common physical model. For such a gas the enthalpy is

$$h(\rho) = \int_0^\rho k\gamma\sigma^{\gamma-2}\,\mathrm{d}\sigma = \frac{k\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\gamma p}{(\gamma-1)\rho},$$

and the internal energy is

$$\epsilon(\rho) = \int_0^\rho k\sigma^{\gamma-2}\,\mathrm{d}\sigma = \frac{k\rho^{\gamma-1}}{\gamma-1} = \frac{p}{(\gamma-1)\rho}.$$

The constant γ is called the *polytropic index* and depends on the number of degrees of freedom of the molecules making up the gas. Specifically, $\gamma = (n_{\mathrm{dof}} + 2)/n_{\mathrm{dof}}$. Diatomic molecues such as N_2 or O_2 have $n_{\mathrm{dof}} = 5$ (3 translational + 1 rotational), so $\gamma = \frac{7}{5} = 1.4$. Thus air has $\gamma \approx 1.4$.

▶ Some physical background for ideal gases:

1. The internal energy density is related to temperature by

$$\epsilon = \frac{c_V T}{\mu},$$

where c_V is the *specific heat at constant volume* (*i.e.*, how much does the internal energy change when absorbing heat in a fixed volume), and μ is the molar mass. For an ideal gas, $c_V = \frac{1}{2} n_{\mathrm{dof}} R$, where $R \approx 8.3\,\mathrm{J\,mol^{-1}\,K^{-1}}$ is the *universal gas constant*.

2. Thermodynamics tells us that the *entropy*, S, of an ideal gas satisfies

$$S = c_V \log\left(p\rho^{-\gamma}\right) + \text{constant}.$$

In an *adiabatic* model, we assume that no heat can be transferred between fluid particles and $\mathrm{D}S/\mathrm{D}t = 0$. It follows that $p\rho^{-\gamma}$ remains constant for each fluid particle. A polytropic gas is a special case called *isentropic* where $p\rho^{-\gamma}$ is assumed to be the same constant everywhere.

Proof of Proposition 4.1. We consider each term separately, starting with the first term (kinetic energy). Since the volume is fixed, we can take the derivative inside the integral to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}\rho|\boldsymbol{u}|^2\,\mathrm{d}V = \int_V \frac{1}{2}\frac{\partial}{\partial t}\left(\rho|\boldsymbol{u}|^2\right)\,\mathrm{d}V \tag{4.6}$$

$$= \int_V \left(\frac{1}{2}|\boldsymbol{u}|^2\frac{\partial\rho}{\partial t} + \rho\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t}\right)\,\mathrm{d}V \tag{4.7}$$

$$= \int_V \left[\frac{1}{2}|\boldsymbol{u}|^2\frac{\partial\rho}{\partial t} + \rho\boldsymbol{u}\cdot\left(-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{1}{\rho}\nabla p\right)\right]\,\mathrm{d}V. \tag{4.8}$$

Now we use the identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$ to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}\rho|\boldsymbol{u}|^2\,\mathrm{d}V = \int_V \left(\frac{1}{2}|\boldsymbol{u}|^2\frac{\partial\rho}{\partial t} - \rho\boldsymbol{u}\cdot\nabla\left(\frac{1}{2}|\boldsymbol{u}|^2\right) - \boldsymbol{u}\cdot\nabla p\right)\,\mathrm{d}V \tag{4.9}$$

$$= \int_V \left(\frac{1}{2}|\boldsymbol{u}|^2\frac{\partial\rho}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2\nabla\cdot(\rho\boldsymbol{u}) - \boldsymbol{u}\cdot\nabla p\right)\,\mathrm{d}V - \oint_{\partial V}\frac{1}{2}|\boldsymbol{u}|^2\rho\boldsymbol{u}\cdot\,\mathrm{d}\boldsymbol{S} \tag{4.10}$$

$$= -\int_V \boldsymbol{u}\cdot\nabla p\,\mathrm{d}V. \tag{4.11}$$

Here we used the divergence theorem and the continuity equation. Notice that (4.11) holds whether or not the fluid is barotropic.

Now consider the internal energy term. From the chain rule and the definition of ϵ, we have $\dfrac{\partial\epsilon}{\partial t} =$

$\epsilon'(\rho)\dfrac{\partial \rho}{\partial t} = \dfrac{p}{\rho^2}\dfrac{\partial \rho}{\partial t}$ so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\epsilon \,\mathrm{d}V = \int_V \left(\epsilon\frac{\partial \rho}{\partial t} + \rho\frac{\partial \epsilon}{\partial t}\right)\,\mathrm{d}V \tag{4.12}$$
$$= \int_V \left(\epsilon + \frac{p}{\rho}\right)\frac{\partial \rho}{\partial t}\,\mathrm{d}V \tag{4.13}$$
$$= -\int_V \left(\epsilon + \frac{p}{\rho}\right)\nabla\cdot(\rho\boldsymbol{u})\,\mathrm{d}V \tag{4.14}$$
$$= \int_V \rho\boldsymbol{u}\cdot\nabla\left(\epsilon + \frac{p}{\rho}\right)\,\mathrm{d}V - \oint_{\partial V}\left(\epsilon + \frac{p}{\rho}\right)\rho\boldsymbol{u}\cdot\,\mathrm{d}\boldsymbol{S} \tag{4.15}$$
$$= \int_V \rho\boldsymbol{u}\cdot\left(\epsilon'(\rho)\nabla\rho + \frac{P'(\rho)}{\rho}\nabla\rho - \frac{p}{\rho^2}\nabla\rho\right)\,\mathrm{d}V \tag{4.16}$$
$$= \int_V \rho\boldsymbol{u}\cdot\left(\frac{p}{\rho^2}\nabla\rho + \frac{P'(\rho)}{\rho}\nabla\rho - \frac{p}{\rho^2}\nabla\rho\right)\,\mathrm{d}V \tag{4.17}$$
$$= \int_V \rho\boldsymbol{u}\cdot\frac{P'(\rho)}{\rho}\nabla\rho\,\mathrm{d}V \tag{4.18}$$
$$= \int_V \boldsymbol{u}\cdot\nabla p\,\mathrm{d}V. \tag{4.19}$$

This is equal and opposite to (4.11), hence the total energy E is conserved. □

► The opposite of barotropic is *baroclinic*, meaning that p is not a function only of ρ. This means that surfaces of constant p and ρ do not coincide, so $\nabla p \times \nabla\rho$ is large. Areas of high baroclinicity in the Earth's atmosphere are associated with the formation of cyclones.

4.2 Sound waves

The most important new feature of a compressible fluid is that it supports pressure oscillations, or *sound waves*. We will begin by studying linear (small amplitude) sound waves.

We start from the unforced compressible barotropic Euler equations:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0, \tag{4.20}$$
$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p, \tag{4.21}$$
$$p = P(\rho). \tag{4.22}$$

We will consider disturbances around a *basic state* with uniform density ρ_0, uniform pressure $p_0 = P(\rho_0)$, and zero velocity. Replacing $\rho \to \rho + \rho_0$ and $p \to p + p_0$ in (4.20) and (4.21) gives

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) + \rho_0\nabla\cdot\boldsymbol{u} = 0, \tag{4.23}$$
$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho+\rho_0}\nabla p, \tag{4.24}$$
$$p + p_0 = P(\rho + \rho_0). \tag{4.25}$$

4.2.1 Linearisation

To linearise these equations, we use Taylor expansion to write

$$\frac{1}{\rho+\rho_0} = \frac{1}{\rho_0} - \frac{\rho}{\rho_0^2} + \frac{\rho^2}{\rho_0^3} + \dots \tag{4.26}$$

and also

$$P(\rho+\rho_0) = P(\rho_0) + \rho P'(\rho_0) + \frac{\rho^2}{2!}P''(\rho_0) + \dots \tag{4.27}$$

$$= p_0 + \rho P'(\rho_0) + \frac{\rho^2}{2!}P''(\rho_0) + \dots, \tag{4.28}$$

so that the linearised equations are

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{u} = 0, \tag{4.29}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p, \tag{4.30}$$

$$p = P'(\rho_0)\rho. \tag{4.31}$$

We can derive an equation for the pressure perturbation p by differentiating (4.29) with respect to time:

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0 \nabla \cdot \frac{\partial \boldsymbol{u}}{\partial t} = \Delta p, \tag{4.32}$$

which implies

$$\boxed{\frac{\partial^2 p}{\partial t^2} = c_0^2 \Delta p \quad \text{with } c_0 = \sqrt{P'(\rho_0)}.} \tag{4.33}$$

This is the three-dimensional wave equation and c_0 is called the *sound speed*.

Example → Polytropic gas again, $P(\rho) = k\rho^\gamma$.
The sound speed is $c_0 = \sqrt{k\gamma\rho_0^{\gamma-1}} = \sqrt{\gamma(\gamma-1)\epsilon(\rho_0)}$. So, for such a gas, the sound speed is only a function of temperature.

► To calculate the speed for a given temperature, use $\epsilon = c_V T/\mu$, $\gamma = (n_{\rm dof}+2)/n_{\rm dof}$ and $c_V = \frac{1}{2}n_{\rm dof}R$, to see that $c_0 = \sqrt{\gamma RT/\mu}$.

For air, $\gamma = 1.4$ and $\mu = 28.97\,\mathrm{g\,mol^{-1}}$, so $c_0 = \begin{cases} 332\,\mathrm{m\,s^{-1}} & \text{at } T = 273\,\mathrm{K} \\ 344\,\mathrm{m\,s^{-1}} & \text{at } T = 293\,\mathrm{K}. \end{cases}$

► Newton got the speed of sound in air wrong. He estimated $c_0 \approx 290\,\mathrm{m\,s^{-1}}$ because he assumed the air to be *isothermal*, meaning that $T = \text{const.}$ so $P(\rho) = k\rho$, rather than polytropic.

To implement boundary conditions, it is more convenient to work in terms of the velocity. Taking the curl of (4.30) shows that sound waves do not change the vorticity. So if $\boldsymbol{\omega} = \mathbf{0}$ initially, we can write the velocity for all time in terms of an *acoustic velocity potential* $\phi(\boldsymbol{x}, t)$, where

$$\boldsymbol{u} = \nabla\phi. \tag{4.34}$$

To find the equation satisfied by ϕ, note that (4.30) gives

$$\nabla\frac{\partial \phi}{\partial t} = -\nabla\Big(c_0^2\frac{1}{\rho_0}\rho\Big). \tag{4.35}$$

Differentiating with respect to time and substituting (4.29),

$$\nabla\frac{\partial^2 \phi}{\partial t^2} = -\nabla\left(c_0^2\frac{1}{\rho_0}\frac{\partial \rho}{\partial t}\right) = \nabla\Big(c_0^2\Delta\phi\Big). \tag{4.36}$$

Ignoring the arbitrary function of time (which does not affect $\boldsymbol{u}$), we have

$$\boxed{\frac{\partial^2 \phi}{\partial t^2} = c_0^2\Delta\phi.} \tag{4.37}$$

In other words, ϕ satisfies the same wave equation as p.

► Exercise: show that the density ρ satisfies this wave equation too.

So in acoustics we typically solve for ϕ, then compute $\boldsymbol{u}$ and p by

$$\boldsymbol{u} = \nabla\phi, \quad p = -\rho_0\frac{\partial \phi}{\partial t}. \tag{4.38}$$

4.2.2 One-dimensional solutions

If we assume that $\phi = \phi(x,t)$, then (4.37) reduces to

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \frac{\partial^2 \phi}{\partial x^2}. \tag{4.39}$$

Differentiation and substitution shows that any travelling plane wave of the form

$$\phi(x,t) = X(x - c_0 t) \tag{4.40}$$

satisfies (4.39).

In fact, the general solution to (4.39) may be written as a superposition of left-moving and right-moving plane waves. To see this, apply the change of variables

$$u(x,t) = x - c_0 t, \qquad v(x,t) = x + c_0 t, \tag{4.41}$$

to transform (4.39) into

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0. \tag{4.42}$$

Integrating with respect to first u then v, we find that the general solution is a superposition

$$\phi(x,t) = F(u) + G(v) \tag{4.43}$$

$$= F(x - c_0 t) + G(x + c_0 t). \tag{4.44}$$

In other words, a superposition of left- and right-moving components. Another way of looking at it is that

$$F \text{ is constant along lines } x - c_0 t = x_0, \tag{4.45}$$

$$G \text{ is constant along lines } x + c_0 t = x_0. \tag{4.46}$$

So information propagates along these lines, at speed c_0.

► The curves $x \pm c_0 t = x_0$ are called *characteristics*. We will see later how they are useful for nonlinear equations.

We can use (4.44) to solve the initial value problem for (4.39).

Proposition 4.2 (D'Alembert's solution)**.** *The initial value problem*

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \frac{\partial^2 \phi}{\partial x^2}, \qquad \phi(x,0) = \phi_0(x), \qquad \frac{\partial \phi}{\partial t}(x,0) = \dot{\phi}_0(x)$$

has solution

$$\phi(x,t) = \frac{1}{2}\Big[\phi_0(x - c_0 t) + \phi_0(x + c_0 t)\Big] + \frac{1}{2c_0}\int_{x-c_0 t}^{x+c_0 t} \dot{\phi}_0(s)\,\mathrm{d}s.$$

► Note that these initial conditions are equivalent to specifying both $\boldsymbol{u}$ and p as functions of x.

Proof. The first initial condition is

$$\phi_0(x) = F(x) + G(x), \tag{4.47}$$

The second condition is

$$\dot{\phi}_0(x) = -c_0 F'(x) + c_0 G'(x), \tag{4.48}$$

$$\implies \int_a^x \dot{\phi}_0(s)\,\mathrm{d}s = -c_0 F(x) + c_0 G(x), \tag{4.49}$$

where a is an arbitrary point. Solving (4.47) and (4.49) simultaneously gives

$$F(x) = \frac{1}{2}\phi_0(x) - \frac{1}{2c_0}\int_a^x \dot{\phi}_0(s)\,\mathrm{d}s, \tag{4.50}$$

$$G(x) = \frac{1}{2}\phi_0(x) + \frac{1}{2c_0}\int_a^x \dot{\phi}_0(s)\,\mathrm{d}s. \tag{4.51}$$

Thus we arrive at the required expression. □

Example → Gaussian pulse $u_x(x,0) = 0$, $p(x,0) = \rho_0 \mathrm{e}^{-x^2/a^2}$.
This is a (crude) model for a bursting balloon: initially there is an excess of pressure inside the balloon, but all of the air is at rest.

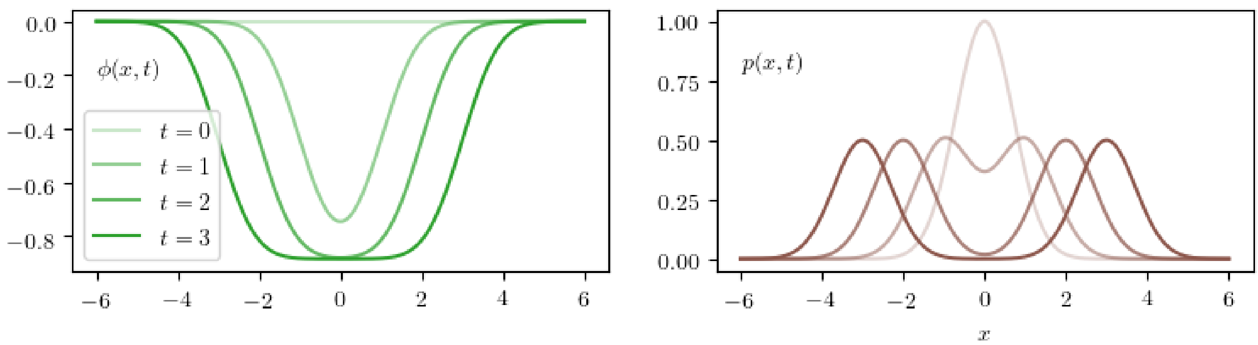
These initial conditions correspond to $\phi_0(x) = 0$, $\dot{\phi}_0(x) = -\mathrm{e}^{-x^2/a^2}$ in Proposition 4.2, so the solution is

$$\begin{aligned}\phi(x,t) &= -\frac{1}{2c_0}\int_{x-c_0t}^{x+c_0t} \mathrm{e}^{-s^2/a^2}\,\mathrm{d}s\\ &= -\frac{1}{2c_0}\int_0^{x+c_0t} \mathrm{e}^{-s^2/a^2}\,\mathrm{d}s + \frac{1}{2c_0}\int_0^{x-c_0t} \mathrm{e}^{-s^2/a^2}\,\mathrm{d}s\\ &= \frac{a\sqrt{\pi}}{4c_0}\left[-\mathrm{erf}\left(\frac{x+c_0t}{a}\right) + \mathrm{erf}\left(\frac{x-c_0t}{a}\right)\right].\end{aligned}$$

Differentiating, we see that the corresponding pressure perturbation is

$$p(x,t) = -\rho_0\frac{\partial\phi}{\partial t} = \frac{\rho_0}{2}\exp\left[-\frac{(x+c_0t)^2}{a^2}\right] + \frac{\rho_0}{2}\exp\left[-\frac{(x-c_0t)^2}{a^2}\right].$$

So, as expected from the general solution (4.44), the initial disturbance lauches two identical pulses of the same shape moving in opposite directions at constant speed c_0 – in the picture below I took $a = c_0 = \rho_0 = 1$:



Example → Numerical solution of the nonlinear equations.
To illustrate what happens in the full nonlinear equations, I have solved them numerically for the Gaussian pulse initial condition, using the `Lare2D` code by Tony Arber[1].

The code solves the 2D Euler equations for an ideal gas, so I used the domain $x \in [-6, 6]$ and $y \in [-0.1, 0.1]$, discretizing with a regular mesh of 256 cells in x and 2 in y (because there is no variation in y in this 1D problem).

The `Lare2D` code works in terms of $\boldsymbol{u}$, ρ and ϵ (internal energy), so I need to set the initial conditions in these variables. I set $\boldsymbol{u}(x,0) = \boldsymbol{0}$, and

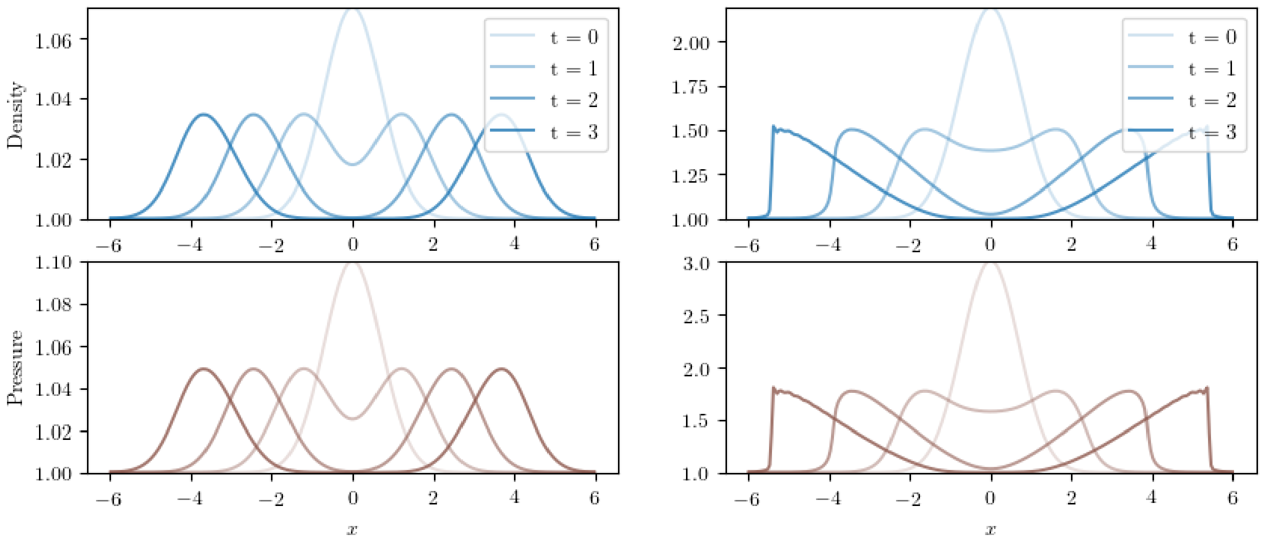
$$\rho(x,0) = \left(1 + \rho_1\mathrm{e}^{-x^2}\right)^{1/\gamma}.$$

[1] Written in MPI-Fortran and available from `https://warwick.ac.uk/fac/sci/physics/research/cfsa/people/tda/larexd/`. This 1D simulation runs in a few seconds on my desktop machine.

Then $p(x,0) = \rho^\gamma(x,0) = 1 + \rho_1 e^{-x^2}$, which is equivalent to the previous example. (The constant 1 ensures that the background density is non-zero, otherwise the code will crash!) To be consistent, we need to set the initial condition

$$\epsilon(x,0) = \frac{p(x,0)}{(\gamma-1)\rho(x,0)} = \frac{\big[\rho(x,0)\big]^{\gamma-1}}{\gamma-1}.$$

I took $\gamma = 1.4$ (for air). The plots below show the evolution of ρ and p for two values of ρ_1, corresponding to small- and large-amplitude perturbations.



Notice that the sound speed corresponding to the background density $\rho = 1$ is $c = \sqrt{\gamma\rho^{\gamma-1}} = \sqrt{\gamma} \approx 1.18$. So this is the (approximate) speed of the small-amplitude pulse. Similarly the linearised equations say that $p = c_0^2\rho$, and indeed the amplitude of the initial pressure perturbation is about 1.4 times that of the density perturbation.

However, in this nonlinear calculation the sound speed c can vary in space and time. In the large-amplitude case, the effect is to make the Gaussian pulses steepen, forming “shocks” (the loud bang we hear when the balloon bursts).

▶ We will explain this nonlinear steepening in Section 4.3.

Now consider a finite domain $D = \{x : 0 < x < L\}$. As for water waves, we can then look for standing wave solutions of the form

$$\phi(x,t) = X(x)\sin(\omega t). \tag{4.52}$$

Substituting this into (4.39) gives the ODE

$$-\omega^2 X = c_0^2 X''. \tag{4.53}$$

The solution ϕ must satisfy suitable boundary conditions at $x = 0$ and $x = L$, which depend on the specific problem.

Example → Model for a flute.

The simplest model of a flute is a tube with two open ends where $p = 0$. Since $p = -\rho_0\dfrac{\partial\phi}{\partial t}$, this means that we need $X(0) = X(L) = 0$. The general solution of (4.53) is

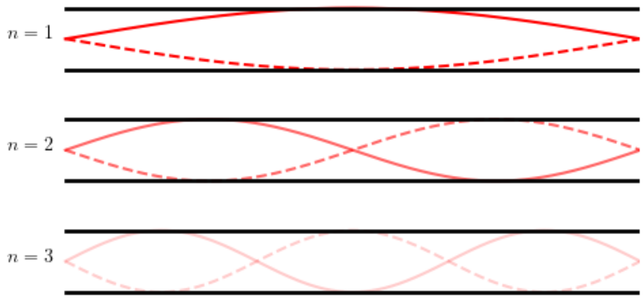
$$X(x) = A\sin(kx) + B\cos(kx), \quad \text{where } k = \frac{\omega}{c_0}.$$

The boundary condition $X(0) = 0$ implies that $B = 0$, so $X(x) = A\sin(kx)$. The condition $X(L) = 0$ then requires

$$A\sin(kL) = 0 \qquad \implies k = \frac{n\pi}{L} \quad \text{for } n \in \mathbb{Z}.$$

So there is a discrete spectrum of possible modes with $\omega = \dfrac{n\pi c_0}{L}$. Each mode has the form

$$\phi(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c_0 t}{L}\right).$$



The $n = 1$ mode is called the *fundamental* and higher n modes are called *overtones*. Experienced players can "overblow" so as to excite the second harmonic (an octave higher) rather than the fundamental.

Note that this solution does not contradict (4.44), because it may be written (using trig identities) in the alternative form

$$\phi(x,t) = \frac{1}{2}\cos\left(\frac{n\pi}{L}(x - c_0 t)\right) + \frac{1}{2}\cos\left(\frac{n\pi}{L}(x + c_0 t)\right).$$

At $x = 0, L$ the two travelling waves cancel each other out so that $\phi = 0$ and the boundary condition is satisfied.

▶ How does this measure up to a real flute? With $c_0 = 350\,\mathrm{m\,s^{-1}}$ (warm, moist air!) and $L = 0.66\,\mathrm{m}$, our model predicts that the fundamental mode should have frequency $\dfrac{\omega}{2\pi} = \dfrac{c_0}{2L} = 265.2\,\mathrm{Hz}$. In reality, a flute is constructed so that the fundamental mode is "middle C" at $261.6\,\mathrm{Hz}$. (The remaining error mostly comes from the fact that p is not quite zero at the ends – in reality, it has to match some exterior flow.)

4.2.3 Three-dimensional solutions

In general, the solution to (4.37) will be a superposition of plane waves travelling in different directions. Each component will have the form

$$\phi(\boldsymbol{x},t) = F(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) \tag{4.54}$$

for some fixed vector $\boldsymbol{k}$ whose direction represents the direction of motion. Differentiating this expression, we have

$$\frac{\partial^2\phi}{\partial t^2} = \omega^2 F''(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t), \tag{4.55}$$

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = (k_x^2 + k_y^2 + k_z^2)F''(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t), \tag{4.56}$$

so this will be a solution to (4.37) providing that ω satisfies the dispersion relation

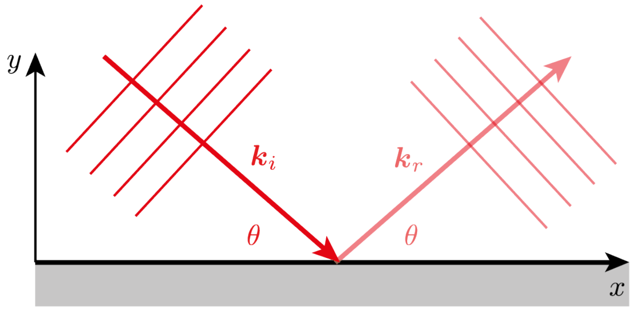
$$\omega^2 = c_0^2|\boldsymbol{k}|^2. \tag{4.57}$$

The corresponding velocity perturbation is

$$\boldsymbol{u}(\boldsymbol{x},t) = \nabla\phi = \boldsymbol{k}F'(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t), \tag{4.58}$$

so we see that the oscillations are in the direction of propagation, $\boldsymbol{k}$. In other words, sound waves are *longitudinal.*

Example → Reflection of a plane wave $\phi_i(x, y, t) = \exp\left(ik_x x - ik_y y - i\omega t\right)$ at a rigid wall, $y = 0$. This is an echo. First we will use separation of variables, then we will show an easier "trick".



Method 1 – Separation of variables. Start from the ansatz $\phi(x, y, t) = X(x)Y(y)\exp(-i\omega t)$. Inserting this into (4.37) gives

$$-\omega^2 = c_0^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right)$$

which separates again to give

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{\omega^2}{c_0^2} = k_x^2.$$

The solutions have the form

$$X(x) = A\exp(ik_x x) + B\exp(-ik_x x), \qquad Y(y) = C\exp(ik_y y) + D\exp(-ik_y y),$$

where $k_y^2 = \dfrac{\omega^2}{c_0^2} - k_x^2$. The constants A, B, C, D are then determined by boundary conditions. Firstly at the wall we need to impose $u_y = \dfrac{\partial\phi}{\partial y} = 0$, which requires $Y'(0) = 0$. It follows that

$$ik_y C - ik_y D = 0 \qquad \Longrightarrow \; C = D.$$

Secondly, as $x \to -\infty$ we should obtain only the incident wave ϕ_i, travelling to the right. Thus $B = 0$. Matching the overall normalisation of the incident wave, we therefore have

$$\begin{aligned}\phi(x, y, t) &= \exp(ik_x x - i\omega t)\Big[\exp(ik_y y) + \exp(-ik_y y)\Big] \\ &= \exp(ik_x x - ik_y y - i\omega t) + \underbrace{\exp(ik_x x + ik_y y - i\omega t).}_{\text{reflected wave}}\end{aligned}$$

If θ is the *angle of incidence*, so that the wavevector of the incident wave is $\boldsymbol{k}_i = \sqrt{k_x^2 + k_y^2}(\cos\theta\boldsymbol{e}_x - \sin\theta\boldsymbol{e}_y)$, then we see that the wave vector of the reflected wave is $\boldsymbol{k}_r = \sqrt{k_x^2 + k_y^2}(\cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_y)$. This recovers the well-known fact that angle of reflection equals angle of incidence.

Method 2 – Method of images. A mathematical trick for arriving at the solution more easily is to recognise that we can achieve $u_y = 0$ at the wall by superimposing the incident wave ϕ_i with an *image* wave $\widetilde{\phi}$ that is the reflection of ϕ_i in the wall $(y \to -y)$. Thus we postulate directly the solution

$$\phi(x, y, t) = \exp(ik_x x - ik_y y - i\omega t) + \exp(ik_x x + ik_y y - i\omega t).$$

▶ The method of images works because the wave equation is linear, so we can superimpose solutions. It also relies on the uniqueness of solutions (see the PDEs course).

We saw in 1-d that a domain of finite *length* supports a discrete spectrum of standing wave modes. Similarly, a domain of finite *width*, called a *waveguide*, supports a discrete spectrum of propagating modes.

Example → Planar waveguide $\{(x, y) : -\infty < x < \infty, 0 < y < a\}$.
We look for a solution propagating in x of the form $\phi(x, y, t) = Y(y)\exp(ik_x x - i\omega t)$. Substituting into (4.37) gives

$$-\omega^2 Y = c_0^2\Big(Y'' - k_x^2 Y\Big) \quad \Longrightarrow Y'' = -k_y^2 Y \quad \text{where } k_y^2 = \frac{\omega^2}{c_0^2} - k_x^2.$$

The boundary conditions require $u_y = 0$ on $y = 0$ and $y = a$, so we need $Y'(0) = Y'(a) = 0$. This shows that

$$Y(y) = \cos(k_y y), \quad \text{with } k_y = \frac{n\pi}{a} \text{ for } n = 0, 1, 2, \ldots.$$

Again, there is a discrete spectrum of modes. The allowed modes therefore take the form

$$\phi(x, y, t) = \cos\left(\frac{n\pi y}{a}\right)\exp(ik_x x - i\omega t),$$

provided that ω satisfies the dispersion relation

$$\omega = c_0\sqrt{k_x^2 + \frac{n^2\pi^2}{a^2}}.$$

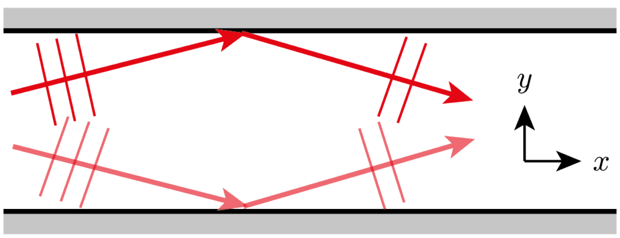
Notice the following:

1. A plane wave with no y-dependence (the $n = 0$ mode) can propagate for any ω. But a mode with $n > 0$ can exist only for $\omega > c_0\sqrt{n^2\pi^2/a^2}$. Thus for $\omega \le c_0\sqrt{\pi^2/a^2}$, called the *cutoff frequency*, only a plane wave can propagate.

2. The phase speed in the x-direction, ω/k_x, is a function of n, so sound waves of different frequencies progress along the waveguide at different speeds. (This explains why the speech of someone shouting down a long corridor gets garbled before it reaches the listener.)

3. It appears that the waves are travelling faster than the speed of sound, since $\omega/k_x > c_0$. However, the mode is actually made up of two travelling waves travelling at an angle and bouncing off the walls. To see this, note that

$$\begin{aligned}\cos\left(\frac{n\pi y}{a}\right)\exp(ik_x x - i\omega t) &= \frac{\exp\left(\frac{in\pi y}{a}\right) + \exp\left(\frac{-in\pi y}{a}\right)}{2}\exp(ik_x x - i\omega t)\\ &= \frac{1}{2}\exp\left(\frac{in\pi y}{a} + ik_x x - i\omega t\right) + \frac{1}{2}\exp\left(-\frac{in\pi y}{a} + ik_x x - i\omega t\right).\end{aligned}$$



Each of these waves has wavenumber $|\boldsymbol{k}| = \sqrt{k_x^2 + \frac{n^2\pi^2}{a^2}}$, so their phase speed is $\frac{\omega}{|\boldsymbol{k}|} = c_0$, as expected. The waves bounce off the walls at angle $\alpha = \pm\arctan\left(\frac{k_y}{k_x}\right) = \pm\arctan\left(\frac{n\pi}{k_x a}\right)$.

Similar behaviour is found in a two-dimensional square tube (see problem sheet) – here we will look at a cylindrical tube.

Example → Cylindrical waveguide $\{(r, \theta, z) : 0 < r < a, 0 < \theta < 2\pi, -\infty < z < \infty\}$.
We will look for axisymmetric solutions of the form $\phi(r, z, t) = R(r)\exp(ik_z z - i\omega t)$. In cylindrical coordinates, (4.37) then gives

$$-\omega^2 R = c_0^2 \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rR') - k_z^2 R \right).$$

This ODE differs from the Cartesian case. Rearranging, we have

$$R'' + \frac{1}{r}R' + \left(\frac{\omega^2}{c_0^2} - k_z^2 \right) R = 0.$$

This is Bessel's equation of order 0 in disguise. If we change variable to $s = \left(\frac{\omega^2}{c_0^2} - k_z^2 \right)^{1/2} r$, then we find that $R(s)$ satisfies

$$\frac{\mathrm{d}^2 R}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}R}{\mathrm{d}s} + R = 0.$$

So the solution is the Bessel function $R(s) = J_0(s)$ (the other type of Bessel function $Y_n(s)$ is no good as it is unbounded as $s \to 0$). Therefore

$$R(r) = J_0\left[\left(\frac{\omega^2}{c_0^2} - k_z^2 \right)^{1/2} r \right]$$

We still need to impose the boundary condition $u_r = 0$ on $r = a$, meaning $R'(a) = 0$. So we need

$$J_0'\left[\left(\frac{\omega^2}{c_0^2} - k_z^2 \right)^{1/2} a \right] = 0.$$

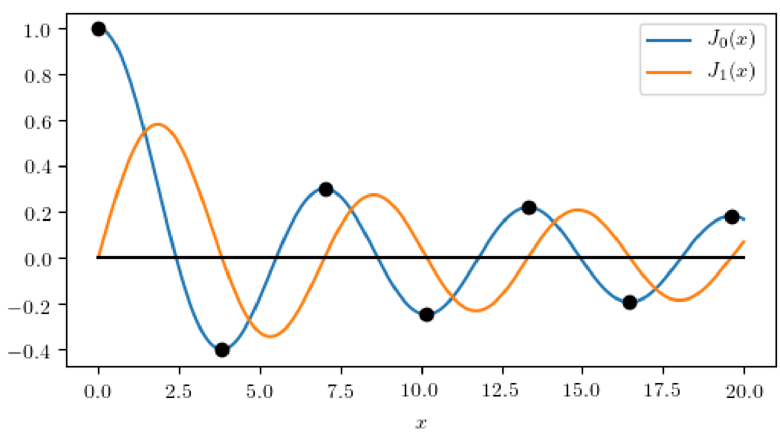
Because J_0 has a finite sequence of turning points, this will give us a discrete spectrum of ω, just like the boundary condition in the planar waveguide. Let j_n be the nth turning point of J_0, so that

$$\left(\frac{\omega^2}{c_0^2} - k_z^2 \right)^{1/2} a = j_n, \quad n = 0, 1, \ldots.$$

and hence

$$\phi(r, z, t) = J_0\left(\frac{j_n r}{a} \right) \exp(ik_z z - i\omega t), \quad \text{where } \omega = c_0\sqrt{k_z^2 + \frac{j_n^2}{a^2}}.$$

The plot below shows the first few turning points of J_0 (black dots):



As for the planar waveguide, $j_0 = 0$ so $n = 0$ corresponds to a plane wave with no r-dependence. Again the modes with $n > 0$ have a cutoff frequency $\omega = c_0\sqrt{j_n^2/a^2}$.

▶ Notice that J_0 is rather like cosine, while J_1 is rather like sine, but with a radial dependence.

▶ Computing the above figure: there is a useful identity $J_0'(z) = -J_1(z)$, so turning points of J_0 are actually roots of J_1. You can compute the Bessel roots in Python using `scipy.special.jn_zeros`.

4.3 Nonlinearity

In this section we will examine the effect of nonlinearity in the Euler equations, using the method of characteristics. We will consider flow in only one spatial dimension, $\boldsymbol{u} = u(x,t)\boldsymbol{e}_x$.

To illustrate the idea, start by looking only at the inertial terms in the momentum equation (the left-hand side),

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{4.59}$$

called the *inviscid Burgers equation*.

Suppose we have a curve $x(t)$ in the (x,t)-plane where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\big(x(t),t\big), \qquad x(0) = x_0. \tag{4.60}$$

Then along such a curve (4.59) may be written as

$$\frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x} = 0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = 0. \tag{4.61}$$

Such curves, along which a PDE reduces to an ODE, are called *characteristics*.

Example → Solve the inviscid Burgers equation with initial condition $u(x,0) = u_0(x) = x$.
Since u is constant along the characteristics, and they have slope $\dfrac{\mathrm{d}x}{\mathrm{d}t} = u$, they must be straight lines.
Integrating (4.60) gives

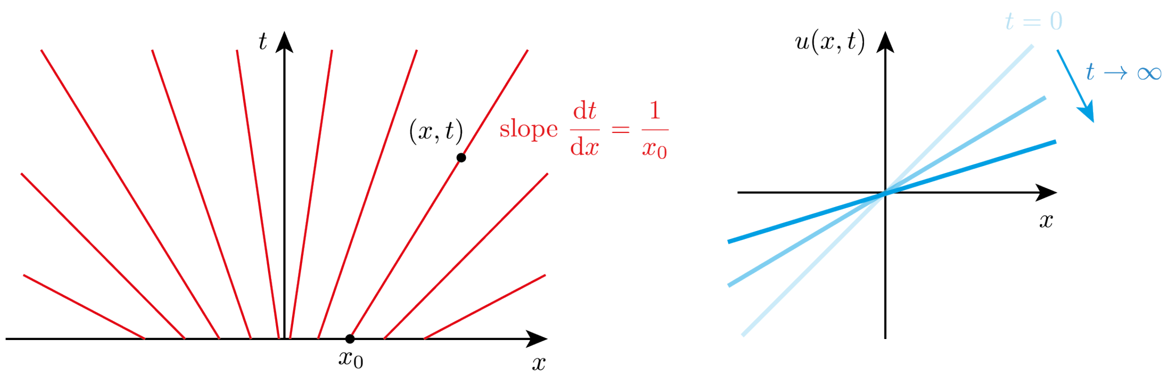
$$x - x_0 = u\big(x_0, 0\big)t = u_0\big(x_0\big)t = x_0 t.$$

In this case we can invert to find the x-intercept of the characteristic passing through (x,t), finding

$$x_0 = \frac{x}{1+t}.$$

Equation (4.61) then gives

$$u(x,t) = u_0\left(\frac{x}{1+t}\right) = \frac{x}{1+t}.$$



If the fluid to the left moves *faster* than the fluid to the right, then it will "catch up" and the characteristics will intersect.

Example → Solve the inviscid Burgers equation with initial condition $u_0(x) = -x$.
This time the characteristics are the lines

$$x - x_0 = -x_0 t,$$

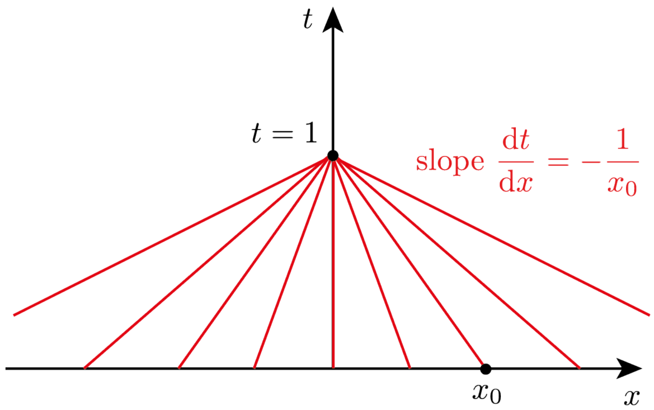
so

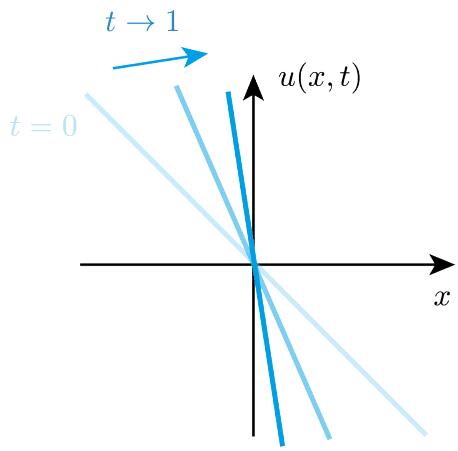
$$x_0 = \frac{x}{1-t}.$$

The solution is then

$$u(x,t) = \frac{x}{t-1},$$

which has a finite-time singularity as $t \to 1$. At this time, all of the characteristics intersect at $x = 0$, so that the solution is not well-defined there.





We could also have computed the intersection time by looking for points where the characteristic curve x is independent of x_0, so that

$$\frac{\partial x}{\partial x_0} = 0 \quad \iff \quad 1 - t = 0 \quad \iff \quad t = 1.$$

▶ This is essentially the reason why the fluid equations are "nasty": they contain this nonlinear term that wants to make the solution blow up!

Now let us put back in the other terms in the 1-d Euler equations. For simplicity we will consider a polytropic gas $p = P(\rho) = k\rho^\gamma$. So the continuity and momentum equations are

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{4.62}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{k\gamma\rho^{\gamma-1}}{\rho}\frac{\partial \rho}{\partial x} = 0, . \tag{4.63}$$

We will see that these may be reduced to two sets of ODEs if we first eliminate ρ in favour of the variable

$$c(\rho) = \sqrt{P'(\rho)} = \sqrt{k\gamma\rho^{\gamma-1}}. \tag{4.64}$$

We will see later that it still makes sense to call this the *sound speed*, although it now varies in space. To eliminate ρ from (4.62) and (4.63), we use the fact that

$$\frac{\partial c}{\partial t} = c'(\rho)\frac{\partial \rho}{\partial t} = \sqrt{k\gamma}\frac{\gamma-1}{2}\rho^{(\gamma-3)/2}\frac{\partial \rho}{\partial t} = \frac{\gamma-1}{2}\frac{c}{\rho}\frac{\partial \rho}{\partial t} \qquad \implies \frac{\partial \rho}{\partial t} = \frac{2\rho}{(\gamma-1)c}\frac{\partial c}{\partial t}. \tag{4.65}$$

In the same way,

$$\frac{\partial c}{\partial x} = c'(\rho)\frac{\partial \rho}{\partial x} \qquad \Longrightarrow \frac{\partial \rho}{\partial x} = \frac{2\rho}{(\gamma-1)c}\frac{\partial c}{\partial x}. \tag{4.66}$$

Substituting these into (4.62) and (4.63) we find

$$c\frac{\partial u}{\partial x} + \frac{2}{\gamma-1}\frac{\partial c}{\partial t} + \frac{2u}{\gamma-1}\frac{\partial c}{\partial x} = 0, \tag{4.67}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2c}{\gamma-1}\frac{\partial c}{\partial x} = 0. \tag{4.68}$$

The trick is then to add or subtract these two equations, to find

$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)\left(u + \frac{2c}{\gamma-1}\right) = 0, \tag{4.69}$$

$$\left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)\left(u - \frac{2c}{\gamma-1}\right) = 0. \tag{4.70}$$

Thus

$$F_+ = u + \frac{2c}{\gamma-1} \text{ is constant on curves } x_+(t) \text{ satisfying } \frac{\mathrm{d}x_+}{\mathrm{d}t} = u + c, \tag{4.71}$$

$$F_- = u - \frac{2c}{\gamma-1} \text{ is constant on curves } x_-(t) \text{ satisfying } \frac{\mathrm{d}x_-}{\mathrm{d}t} = u - c. \tag{4.72}$$

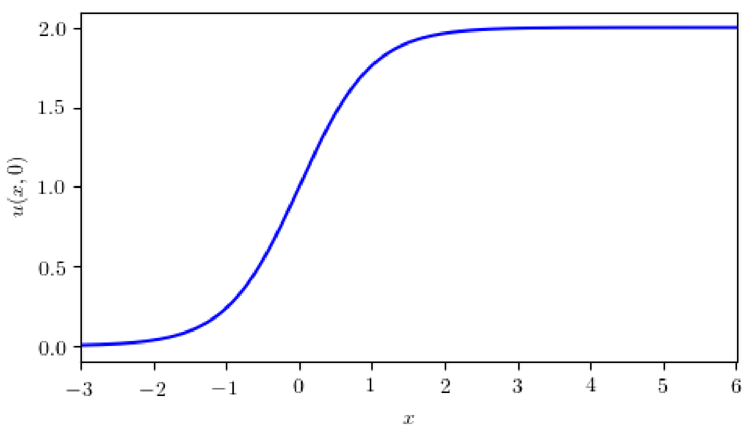
The functions F_+ and F_- are called *Riemann invariants*. Now there are two families of characteristics, known as the $+$ and $-$ characteristics. They need no longer be straight lines since u and c need no longer be constant along them.

▶ Named after the (famous) mathematician Bernhard Riemann who wrote a paper on gas dynamics, *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, in 1860.

▶ Although this derivation was rather *ad hoc*, there is a more systematic procedure for finding characteristics of hyperbolic systems like this, studied in more detail in the PDEs course.

In general it is hard to find closed-form solutions for the characteristics. However, it is possible in some cases, called *simple waves*, in which one of the Riemann invariants is constant across a whole region.

Example → Expansion wave with $u(x,0) = U_0\big[1 + \tanh(x_0)\big], \quad c(x,0) = c_0 + \frac{1}{2}(\gamma-1)u(x,0)$.
This initial velocity profile looks as follows:



At $t = 0$, the Riemann invariants have values

$$F_+(x,0) = u(x,0) + \frac{2c(x,0)}{\gamma-1} = 2u(x,0) + \frac{2c_0}{\gamma-1},$$

$$F_-(x,0) = u(x,0) - \frac{2c(x,0)}{\gamma-1} = -\frac{2c_0}{\gamma-1}.$$

Since F_- is constant along every $-$ characteristic, it must therefore be constant everywhere in $t > 0$.

The other Riemann invariant, F_+, is constant along $+$ characteristics, but takes different values on each. In particular, $F_+(x,t) = F_+\big(x_0, 0\big)$, where x_0 is the x-intercept of a given $+$ characteristic. So along this characteristic we have

$$u = \frac{F_+(x_0,0) + F_-}{2} = u(x_0, 0),$$

which implies that u is constant along each $+$ characteristic. Similarly, along a $+$ characteristic we have

$$c = \frac{\gamma - 1}{4}\big(F_+(x_0,0) - F_-\big) = c_0 + \frac{\gamma-1}{2}u(x_0,0),$$

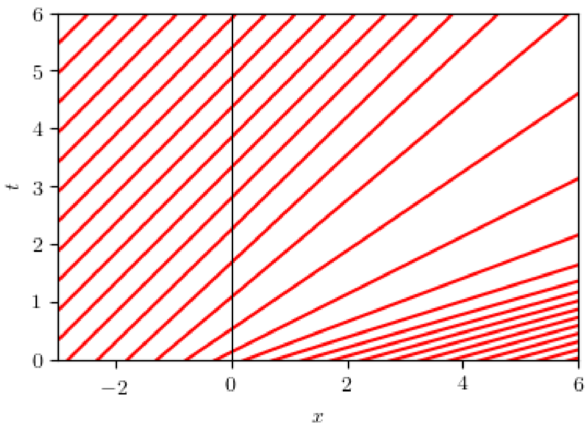
so c is also constant along a $+$ characteristic. Therefore the $u + c$ is constant and the $+$ characteristics are straight lines by (4.71). They have the form

$$x_+ - x_0 = (u+c)t = \left[c_0 + \frac{\gamma+1}{2}u(x_0,0)\right]t.$$

For our specific initial conditions,

$$x_+ - x_0 = \left\{c_0 + \frac{\gamma+1}{2}U_0\big[1 + \tanh(x_0)\big]\right\}t. \qquad (*)$$

The $+$ characteristics look as follows (for $c_0 = U_0 = 1$ and $\gamma = \frac{5}{3}$):



From (*), with these parameters, we see that as $x_0 \to \infty$, the characteristic slopes are $\dfrac{\mathrm{d}t}{\mathrm{d}x} \to \dfrac{3}{11}$, while in the limit $x_0 \to -\infty$, we have $\dfrac{\mathrm{d}t}{\mathrm{d}x} \to 1$. This agrees with the plot.

To find the solutions for $u(x,t)$ and $c(x,t)$ explicitly, we would need to invert (*) to find $x_0(x,t)$, which needs to be done numerically. Nevertheless, we can see that the solution will take the form of an *expansion wave*, and that it will exist for all time.

The $-$ characteristics are not straight lines, and would also have to be found numerically. But in this case we don't need to know them to find the solution.

► In the above example, since u is constant on $+$ characteristics, the solution will (formally) take the form $u(x,t) = F\big(x - [u+c]t\big)$. In other words, it is a disturbance moving at speed c relative to the fluid. Thus we are justified in calling c the sound speed. If U_0 is small, then $c \to c_0$, the sound speed from linear theory.

Just like with the inviscid Burgers equation, if we have a region of fast flow catching up a region of slow flow, the characteristics will eventually intersect.

Example → Compression wave with $u(x,0) = U_0\big[1 - \tanh(x_0)\big]$, $\quad c(x,0) = c_0 + \frac{1}{2}(\gamma-1)u(x,0)$.
This is the same as the previous example but the + characteristics are now

$$x_+ - x_0 = \left\{c_0 + \frac{\gamma+1}{2}U_0\big[1 - \tanh(x_0)\big]\right\}t.$$

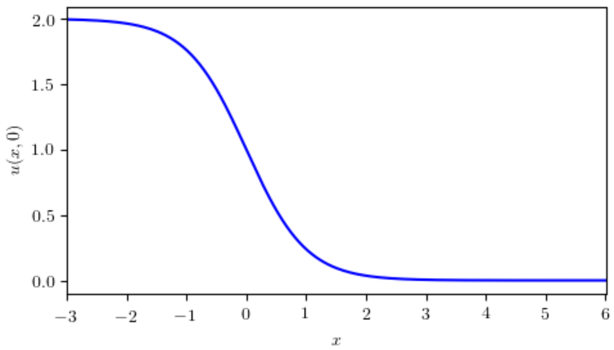
Now there is a region of faster flow to the left so the wave will steepen until the + characteristics intersect at some finite time. We can find this time using the fact that

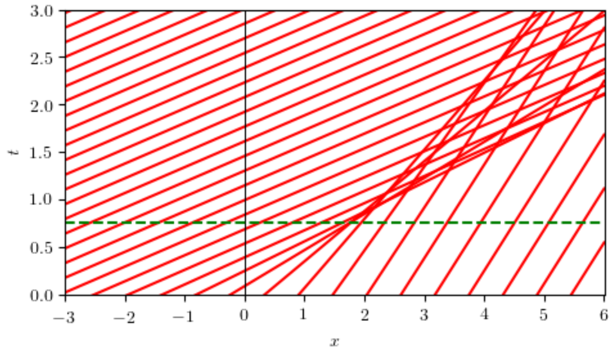
$$\frac{\partial x_+}{\partial x_0} = 0 \quad \iff 1 - \frac{\gamma+1}{2}U_0\,\mathrm{sech}^2(x_0)t = 0 \quad \implies t = \frac{2}{(\gamma+1)U_0}\cosh^2(x_0).$$

This gives the time of intersection for the characteristic starting at $x = x_0$. The earliest intersection will be

$$t_s = \min_{x_0}\left[\frac{2}{(\gamma+1)U_0}\cosh^2(x_0)\right] = \frac{2}{(\gamma+1)U_0},$$

shown by the dashed green line below.





► After the characteristics have intersected, we can no longer find a classical solution of the Euler equations. There is a large body of theory dedicated to finding *weak solutions*, which satisfy the equations except along curves called *shocks* where u and ρ (or c) can be discontinuous. The properties of shocks are constrained by the underlying integral conservation laws (mass, momentum, energy), but this is beyond the scope of this course.

5 Instability

The basic question is simple: how robust is a given equilibrium solution to small disturbances? In fluid mechanics, this can help explain why some equilibria are observed in nature while others are not. It can also explain why simple flows often break down and become turbulent.

We say that a given flow is *stable* if all perturbations which are small initially remain small for all time. It is *unstable* if at least one perturbation which is initially small grows to become large. Mathematically, a flow is *stable* if, for all $\epsilon > 0$ there exists $\delta(\epsilon)$ such that

$$|\boldsymbol{u}(\boldsymbol{x}, 0)|,\ |\rho(\boldsymbol{x}, 0)|,\ |p(\boldsymbol{x}, 0)| < \delta \tag{5.1}$$

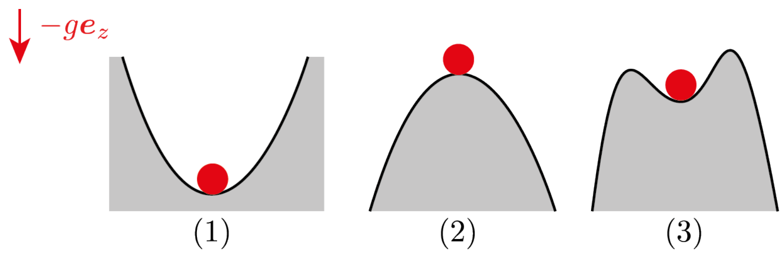
guarantees

$$|\boldsymbol{u}(\boldsymbol{x}, t)|,\ |\rho(\boldsymbol{x}, t)|,\ |p(\boldsymbol{x}, t)| < \epsilon \quad \text{for all } t > 0. \tag{5.2}$$

Since the definition is concerned with small perturbations, it makes sense to investigate *linear stability* where we consider behaviour under the linearised equations of motion.

Unfortunately, the conclusion of a linear stability analysis does not always transfer to the full nonlinear equations.

Example → Analogy of a particle in a potential.
In the sketches below, (1) is linearly and nonlinearly stable, (2) is linearly unstable, and (3) is linearly stable but nonlinearly unstable (*i.e.*, unstable to large enough disturbances).



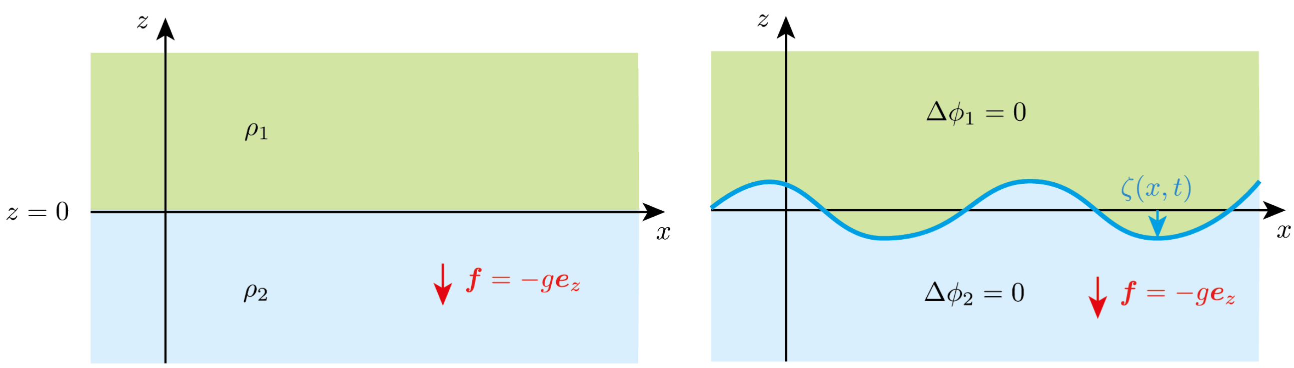
► In this course, we will consider only some classic case studies of linear stability.

5.1 Rayleigh-Taylor instability

This is an instability that occurs at an interface between two fluids of different density. We suppose the two fluids are incompressible, with initial density

$$\rho(x, z, 0) = \begin{cases} \rho_1 & z > 0, \\ \rho_2 & z < 0, \end{cases} \tag{5.3}$$

and downward gravitational body force as shown in the left-hand picture:



The right-hand picture shows the situation after the interface is disturbed, where $z = \zeta(x,t)$ is the interface (so $\zeta(x,0) = 0$). The governing equations are similar to those for water waves, except that we don't have $p = 0$ above the interface. So

1. Assuming irrotational flow, we have $\boldsymbol{u}_1 = \nabla\phi_1$ and $\boldsymbol{u}_2 = \nabla\phi_2$ with

$$\Delta\phi_1 = 0 \quad \text{for } z > \zeta, \qquad \Delta\phi_2 = 0 \quad \text{for } z < \zeta. \tag{5.4}$$

2. As $z \to \infty$ we assume $\boldsymbol{u}_1, \phi_1 \to 0$, and as $z \to -\infty$, we assume $\boldsymbol{u}_2, \phi_2 \to 0$.

3. The kinematic boundary condition holds at the interface, so

$$\frac{\partial\phi_1}{\partial z} = \frac{\mathrm{D}\zeta}{\mathrm{D}t} = \frac{\partial\phi_2}{\partial z} \quad \text{on } z = \zeta. \tag{5.5}$$

4. Bernoulli's theorem for potential flow tells us that

$$\nabla\Big(\frac{\partial\phi}{\partial t} + \frac{p}{\rho_0} + \frac{1}{2}|\nabla\phi|^2 + gz\Big) = 0, \tag{5.6}$$

$$\iff \quad \nabla\Big(\rho_0\frac{\partial\phi}{\partial t} + p + \frac{\rho_0}{2}|\nabla\phi|^2 + \rho_0 gz\Big) = 0. \tag{5.7}$$

throughout the fluid, where ϕ, p and ρ_0 are the local velocity potential, pressure and density. Unlike for water waves we do not assume $p = 0$ at the interface, but we do assume that p is continuous there, so

$$\rho_1\frac{\partial\phi_1}{\partial t} + \frac{\rho_1}{2}|\nabla\phi_1|^2 + g\rho_1\zeta = \rho_2\frac{\partial\phi_2}{\partial t} + \frac{\rho_2}{2}|\nabla\phi_2|^2 + g\rho_2\zeta \quad \text{at } z = \zeta. \tag{5.8}$$

This needs tightening up. The Bernoulli functions differ by $f(t) = f_1(t) - f_2(t)$. At $t = 0$, we have $\rho_1 f_1 = \rho_2 f_2$ so they indeed cancel. How to argue that they cancel still at later times? (Maybe from conditions at $|z| \to \infty$??

Now we linearise the equations and consider small disturbances to the steady state $\zeta = 0$. Similar to water waves, linearising these equations leads to

$$\Delta\phi_1 = 0 \quad \text{for } z > 0, \tag{5.9}$$

$$\Delta\phi_2 = 0 \quad \text{for } z < 0, \tag{5.10}$$

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_2}{\partial z} \quad \text{at } z = 0, \tag{5.11}$$

$$\rho_1\frac{\partial\phi_1}{\partial t} + g\rho_1\zeta = \rho_2\frac{\partial\phi_2}{\partial t} + g\rho_2\zeta \quad \text{at } z = 0. \tag{5.12}$$

As for water waves, try the ansatz

$$\phi_1(x,z,t) = \hat{\phi}_1(z)\mathrm{e}^{i(kx-\omega t)}. \tag{5.13}$$

Substituting this into (5.9) and using the boundary condition $\phi_1 \to 0$ as $z \to \infty$ shows that

$$\hat{\phi}_1(z) = a_1\mathrm{e}^{-kz}. \tag{5.14}$$

Similarly,

$$\phi_2(x,z,t) = a_2\mathrm{e}^{kz}\mathrm{e}^{i(kx-\omega t)}. \tag{5.15}$$

For the interface, let us try a similar ansatz $\zeta(x,t) = \zeta_0\mathrm{e}^{i(kx-\omega t)}$. Substituting this into (5.11) gives

$$-i\omega\zeta_0 = -ka_1 = ka_2. \tag{5.16}$$

So we need $a_1 = -a_2 = a$ and

$$\zeta(x,t) = \frac{ka}{i\omega}\mathrm{e}^{i(kx-\omega t)}. \tag{5.17}$$

Finally, substituting our expressions for ϕ_1, ϕ_2 and ζ into (5.12) gives

$$-i\omega\rho_1 a + g\rho_1\frac{ka}{i\omega} = i\omega\rho_2 a + g\rho_2\frac{ka}{i\omega} \tag{5.18}$$

$$\iff \boxed{\omega^2 = \frac{g(\rho_2-\rho_1)k}{\rho_1+\rho_2}.} \tag{5.19}$$

This is the dispersion relation for linear waves at such an interface.

► Notice that when $\rho_1 = 0$ this reduces to $\omega^2 = gk$, which was the dispersion relation for waves on the surface of deep water.

As for surface gravity waves ($\rho_1 = 0$) the system will support linear waves, provided that $\rho_2 > \rho_1$ (so that ω is real). But if $\rho_2 < \rho_1$ (denser fluid on top), then

$$\omega = \pm i\sigma, \qquad \text{with } \sigma = \sqrt{\frac{g(\rho_1-\rho_2)k}{\rho_1+\rho_2}}. \tag{5.20}$$

The corresponding perturbations will take the form $\zeta(x,t) \sim \mathrm{e}^{\pm\sigma t}$, and will therefore exhibit exponential growth, with *growth rate* σ. This is called the *Rayleigh-Taylor instability.*

► A simple demonstration of this is to turn a glass of water upside down!

Example → Numerical solution of the nonlinear equations.
The full Euler equations have to be solved numerically for this problem, so I will illustrate here using the `Lare2D` code by Tony Arber[2].

Here I discretized the Euler equations on the rectangular domain $x \in [-\frac{1}{2}, \frac{1}{2}]$, $z \in [-2, 2]$ using a regular mesh with 512 points in x and 1024 in z. I imposed periodic boundary conditions in x and set $\boldsymbol{u} = \boldsymbol{0}$ on the z-boundaries. I set $g = 1$.

The `Lare2D` code works in terms of $\boldsymbol{u}$, ρ and ϵ (internal energy), so initial conditions were required for these variables. I simply set $\boldsymbol{u}(x,z,0) = \boldsymbol{0}$. The density was initialized with a weak sinusoidal perturbation of the form

$$\rho(x,z,0) = \begin{cases} \rho_1 & z > -\frac{1}{20}\cos(2\pi x), \\ \rho_2 & z < -\frac{1}{20}\cos(2\pi x). \end{cases}$$

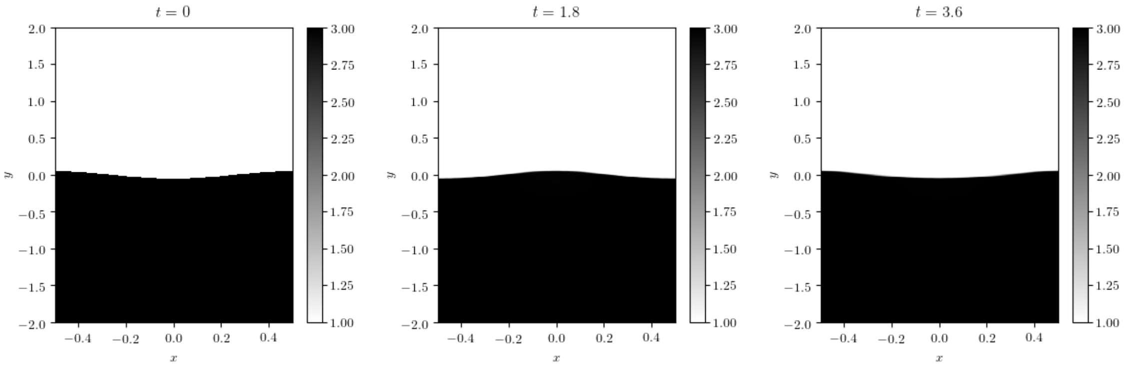
The internal energy then had to be set to ensure that the initial condition was in hydrostatic equilibrium with this density profile. Last term we found that hydrostatic equilibrium requires a pressure $p = p_0 - g\rho z$, so the corresponding internal energy should be

$$\epsilon = \frac{p}{(\gamma-1)\rho} = \frac{1}{(\gamma-1)\rho}\Big(p_0 - g\rho z\Big).$$

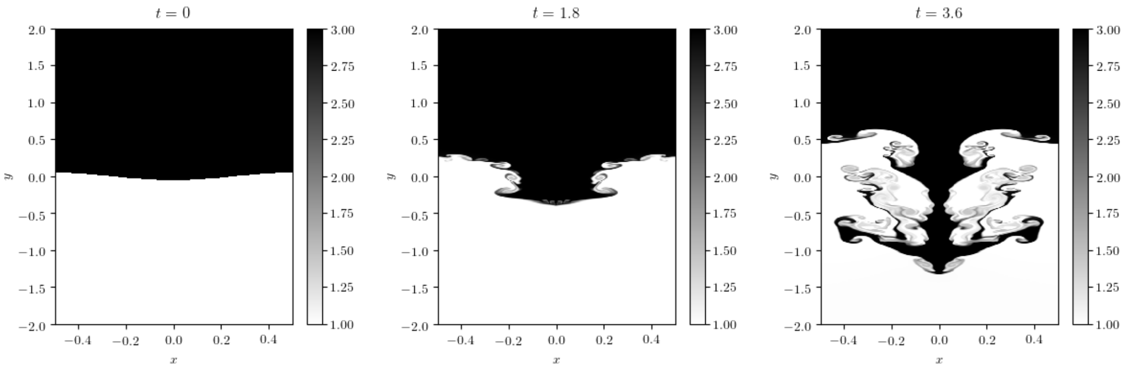
For the record, I took $\gamma = \frac{5}{3}$ and chose $p_0 = 10$ to ensure that $p > 0$ throughout the domain.

1. Stable ($\rho_1 = 1$, $\rho_2 = 3$). Here the interface just oscillates at the frequency of the perturbation.

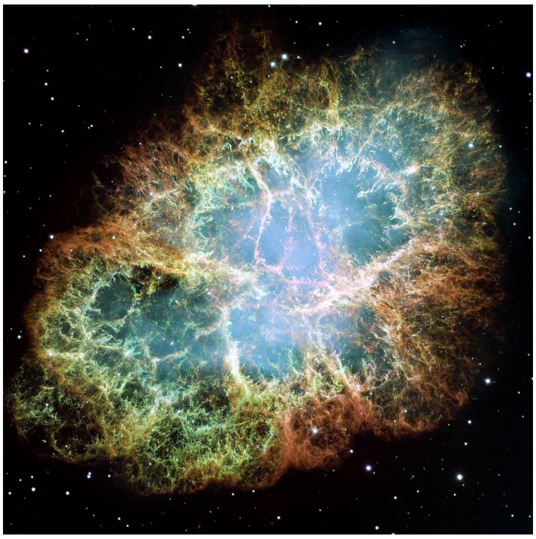
[2]Written in MPI-Fortran and available from `https://warwick.ac.uk/fac/sci/physics/research/cfsa/people/tda/larexd/`. These simulations run in a few minutes on my desktop machine.



2. Unstable ($\rho_1 = 3$, $\rho_2 = 1$). Here we observe the Rayleigh-Taylor instability. Notice how the disturbances develop first at small scales, because small scales (large k) have the largest growth rate σ.



► Although Lord Rayleigh originally investigated the problem described above, it is named also for G. I. Taylor because he recognised that this instability will occur more generally whenever a dense fluid is accelerated into a less dense fluid. For example, it is responsible for the filamentary structure in supernova remnants such as the Crab nebula (image: NASA and STScI):



Here denser gas is pushing outward, driven not by gravity but by the blast wave from the explosion.

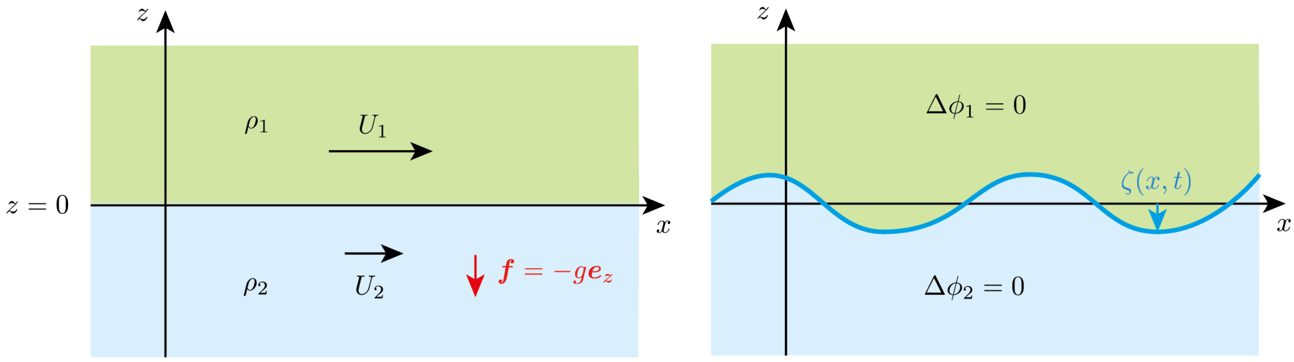
5.2 Kelvin-Helmholtz instability

This is another instability that occurs at an interface between two fluids, when the fluids are moving horizontally at different speeds. The basic flow is no longer zero but now a *vortex sheet*

$$\boldsymbol{u}(x,z,0) = \begin{cases} U_1\boldsymbol{e}_x & z>0, \\ U_2\boldsymbol{e}_x & z<0, \end{cases} \tag{5.21}$$

where we assume U_1 and U_2 are constants (for simplicity). Again we assume that both fluids are inviscid and incompressible with density

$$\rho(x,z,0) = \begin{cases} \rho_1 & z>0, \\ \rho_2 & z<0. \end{cases} \tag{5.22}$$



Denoting the interface again as $z=\zeta(x,t)$, with $\zeta(x,0)=0$, the governing equations are similar to Rayleigh-Taylor:

1. We again assume that the perturbed flow is irrotational, so that $\boldsymbol{u}_1 = U_1\boldsymbol{e}_x + \nabla\phi_1$ and $\boldsymbol{u}_2 = U_2\boldsymbol{e}_x + \nabla\phi_2$, with
$$\Delta\phi_1 = 0 \quad \text{for } z>\zeta, \qquad \Delta\phi_2 = 0 \quad \text{for } z<\zeta. \tag{5.23}$$
▶ In assuming the disturbance to be irrotational, we are following Lord Kelvin's original analysis. It means that we are only considering the stability of irrotational disturbances. So we can prove instability but we cannot prove stability since it could be that all irrotational perturbations are linearly stable but some rotational one is not.

2. As $z\to\infty$ we assume $\boldsymbol{u}_1 \to U_1\boldsymbol{e}_x$, so assume $\phi_1\to 0$. Similarly, as $z\to-\infty$, we assume $\boldsymbol{u}_2\to U_2\boldsymbol{e}_x$ and $\phi_2\to 0$.

3. The kinematic boundary condition is the same as before, so
$$\frac{\partial\phi_1}{\partial z} = \frac{\mathrm{D}\zeta}{\mathrm{D}t} = \frac{\partial\phi_2}{\partial z} \quad \text{on } z=\zeta. \tag{5.24}$$

4. To apply the Bernoulli theorem, we need to write $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in terms of an overall velocity potential, so $\boldsymbol{u}_1 = \nabla\big(U_1x+\phi_1\big)$ and $\boldsymbol{u}_2 = \nabla\big(U_2x+\phi_2\big)$. So now at $z=\zeta$, with continuity of pressure, we have

$$\rho_1\frac{\partial\phi_1}{\partial t} + \frac{\rho_1}{2}\big|\nabla(U_1x+\phi_1)\big|^2 + g\rho_1\zeta = \rho_2\frac{\partial\phi_2}{\partial t} + \frac{\rho_2}{2}\big|\nabla(U_2x+\phi_2)\big|^2 + g\rho_2\zeta, \tag{5.25}$$

$$\iff \rho_1\left(\frac{\partial\phi_1}{\partial t} + \frac{U_1^2}{2} + U_1\frac{\partial\phi_1}{\partial x} + \frac{1}{2}|\nabla\phi_1|^2 + g\zeta\right) = \rho_2\left(\frac{\partial\phi_2}{\partial t} + \frac{U_2^2}{2} + U_2\frac{\partial\phi_2}{\partial x} + \frac{1}{2}|\nabla\phi_2|^2 + g\zeta\right). \tag{5.26}$$

We can simplify this further by considering the initial equilibrium. Putting in $\phi_1 = \phi_2 = \zeta = 0$ shows that we must have

$$\rho_1 \frac{U_1^2}{2} = \rho_2 \frac{U_2^2}{2}. \tag{5.27}$$

Thus the condition at $z = \zeta$ is

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} + \frac{1}{2}|\nabla \phi_1|^2 + g\zeta \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} + \frac{1}{2}|\nabla \phi_2|^2 + g\zeta \right). \tag{5.28}$$

The linearisation is now modified. As before, we have

$$\Delta\phi_1 = 0 \quad \text{for } z > 0, \tag{5.29}$$
$$\Delta\phi_2 = 0 \quad \text{for } z < 0. \tag{5.30}$$

However, the kinematic boundary condition is modified because we get an extra linear term from $\frac{\mathrm{D}\zeta}{\mathrm{D}t} \approx \frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x}$, and similarly for U_2. So the linearised form of this equation is now

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \phi_1}{\partial z} - U_1 \frac{\partial \zeta}{\partial x} = \frac{\partial \phi_2}{\partial z} - U_2 \frac{\partial \zeta}{\partial x} \quad \text{at } z = 0. \tag{5.31}$$

Linearising (5.28) gives

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} + g\zeta \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} + g\zeta \right). \tag{5.32}$$

As for Rayleigh-Taylor, we can satisfy (5.29) and (5.30), as well as the boundary conditions at $z \to \pm\infty$ with

$$\phi_1(x, z, t) = a_1 \mathrm{e}^{-kz} \mathrm{e}^{i(kx - \omega t)}, \qquad \phi_2(x, z, t) = a_2 \mathrm{e}^{kz} \mathrm{e}^{i(kx - \omega t)}. \tag{5.33}$$

Now trying our ansatz $\zeta(x, t) = \zeta_0 \mathrm{e}^{i(kx - \omega t)}$ in the new (5.31) gives

$$-i\omega\zeta_0 = -ka_1 - ik\zeta_0 U_1 = ka_2 - ik\zeta_0 U_2, \tag{5.34}$$

so we have

$$a_1 = i\zeta_0 \left(\frac{\omega}{k} - U_1 \right), \qquad a_2 = i\zeta_0 \left(U_2 - \frac{\omega}{k} \right). \tag{5.35}$$

Putting this in (5.32) then gives

$$\rho_1 \Big(-i\omega a_1 + ikU_1 a_1 + g\zeta_0 \Big) = \rho_2 \Big(-i\omega a_2 + ikU_2 a_2 + g\zeta_0 \Big) \tag{5.36}$$

$$\iff \quad \rho_1 \left(\frac{\omega}{k} - U_1 \right) (\omega - U_1 k) + \rho_1 g = \rho_2 \left(U_2 - \frac{\omega}{k} \right) (\omega - U_2 k) + \rho_2 g \tag{5.37}$$

$$\iff \quad \boxed{\Big[\rho_1 + \rho_2\Big]\omega^2 - 2\Big[(\rho_1 U_1 + \rho_2 U_2)k\Big]\omega + \Big[(\rho_1 U_1^2 + \rho_2 U_2^2)k^2 + (\rho_1 - \rho_2)gk\Big] = 0.} \tag{5.38}$$

This is the dispersion relation, which is a quadratic in ω.

► Notice that, when $U_1 = U_2 = 0$, this reduces to the Rayleigh-Taylor dispersion relation (5.19), as it should.

Instability requires a positive imaginary part of ω for (at least) one of the solutions to the quadratic. This will occur if and only if $b^2 - 4ac < 0$, namely

$$4(\rho_1 U_1 + \rho_2 U_2)^2 k^2 - 4(\rho_1 + \rho_2)\Big[(\rho_1 U_1^2 + \rho_2 U_2^2)k^2 + (\rho_1 - \rho_2)gk\Big] < 0 \tag{5.39}$$

$$\iff \quad k > \frac{g(\rho_2^2 - \rho_1^2)}{\rho_1 \rho_2 (U_1 - U_2)^2}. \tag{5.40}$$

This is called *Kelvin-Helmholtz* instability, and you can see that it requires $U_1 \neq U_2$. If $\rho_2 > \rho_1$, then the flow is stable below a certain value of k.

▶ In a sense, the instability is not surprising, since we started with a δ-function of vorticity.

Example → Numerical solution of the nonlinear equations.
I again solved the Euler equations using `Lare2D`. The domain is $x \in [-2, 2]$ and $z \in [-\frac{1}{2}, \frac{1}{2}]$, with periodic boundary conditions in x. I chose $U_1 = 1$ and $U_2 = 0$, so the boundary conditions in z are $\boldsymbol{u}(x, \frac{1}{2}) = \boldsymbol{e}_x$, $\boldsymbol{u}(x, -\frac{1}{2}) = \boldsymbol{0}$. For simplicity, I set $\rho_1 = \rho_2 = 1$ and $g = 0$. The internal energy ϵ was then initialised in the same way as before.

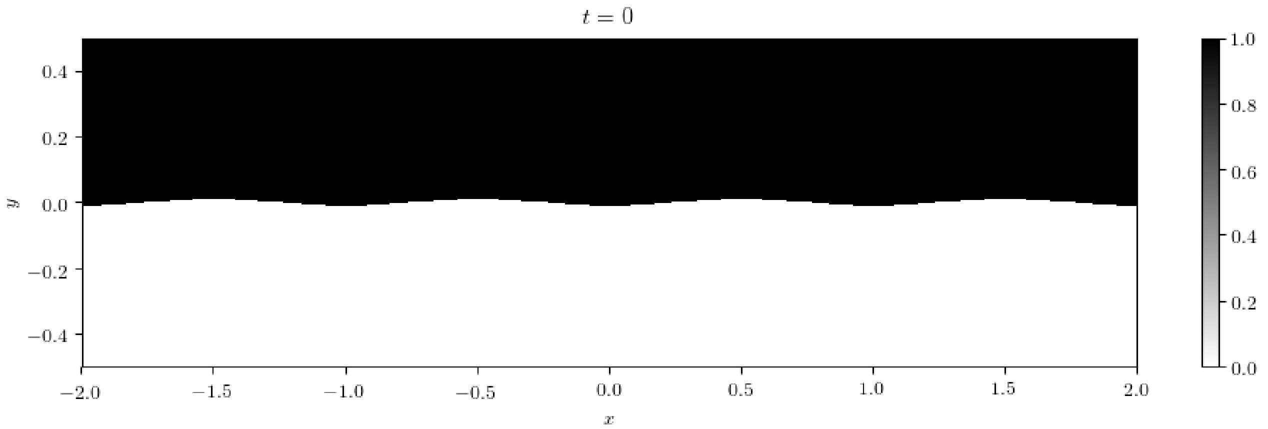
To initialize the perturbation I set

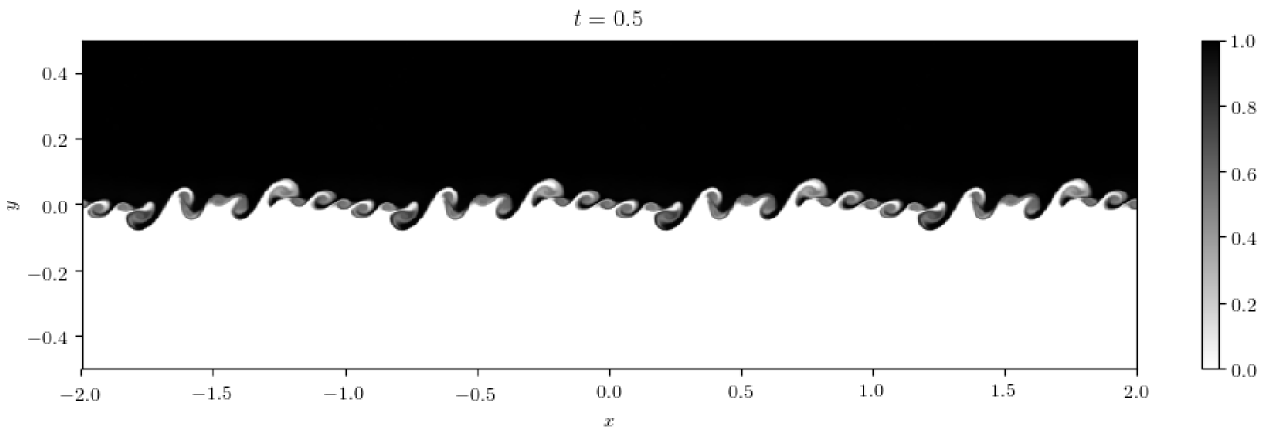
$$\boldsymbol{u}(x, z, 0) = \begin{cases} \boldsymbol{e}_x & z > -\frac{1}{100}\cos(2\pi x), \\ \boldsymbol{0} & z < -\frac{1}{100}\cos(2\pi x). \end{cases}$$

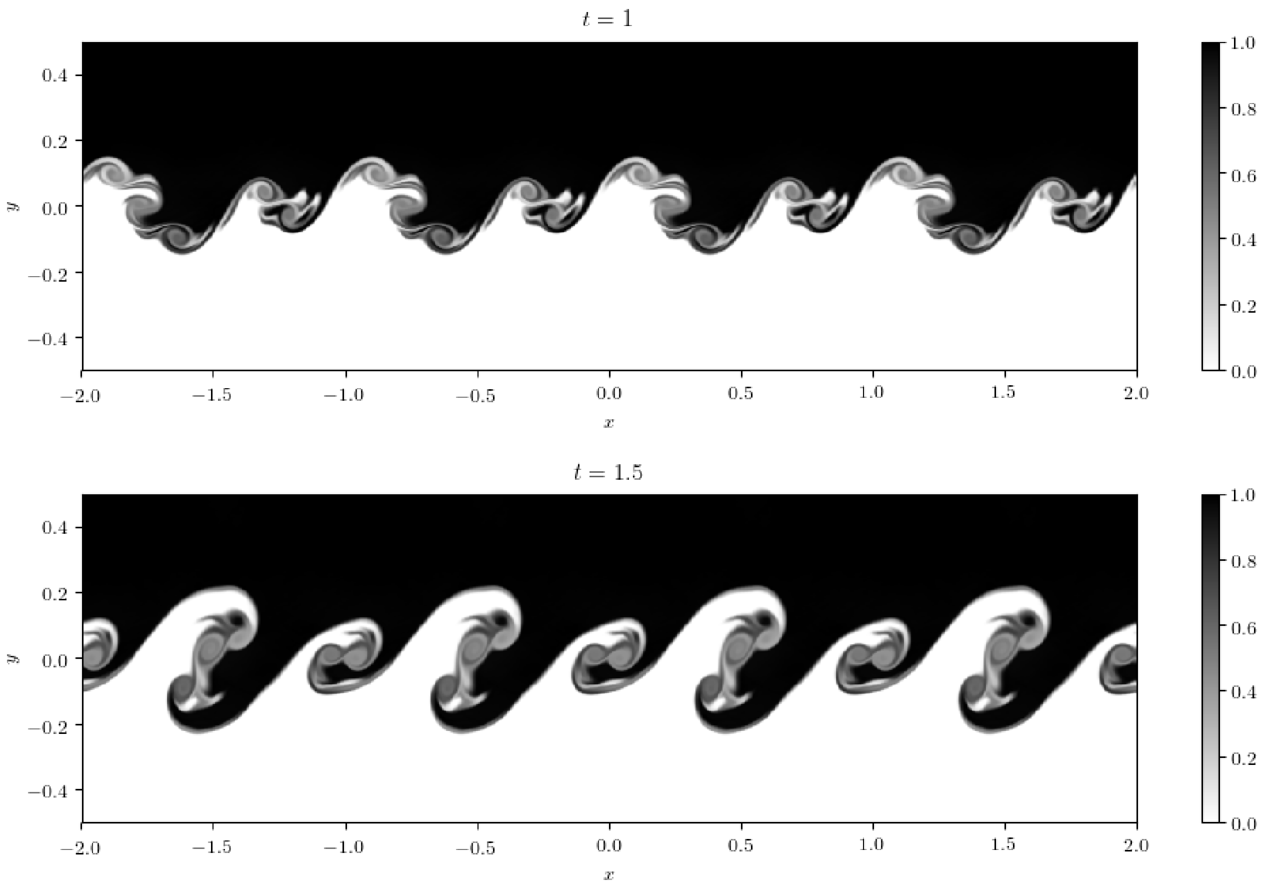
To track the interface visually I evolved also a *passive scalar* field $s(x, z, t)$, which started with the distribution

$$s(x, z, 0) = \begin{cases} 1 & z > -\frac{1}{100}\cos(2\pi x), \\ 0 & z < -\frac{1}{100}\cos(2\pi x) \end{cases}$$

and was evolved according to $\dfrac{\mathrm{D}s}{\mathrm{D}t} = 0$. This has no effect on the evolution but allows us to follow the interface. The instability breaks the vortex sheet up into discrete vortices (or “rolls”), as shown by the evolution of s below:







The grey areas where $0 < s < 1$ occur because numerical error is smoothing out large gradients of s, inevitable due to the finite evolution. The true solution of the Euler equations would always have either $s = 0$ or $s = 1$ everywhere. In real fluids, however, viscosity will act in the same way to smooth out these discontinuities once they become thin enough. This is the subject of the next chapter…

▶ The Kelvin-Helmholtz instability is sometimes seen in cloud layers in the Earth's atmosphere, and may be observed on the gas giant planets Jupiter and Saturn.

5.3 Turbulence

The complex, chaotic motion that results from instabilities such as Rayleigh-Taylor or Kelvin-Helmholtz is called *turbulence*. Although the flow still solves the same fluid equations, the sensitive dependence on initial conditions means that it cannot be computed in practice for long times. Instead, it needs to be modelled statistically (beyond the scope of this course).

▶ Richard Feynman described turbulence as "the most important unsolved problem in classical physics." It remains a major area of research.

6 Dynamics of viscous fluids

So far we have studied only ideal fluids with no *viscosity*, meaning no friction (shear stresses) between neighbouring fluid elements. But clearly viscosity is sometimes important.

Example → Different fluids have different viscosity.
Intuitively, oil and honey are more viscous than water or air.

▶ Some fluids are extremely viscous: the pitch in the University of Queensland Pitch Drop Experiment `https://smp.uq.edu.au/pitch-drop-experiment` has a viscosity approximately 2.3×10^{11} times that of water!

Example → Air flowing over water or a solid body.
In an ideal fluid, the flowing air would exert no force on the body/water, because there are no shear stresses. So there could be no wind-generated water waves, and very little drag on cars, swimmers, etc.

6.1 Tensors

In order to derive the equation of motion for a viscous fluid, we will need to know a bit more about tensors.

A rank n *tensor* is an object with n indices whose 3^n components $T_{ijk\ldots}$ transform under a rotation of the coordinate frame according to

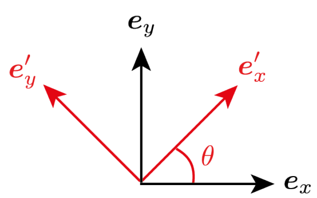
$$T'_{ijk\ldots} = \underbrace{R_{ip}R_{jq}R_{kr}\cdots}_{n \text{ copies of } R} T_{pqr\ldots} \tag{6.1}$$

where R_{ij} is the rotation matrix.

Example → Rank 0 tensors.
A rank 0 tensor has 0 indices and hence $3^0 = 1$ component, so is just a scalar function T. Under a rotation of the coordinates, it does not change [you do not multiply by any rotation matrices].

Example → Rank 1 tensors.
A rank 1 tensor has 1 index and hence $3^1 = 3$ components. So it is a vector T_i. Under a rotation of the coordinates, the new components are $T'_i = R_{ip}T_p$, which is the usual transformation rule for a vector.

For example, let $\boldsymbol{T} = \boldsymbol{e}_x$, and consider rotating the coordinate axes through angle θ about the z-axis to give new axes $\boldsymbol{e}'_x$, $\boldsymbol{e}'_y$.



This corresponds to the rotation matrix $R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so

$$T'_i = R_{ip}T_p = R_{i1} \quad \Longrightarrow \boldsymbol{T} = \cos\theta \boldsymbol{e}'_x - \sin\theta \boldsymbol{e}'_y.$$

Example → Rank 2 tensors.
A rank 2 tensor has 2 indices, T_{ij} and hence $3^2 = 9$ components. It is a 3×3 matrix but with a rule (6.1) about how the components behave under a coordinate rotation.

The simplest example of a (non-zero) rank 2 tensor is the Kronecker delta, δ_{ij}. It has components

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Under a coordinate rotation, the new components of this tensor would be

$$\delta'_{ij} = R_{ip}R_{jq}\delta_{pq} = R_{ip}R_{jp} = \left[RR^\top\right]_{ij} = \left[\mathbb{I}\right]_{ij} = \delta_{ij},$$

where I used the fact that rotation matrices are orthogonal ($R^\top = R^{-1}$). So we see that the components of this particular tensor are unchanged under rotation of the coordinate axes.

A tensor (of any rank) whose components are unchanged under rotation of the coordinate axes is called *isotropic*.

Example → The stress tensor.
Another example of a rank 2 tensor is the stress tensor $\boldsymbol{\sigma}$. For an ideal fluid, we saw last term that $\sigma_{ij} = -p\delta_{ij}$, so this is also isotropic. In other words, there is no "preferred direction". This will no longer be true for a viscous fluid.

▶ In fact, any isotropic rank 2 tensor must have the form $\alpha\delta_{ij}$ [see problem sheet].

Example → Rank 3 tensors.
A rank 3 tensor has 3 indices, T_{ijk} and hence $3^3 = 27$ components. The most familiar rank 3 tensor is the antisymmetric tensor

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two of } i, j, k \text{ are equal,} \\ 1 & \text{if } ijk \text{ is an even permutation of 123,} \\ -1 & \text{if } ijk \text{ is an odd permutation of 123.} \end{cases}$$

Given two vectors $\boldsymbol{a}$, $\boldsymbol{b}$, the product $\epsilon_{ijk}a_jb_k$ gives a rank 1 tensor with 1 remaining index, namely a vector with components $\big(\boldsymbol{a} \times \boldsymbol{b}\big)_i$. Given another vector $\boldsymbol{c}$, the product $\epsilon_{ijk}c_ia_jb_k$ leaves no remaining indices, and gives the scalar $\boldsymbol{c} \cdot \big(\boldsymbol{a} \times \boldsymbol{b}\big)$.

▶ In fact, ϵ_{ijk} is also isotropic. To see this, start by writing

$$\begin{aligned} \epsilon'_{ijk} &= R_{ip}R_{jq}R_{kr}\epsilon_{pqr} \\ &= R_{i1}R_{j2}R_{k3} + R_{i2}R_{j3}R_{k1} + R_{i3}R_{j1}R_{k2} - R_{i1}R_{j3}R_{k2} - R_{i3}R_{j2}R_{k1} - R_{i2}R_{j1}R_{k3} \\ &= \begin{vmatrix} R_{i1} & R_{j1} & R_{k1} \\ R_{i2} & R_{j2} & R_{k2} \\ R_{i3} & R_{j3} & R_{k3} \end{vmatrix}. \end{aligned}$$

Then consider the properties of the determinant.

To derive the equation of motion for a viscous fluid, we will need the following result.

Proposition 6.1. *The most general form of a rank 4 isotropic tensor is*

$$A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}.$$

▶ In other words, such a tensor has only three degrees of freedom.

Proof. Rather than considering rotation matrices, we can prove this by a geometrical argument. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ be four arbitrary vectors, and consider the scalar

$$s = A_{ijkl}a_i b_j c_k d_l. \tag{6.2}$$

Under a coordinate rotation, the vector components will change to a'_i, b'_j, c'_k, d'_l according to the usual rotation rule. And ordinarily A_{ijkl} would change to A'_{ijkl} so as to keep s invariant. But an isotropic tensor must have $A'_{ijkl} = A_{ijkl}$, so we must choose the components of A_{ijkl} such that $A_{ijkl}a'_i b'_j c'_k d'_l = A_{ijkl}a_i b_j c_k d_l$. Thus the only non-zero components can be those corresponding to combinations $a_i b_j c_k d_l$ that are themselves invariant. The only such combinations consist of scalar products between pairs of the vectors, so there must exist α, β, γ such that

$$A_{ijkl}a_i b_j c_k d_l = \alpha(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{c}\cdot\boldsymbol{d}) + \beta(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) + \gamma(\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}) \tag{6.3}$$

$$= \alpha a_i b_i c_j d_j + \beta a_i c_i b_j d_j + \gamma a_i d_i b_j c_j \tag{6.4}$$

$$= a_i b_j c_k d_l\big(\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}\big). \tag{6.5}$$

Since the four vectors are arbitrary, A_{ijkl} must equal the part in parentheses. □

6.2 Viscous stresses

To derive the Navier-Stokes equation for a viscous fluid, we need to go back and look at the stress tensor in more generality.

Recall the integral equation for conservation of momentum,

$$\int_{D_t}\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\,\mathrm{d}V = \int_{D_t}\rho\boldsymbol{f}\,\mathrm{d}V + \oint_{\partial D_t}\boldsymbol{\sigma}\cdot\,\mathrm{d}\boldsymbol{S}. \tag{6.6}$$

Applying the divergence theorem to the surface term,

$$\int_{D_t}\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\,\mathrm{d}V = \int_{D_t}\left(\rho\boldsymbol{f} + \nabla\cdot\boldsymbol{\sigma}^\top\right)\mathrm{d}V, \tag{6.7}$$

where $\nabla\cdot\boldsymbol{\sigma}^\top$ is a vector with components $\left(\nabla\cdot\boldsymbol{\sigma}^\top\right)_i = \dfrac{\partial\sigma_{ij}}{\partial x_j}$. Without making any assumptions about the stress tensor $\boldsymbol{\sigma}$, we know that (6.7) has to be satisfied for any D_t, implying the *Cauchy momentum equation*

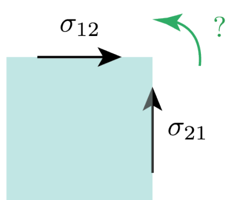
$$\boxed{\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \rho\boldsymbol{f} + \nabla\cdot\boldsymbol{\sigma}^\top.} \tag{6.8}$$

► Ideal fluids correspond to the specific choice $\sigma_{ij} = -p\delta_{ij}$, but a wide variety of fluids, as well as elastic solids, are described by this equation for different $\boldsymbol{\sigma}$.

The viscous stress tensor must have off-diagonal components, describing friction between a fluid element and its neighbours. But even so, these components cannot not all be independent, as we will now show.

Proposition 6.2. *Angular momentum is conserved iff the stress tensor is symmetric,* $\sigma_{ji} = \sigma_{ij}$.

You can see (roughly) why this holds by considering the stresses on a fluid element:



► Note that we are ruling out exotic fluids where angular momentum is not conserved. For example, a suspension of ferromagnetic particles with an applied magnetic field.

Proof. Conservation of angular momentum means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} \boldsymbol{x}\times\rho\boldsymbol{u}\,\mathrm{d}V = \int_{D_t}\boldsymbol{x}\times\rho\boldsymbol{f}\,\mathrm{d}V + \oint_{\partial D_t}\boldsymbol{x}\times(\boldsymbol{\sigma}\cdot\,\mathrm{d}\boldsymbol{S}). \tag{6.9}$$

where $\boldsymbol{x}$ is the position vector from the origin.

Applying the transport theorem to the left-hand side of (6.9) gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t}\boldsymbol{x}\times\rho\boldsymbol{u}\,\mathrm{d}V &= \int_{D_t}\left[\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{x}\times\rho\boldsymbol{u}) + (\boldsymbol{x}\times\rho\boldsymbol{u})\nabla\cdot\boldsymbol{u}\right]\,\mathrm{d}V &(6.10)\\
&= \int_{D_t}\left[\frac{\mathrm{D}\rho}{\mathrm{D}t}(\boldsymbol{x}\times\boldsymbol{u}) + \rho\frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t}\times\boldsymbol{u} + \rho\boldsymbol{x}\times\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + (\rho\boldsymbol{x}\times\boldsymbol{u})\nabla\cdot\boldsymbol{u}\right]\,\mathrm{d}V &(6.11)\\
&= \int_{D_t}\left[\left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{u}\right)\boldsymbol{x}\times\boldsymbol{u} + \rho\frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t}\times\boldsymbol{u} + \rho\boldsymbol{x}\times\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right]\,\mathrm{d}V &(6.12)\\
&= \int_{D_t}\boldsymbol{x}\times\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\,\mathrm{d}V, &(6.13)
\end{aligned}$$

where we used both the continuity equation and the fact that $\dfrac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t} = \boldsymbol{u}$.

Now consider the last term in (6.9). Applying the divergence theorem "backwards" gives

$$\begin{aligned}
\oint_{\partial D_t}\left[\boldsymbol{x}\times(\boldsymbol{\sigma}\cdot\,\mathrm{d}\boldsymbol{S})\right]_i &= \oint_{\partial D_t}\epsilon_{ijk}x_j\sigma_{kl}n_l\,\mathrm{d}S &(6.14)\\
&= \int_{D_t}\frac{\partial}{\partial x_l}\left(\epsilon_{ijk}x_j\sigma_{kl}\right)\,\mathrm{d}V &(6.15)\\
&= \int_{D_t}\epsilon_{ijk}\left(\frac{\partial x_j}{\partial x_l}\sigma_{kl} + x_j\frac{\partial\sigma_{kl}}{\partial x_l}\right)\,\mathrm{d}V &(6.16)\\
&= \int_{D_t}\left(\epsilon_{ijk}\sigma_{kj} + \left[\boldsymbol{x}\times(\nabla\cdot\boldsymbol{\sigma}^\top)\right]_i\right)\,\mathrm{d}V. &(6.17)
\end{aligned}$$

Putting (6.13) and (6.17) into (6.9) gives

$$\int_{D_t}\epsilon_{ijk}x_j\left(\rho\frac{\mathrm{D}u_k}{\mathrm{D}t} - \rho f_k - \left[\nabla\cdot\boldsymbol{\sigma}^\top\right]_k\right)\,\mathrm{d}V = \int_{D_t}\epsilon_{ijk}\sigma_{kj}\,\mathrm{d}V. \tag{6.18}$$

The left-hand-side vanishes by the Cauchy momentum equation, so angular momentum is conserved if and only if

$$\int_{D_t}\epsilon_{ijk}\sigma_{kj}\,\mathrm{d}V = 0. \tag{6.19}$$

This needs to hold for arbitrary D_t, so

$$\begin{aligned}
\epsilon_{ijk}\sigma_{kj} = 0 \quad &\iff \quad \epsilon_{imn}\epsilon_{ijk}\sigma_{kj} = 0 &(6.20)\\
&\iff \quad \left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right)\sigma_{kj} = 0 &(6.21)\\
&\iff \quad \sigma_{nm} - \sigma_{mn} = 0. &(6.22)
\end{aligned}$$

□

In order that σ_{ij} reduces to the ideal expression in the absence of viscosity, we write

$$\sigma_{ij} = -p\delta_{ij} + d_{ij} \tag{6.23}$$

where d_{ij} is called the *deviatoric stress tensor*. Since the diagonal components of $\boldsymbol{\sigma}$ need no longer be equal, we will simply define the pressure p as the *mean* normal stress,

$$p = -\frac{1}{3}\sigma_{ii}. \tag{6.24}$$

It follows that the deviatoric part is traceless,

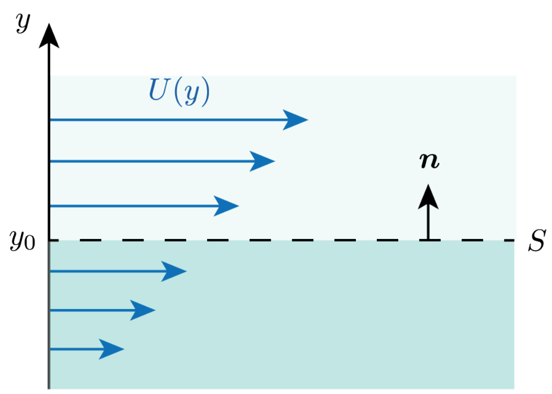
$$d_{ii} = 0. \tag{6.25}$$

▶ Note: some books define p as the thermodynamic pressure and allow $d_{ii} \neq 0$.

6.3 Newtonian fluids

A *Newtonian fluid* is one where the shear stresses depend linearly on velocity gradients. This is the most common model for a viscous fluid.

Example → Shear flow $\boldsymbol{u} = U(y)\boldsymbol{e}_x$ with $U'(y) > 0$.



Consider the stress on the surface S given by $y = y_0$. In an ideal fluid, the forces would all be normal to S, so there would be no transfer of momentum across S. But in reality, kinetic theory dictates that some molecules will diffuse across S, transferring momentum. If a molecule moves from $y < y_0$ to $y > y_0$, it must be accelerated, so there must be a force in the x-direction. In a Newtonian fluid, we approximate the shear stress by the linear relation

$$\sigma_{12} = \mu\frac{\mathrm{d}U}{\mathrm{d}y},$$

where the constant of proportionality μ is the *viscosity*.

Since $\boldsymbol{n} = \boldsymbol{e}_y$ for the volume $y < y_0$, the force $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ on the lower fluid is to the right (speeding it up). For the volume $y > y_0$, the force is to the left (slowing it down) since $\boldsymbol{n} = -\boldsymbol{e}_y$. So viscosity opposes relative motion between different parts of the fluid.

In general, we need to consider the viscous force exerted by all three components of $\boldsymbol{u}$ on surfaces of arbitrary orientation. So a Newtonian fluid is defined by the linear relation

$$d_{ij} = A_{ijkl}\frac{\partial u_l}{\partial x_k} \tag{6.26}$$

for some rank 4 tensor A_{ijkl}. We make the further physical assumption that A_{ijkl} must be isotropic.

▶ This is a bit tricky: we are assuming not that $\dfrac{\partial u_l}{\partial x_k}$ is the same everywhere, but that there is no preferred direction in space for the *relation* between this and d_{ij}. Such isotropy holds for fluids like air or water, but not for (*e.g.*) polymer suspensions with long chain molecules.

Thus for a Newtonian fluid we must have, from Proposition 6.1, that

$$d_{ij} = \alpha \frac{\partial u_k}{\partial x_k} \delta_{ij} + \beta \frac{\partial u_j}{\partial x_i} + \gamma \frac{\partial u_i}{\partial x_j}. \tag{6.27}$$

From Proposition 6.2 it follows that $d_{ji} = d_{ij}$. This implies that $\gamma = \beta$ so

$$d_{ij} = \alpha \frac{\partial u_k}{\partial x_k} \delta_{ij} + \beta \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right). \tag{6.28}$$

Finally, applying (6.25) gives $3\alpha = -2\beta$, so writing $\mu = \beta$ gives

$$d_{ij} = \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right). \tag{6.29}$$

The stress tensor for a Newtonian fluid is therefore

$$\boxed{\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} (\nabla \cdot \boldsymbol{u}) \delta_{ij} \right),} \tag{6.30}$$

and the constant μ is called the *viscosity*.

▶ In *non-Newtonian fluids* such as toothpaste, blood or magma, the stress can't be modelled by the simple linear form (6.26). For example, tomato ketchup has a viscosity that decreases with increasing shear stress (hence why you have to shake it vigorously to get it out of the bottle!).

▶ If the condition $d_{ii} = 0$ is not imposed, then there are two independent coefficients of viscosity, usually written as

$$d_{ij} = \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) + \mu' \frac{\partial u_k}{\partial x_k} \delta_{ij}.$$

6.4 The Navier-Stokes equations

We are finally in a position to write down the famous equations of motion for a viscous fluid. To derive the equation for conservation of momentum, we simply insert the Newtonian stress tensor $\boldsymbol{\sigma}$ into the general Cauchy momentum equation (6.8). We have

$$\begin{aligned}
[\nabla \cdot \boldsymbol{\sigma}^\top]_i &= \frac{\partial \sigma_{ij}}{\partial x_j} && (6.31)\\
&= -\frac{\partial p}{\partial x_i} + \mu \left(\frac{\partial^2 u_j}{\partial x_j \partial x_i} + \frac{\partial^2 u_i}{\partial x_j^2} - \frac{2}{3} \frac{\partial}{\partial x_i} (\nabla \cdot \boldsymbol{u}) \right) && (6.32)\\
&= -\frac{\partial p}{\partial x_i} + \mu \left(\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \frac{\partial^2 u_i}{\partial x_j^2} - \frac{2}{3} \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) && (6.33)\\
&= -\frac{\partial p}{\partial x_i} + \mu \left(\frac{1}{3} \frac{\partial}{\partial x_i} (\nabla \cdot \boldsymbol{u}) + \frac{\partial^2 u_i}{\partial x_j^2} \right), && (6.34)
\end{aligned}$$

so (6.8) gives

$$\boxed{\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \rho \boldsymbol{f} - \nabla p + \frac{\mu}{3} \nabla (\nabla \cdot \boldsymbol{u}) + \mu \Delta \boldsymbol{u}}. \tag{6.35}$$

Thus the *compressible Navier-Stokes equations* are

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, && (6.36)\\
&\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho} \nabla p + \frac{\mu}{3\rho} \nabla (\nabla \cdot \boldsymbol{u}) + \frac{\mu}{\rho} \Delta \boldsymbol{u} && (6.37)\\
&+ \text{equation of state.}
\end{aligned}$$

For the remainder of the course, we will discuss only the *incompressible Navier-Stokes equations*

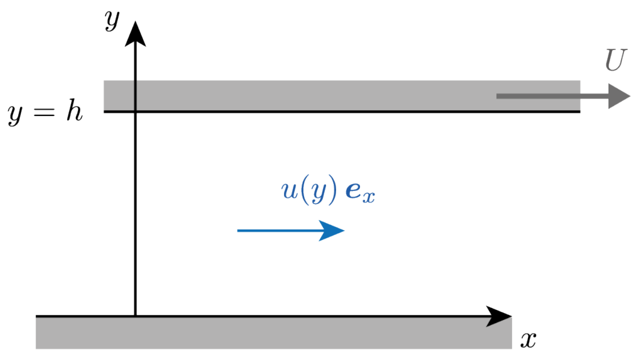
$$\nabla \cdot \boldsymbol{u} = 0, \tag{6.38}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho_0}\nabla p + \frac{\mu}{\rho_0}\Delta \boldsymbol{u}. \tag{6.39}$$

Usually we write $\nu = \mu/\rho_0$, called the *kinematic viscosity*.

The boundary conditions are different to an ideal fluid. For a viscous fluid we need the *no-slip condition* that $\boldsymbol{u} = \boldsymbol{0}$ on a solid boundary ∂V.

Example → Flow between two moving boundaries with $p = 0$ and $\boldsymbol{f} = \boldsymbol{0}$ (*Couette flow*).
Let the boundary $y = 0$ be stationary and the boundary $y = h$ move at constant velocity $U\boldsymbol{e}_x$, and look for a steady flow $\boldsymbol{u} = u(y)\boldsymbol{e}_x$.



In this situation, we have $\nabla \cdot \boldsymbol{u} = 0$, $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}$, and $\nabla p = 0$, so the incompressible Navier-Stokes equations reduce to the single ODE

$$\nu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0,$$

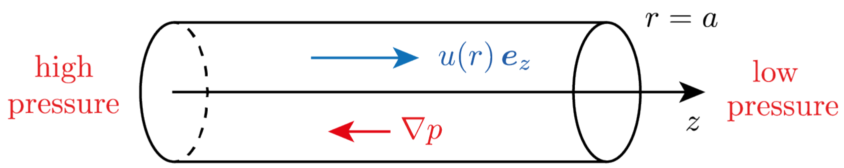
with no-slip boundary conditions $u(0) = 0$ and $u(h) = U$. Integrating twice gives

$$u(y) = C_1 y + C_2,$$

and the boundary conditions imply that $C_2 = 0$ and $C_1 = U/h$. Thus $u(y) = \dfrac{Uy}{h}$.

► Couette flow is the simplest (non-trivial) solution of the Navier-Stokes equations. There is a variation called *Taylor-Couette flow* where the boundaries are cylinders [see problem sheet] – its behaviour (and instability) as you increase the rotation rate are rather complex.

Example $\rightarrow$ Flow in a pipe with $\boldsymbol{f} = \boldsymbol{0}$ (*Poiseuille flow*).
Let the pipe be given by $r < a$ in cylindrical coordinates. We need a pressure gradient to sustain the flow (since the boundary doesn't move), so assume $\nabla p = -P\boldsymbol{e}_z$. We look for a steady solution of the form $\boldsymbol{u} = u(r)\boldsymbol{e}_z$.



We have to be careful in cylindrical coordinates. Firstly,

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}\right)\left(u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z\right) = u\frac{\partial}{\partial z}(u\boldsymbol{e}_z) = 0.$$

Secondly, we have

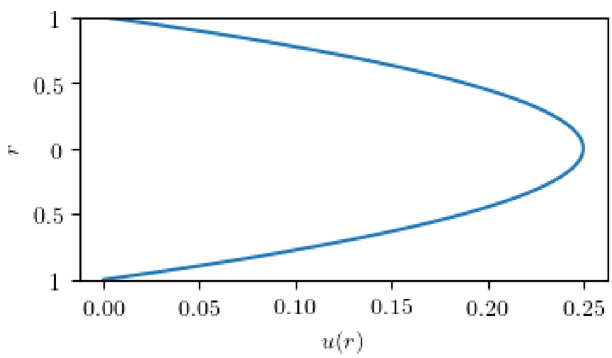
$$\boldsymbol{\Delta u} = \left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)\boldsymbol{e}_r + \left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right)\boldsymbol{e}_\theta + \Delta u_z\boldsymbol{e}_z$$

where $\Delta f = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial\theta^2} + \dfrac{\partial^2 f}{\partial z^2}$. In our case this reduces to $\boldsymbol{\Delta u} = \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\dfrac{\mathrm{d}u}{\mathrm{d}r}\right)\boldsymbol{e}_z$. So the equation of motion gives

$$\begin{aligned} 0 = P + \frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) \qquad &\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = -\frac{Pr}{\mu}, \\ &\Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{Pr}{2\mu} + \frac{C_1}{r}, \\ &\Longrightarrow u(r) = -\frac{Pr^2}{4\mu} + C_1\log(r) + C_2. \end{aligned}$$

Requiring a finite solution at $r = 0$ gives $C_1 = 0$, and the no-slip condition at $r = a$ fixes C_2 so

$$u(r) = \frac{P}{4\mu}(a^2 - r^2).$$



The total flow rate through the tube is $Q = 2\pi\displaystyle\int_0^a ur\,\mathrm{d}r = \frac{\pi P a^4}{8\mu}$. Measuring Q could be used to determine μ.

▶ In practice, Poiseuille flow is observed only for slow enough flow (which we will quantify with the Reynolds number later – indeed, study of this flow was the origin of the Reynolds number). For faster flow, it is unstable and becomes turbulent, although it is not simply a linear instability and remains a topic of research.

6.5 Effects of viscosity

Loosely speaking, viscosity causes vorticity to decay and spread out.

6.5.1 Viscous dissipation

One way to characterise the effect of viscosity on a flow is to look at its effect on kinetic energy. The incompressible Navier-Stokes equations with conservative body force take the form

$$\nabla \cdot \boldsymbol{u} = 0, \tag{6.40}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla U - \frac{1}{\rho_0}\nabla p + \nu \Delta \boldsymbol{u}. \tag{6.41}$$

Recall from last term that the corresponding Euler equations ($\nu = 0$) conserve kinetic energy:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \qquad \text{where } E = \frac{1}{2}\int_V \rho_0 |\boldsymbol{u}|^2 \,\mathrm{d}V. \tag{6.42}$$

The presence of viscosity $\nu \neq 0$, however, leads to the dissipation of this energy.

Proposition 6.3. *In an incompressible viscous fluid with* $\boldsymbol{f} = -\nabla U$*,*

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mu \int_V |\boldsymbol{\omega}|^2 \,\mathrm{d}V.$$

▶ Notice that the energy decreases (provided $\boldsymbol{u}$ remains smooth) until all of the vorticity has been destroyed. If $\boldsymbol{\omega} = \boldsymbol{0}$, then we can write $\boldsymbol{u} = \nabla\phi$, in which case $\Delta \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times \nabla \times \boldsymbol{u} = \boldsymbol{0}$.

▶ In reality, the energy goes into heat, just like normal friction. But we cannot account for this in our incompressible model. [In principle we could extend the energy equation to include both viscosity and compressibility at the same time.]

Proof. Proceeding as we did for the Euler equations, we have

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \rho_0 \int_V \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \,\mathrm{d}V, && (6.43)\\
&= \rho_0 \int_V \boldsymbol{u} \cdot \left(-\nabla U - \frac{1}{\rho_0}\nabla p - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu \Delta \boldsymbol{u}\right) \mathrm{d}V, && (6.44)\\
&= -\rho_0 \int_V \boldsymbol{u} \cdot \nabla \left(U + \frac{p}{\rho_0} + \frac{1}{2}|\boldsymbol{u}|^2\right) \mathrm{d}V + \mu \int_V \boldsymbol{u} \cdot \Delta \boldsymbol{u} \,\mathrm{d}V, && (6.45)\\
&= -\rho_0 \oint_{\partial V} \left(U + \frac{p}{\rho_0} + \frac{1}{2}|\boldsymbol{u}|^2\right) \boldsymbol{u} \cdot \,\mathrm{d}\boldsymbol{S} + \mu \int_V \boldsymbol{u} \cdot \Delta \boldsymbol{u} \,\mathrm{d}V, && (6.46)\\
&= \mu \int_V \boldsymbol{u} \cdot \Delta \boldsymbol{u} \,\mathrm{d}V, && (6.47)
\end{aligned}$$

where we assume as before that $\boldsymbol{u} \cdot \,\mathrm{d}\boldsymbol{S} = 0$ on ∂V. Now use integration by parts:

$$\begin{aligned}
\int_V \boldsymbol{u} \cdot \Delta \boldsymbol{u} \,\mathrm{d}V &= \int_V u_k \frac{\partial^2 u_k}{\partial x_j^2} \,\mathrm{d}V, && (6.48)\\
&= \int_V \left[\frac{\partial}{\partial x_j}\left(u_k \frac{\partial u_k}{\partial x_j}\right) - \left(\frac{\partial u_k}{\partial x_j}\right)^2\right] \mathrm{d}V, && (6.49)\\
&= \oint_{\partial V} u_k \frac{\partial u_k}{\partial x_j} n_j \,\mathrm{d}S - \int_V \left(\frac{\partial u_k}{\partial x_j}\right)^2 \mathrm{d}V, && (6.50)\\
&= -\int_V \left(\frac{\partial u_k}{\partial x_j}\right)^2 \mathrm{d}V, && (6.51)
\end{aligned}$$

using the no-slip boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on ∂V. To show that this has the required expression, it is easiest to go the other way, so

$$\int_V |\nabla \times \boldsymbol{u}|^2 \, \mathrm{d}V = \int_V \epsilon_{ijk}\epsilon_{ilm}\frac{\partial u_k}{\partial x_j}\frac{\partial u_m}{\partial x_l}\,\mathrm{d}V \tag{6.52}$$

$$= \int_V \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)\frac{\partial u_k}{\partial x_j}\frac{\partial u_m}{\partial x_l}\,\mathrm{d}V \tag{6.53}$$

$$= \int_V \left(\frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_j} - \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}\right)\,\mathrm{d}V \tag{6.54}$$

$$= \int_V \left(\frac{\partial u_k}{\partial x_j}\right)^2\mathrm{d}V - \int_V \left[\frac{\partial}{\partial x_j}\left(u_k\frac{\partial u_j}{\partial x_k}\right) - u_k\frac{\partial^2 u_j}{\partial x_j x_k}\right]\mathrm{d}V, \tag{6.55}$$

$$= \int_V \left(\frac{\partial u_k}{\partial x_j}\right)^2\mathrm{d}V - \oint_{\partial V} u_k\frac{\partial u_j}{\partial x_k}n_j\,\mathrm{d}S + \int_V u_k\frac{\partial}{\partial x_k}\left(\nabla\cdot\boldsymbol{u}\right)\mathrm{d}V, \tag{6.56}$$

$$= \int_V \left(\frac{\partial u_k}{\partial x_j}\right)^2\mathrm{d}V. \tag{6.57}$$

In the last step the boundary term vanished again since $\boldsymbol{u} = \boldsymbol{0}$ on ∂V, and the last term vanished by incompressibility ($\nabla \cdot \boldsymbol{u} = 0$). □

Another way to look at this result is to notice that viscous flow is *irreversible*. To see this, suppose that $\boldsymbol{u}$, p solve (6.40) and (6.41). Now let time run backwards, so that

$$\boldsymbol{u} \to -\boldsymbol{u}, \quad p \to p, \quad \frac{\partial}{\partial t} \to -\frac{\partial}{\partial t}, \quad \nabla \to \nabla.$$

Then (6.40) is invariant, but (6.41) becomes

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla U - \frac{1}{\rho_0}\nabla p - \nu\Delta\boldsymbol{u}. \tag{6.58}$$

In other words, it is not invariant: information is lost during the Navier-Stokes evolution, due to the dissipation of kinetic energy. By contrast, we see that the Euler equations (with $\nu = 0$) are invariant under such a time reversal.

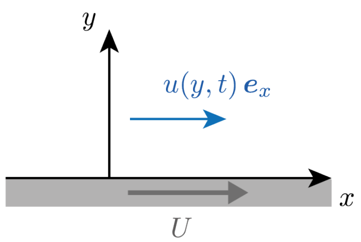
6.5.2 Diffusion of vorticity

Consider now the vorticity equation for a viscous fluid, derived by taking the curl of (6.41). Since the curl commutes with the Laplacian, we get

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega}\times\boldsymbol{u}) = \nu\Delta\boldsymbol{\omega}.} \tag{6.59}$$

This shows that viscosity causes the *diffusion* of vorticity. To illustrate this, we consider an example where vorticity is suddenly generated at one location.

Example → Impulsively moved plane boundary.
Suppose viscous fluid lies at rest in the region $y > 0$ and at $t = 0$ the rigid boundary $y = 0$ is suddenly jerked into motion with constant velocity $U\boldsymbol{e}_x$.



We assume that the flow takes the form $\boldsymbol{u} = u(y,t)\boldsymbol{e}_x$ with $u(y,0) = 0$, and that there is no externally imposed pressure gradient, so that $p = p_0$ (constant) throughout the fluid. The incompressible Navier-Stokes equations reduce to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

with boundary conditions $u(0,t) = U$ (for $t > 0$) and $u \to 0$ as $y \to \infty$, and initial condition $u(y,0) = 0$. We can solve this diffusion equation by seeking a *similarity solution*

$$u(y,t) = f(\eta), \qquad \text{where } \eta = \frac{y}{\sqrt{\nu t}}.$$

By the chain rule,

$$\begin{aligned} \frac{\partial u}{\partial t} &= f'(\eta)\frac{\partial \eta}{\partial t} = -f'(\eta)\frac{y}{2\nu^{1/2}t^{3/2}} = -f'(\eta)\frac{\eta}{2t}, \\ \frac{\partial u}{\partial y} &= f'(\eta)\frac{\partial \eta}{\partial y} = f'(\eta)\frac{1}{\nu^{1/2}t^{1/2}}, \\ \frac{\partial^2 u}{\partial y^2} &= f''(\eta)\left(\frac{\partial \eta}{\partial y}\right)^2 = f''(\eta)\frac{1}{\nu t}. \end{aligned}$$

So in terms of η, the equation becomes an ODE,

$$-\frac{\eta}{2t}f'(\eta) = \frac{1}{t}f''(\eta) \qquad \Longleftrightarrow \qquad f''(\eta) + \frac{\eta}{2}f'(\eta) = 0.$$

Integrating once with an integrating factor gives

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\mathrm{e}^{\eta^2/4}f'(\eta)\right) = 0 \qquad \Longleftrightarrow \qquad f'(\eta) = C_1\mathrm{e}^{-\eta^2/4},$$

and again gives

$$f(\eta) = C_1\int_0^{\eta} \mathrm{e}^{-s^2/4}\,\mathrm{d}s + C_2.$$

Imposing the boundary condition at $y = 0$ (which corresponds to $\eta = 0$) gives $C_2 = U$. The conditions at $y \to \infty$ and $t \to 0$ both correspond (fortunately) to $f \to 0$ as $\eta \to \infty$. So we need

$$C_1\int_0^{\infty} \mathrm{e}^{-s^2/4}\,\mathrm{d}s + U = 0.$$

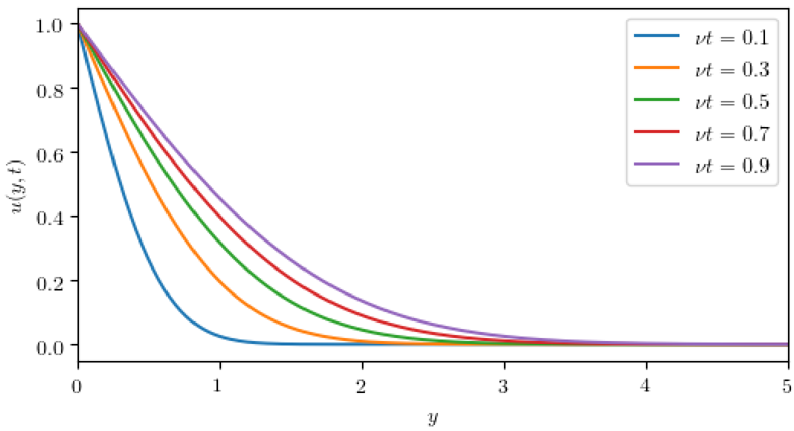
Changing variables to $s' = s/2$ and using the Gaussian integral gives

$$2C_1\frac{\sqrt{\pi}}{2} + U = 0 \qquad \Longleftrightarrow \quad C_1 = -\frac{U}{\sqrt{\pi}}.$$

So the velocity field has the form

$$u(y,t) = U\left[1 - \frac{1}{\sqrt{\pi}}\int_0^{y/\sqrt{\nu t}} \mathrm{e}^{-s^2/4}\,\mathrm{d}s\right].$$

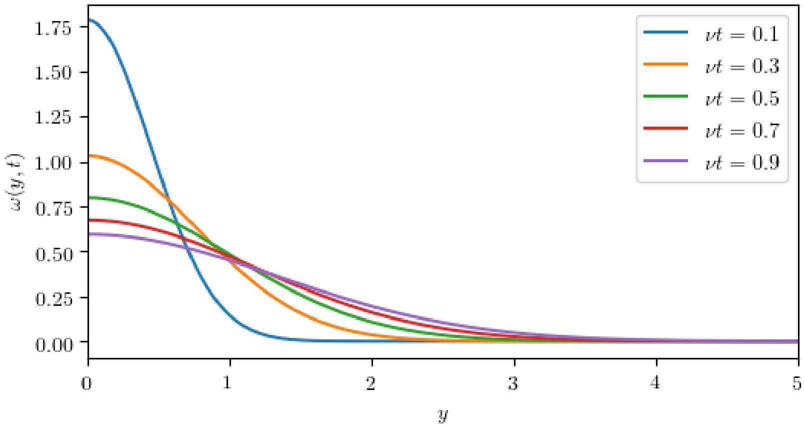
This looks as follows:



Notice that, at time t, the effects of the boundary motion have reached a distance of order $\sqrt{\nu t}$ from the boundary.

Now consider the vorticity. This has the form $\boldsymbol{\omega} = \omega(y,t)\boldsymbol{e}_z$ with

$$\omega(y,t) = -\frac{\partial y}{\partial y} = \frac{U}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right).$$



The viscosity has spread out the vorticity, which was initially concentrated only in a vortex sheet of infinite strength at $y = 0$. The vorticity is generated by the initial impulse, and diffuses into the fluid.

6.6 The Reynolds number

How important is the viscosity for a given flow? This depends not only on the size of ν but also on its size relative to the other terms in the Navier-Stokes equations. So we need to look at the *scaling properties* of the equations.

Let U (constant) denote a *characteristic* (representative) flow speed $|\boldsymbol{u}|$ for the problem of interest, and let L denote a characteristic length scale for this flow. We then change to *dimensionless* variables

$$\boldsymbol{u}' = \frac{1}{U}\boldsymbol{u}, \qquad \boldsymbol{x}' = \frac{1}{L}\boldsymbol{x}, \qquad t' = \frac{U}{L}t. \tag{6.60}$$

Now

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \frac{U}{L}\frac{\partial}{\partial t'}, \qquad \frac{\partial}{\partial x_j} = \frac{\partial x'_k}{\partial x_j}\frac{\partial}{\partial x'_k} = \frac{1}{L}\frac{\partial}{\partial x'_j}, \tag{6.61}$$

so the continuity equation becomes

$$\frac{U}{L}\nabla'\cdot\boldsymbol{u}' = 0 \qquad \Longleftrightarrow \qquad \nabla'\cdot\boldsymbol{u}' = 0. \tag{6.62}$$

and the (unforced) momentum equation becomes

$$\frac{U^2}{L}\frac{\partial\boldsymbol{u}'}{\partial t'} + \frac{U^2}{L}(\boldsymbol{u}'\cdot\nabla')\boldsymbol{u}' = -\frac{1}{\rho_0 L}\nabla' p + \frac{U}{L^2}\nu\Delta'\boldsymbol{u}', \tag{6.63}$$

$$\Longleftrightarrow \qquad \frac{\partial\boldsymbol{u}'}{\partial t'} + (\boldsymbol{u}'\cdot\nabla')\boldsymbol{u}' = -\nabla'\left(\frac{p}{\rho_0 U^2}\right) + \frac{\nu}{UL}\Delta'\boldsymbol{u}'. \tag{6.64}$$

If we define the dimensionless pressure $p' = \dfrac{1}{\rho_0 U^2}p$, then

$$\boxed{\frac{\partial\boldsymbol{u}'}{\partial t'} + (\boldsymbol{u}'\cdot\nabla')\boldsymbol{u}' = -\nabla' p' + \frac{1}{\mathrm{Re}}\Delta'\boldsymbol{u}'.} \tag{6.65}$$

where the only remaining (dimensionless) parameter is the *Reynolds number*

$$\mathrm{Re} = \frac{UL}{\nu}. \tag{6.66}$$

▶ Additional terms lead to additional dimensionless parameters. For example, compressibility introduces the *Mach number* $\mathrm{Ma} = U/c_0$, while non-zero forcing $\boldsymbol{f}$ leads to other parameters, such as the *Froude number* $\mathrm{Fr} = U/\sqrt{gL}$.

▶ The Reynolds number is named after mathematician/engineer Osborne Reynolds, who carried out a famous experiment on pipe flow in 1883, examining the conditions under which laminar Poiseuille flow transitions to turbulence.

It is clear from (6.65) that viscosity will be more important, relative to the other terms, in flows with small Re.

Example → Typical Reynolds numbers.

1. *Flow around an aeroplane wing.* In air at $20°C$ we have $\nu \approx 0.15\,\mathrm{cm}^2\,\mathrm{s}^{-1}$. With $U = 30\,\mathrm{m}\,\mathrm{s}^{-1}$ and $L = 30$m, this gives $\mathrm{Re} \approx 10^8$.
2. *Flow around a car.* With the same ν but $U = 3\,\mathrm{m}\,\mathrm{s}^{-1}$ and $L = 3$m we have $\mathrm{Re} = 10^6$.
3. *Flow around a swimmer.* For water at $20°C$ we have $\nu = 0.01\,\mathrm{cm}^2\,\mathrm{s}^{-1}$, so with $U = 2\,\mathrm{m}\,\mathrm{s}^{-1}$ and $L = 2\,\mathrm{m}$ we have $\mathrm{Re} = 4 \times 10^6$.
4. *Flow around a swimming bacterium.* Again take $\nu = 0.01\,\mathrm{cm}^2\,\mathrm{s}^{-1}$, but take $U = 10^{-4}\,\mathrm{m}\,\mathrm{s}^{-1}$ and $L = 10^{-7}\,\mathrm{m}$ so that $\mathrm{Re} = 10^{-5}$.

▶ Notice that air has higher kinematic viscosity ν than water, because of its lower density. Water has higher μ.

▶ Two flows with the same Re are called *dynamically similar*, because the solutions will be the same under a rescaling of the variables. This is used, for example, in wind tunnel testing of aircraft using scaled down models.

6.6.1 High Reynolds-number limit

As $\mathrm{Re} \to \infty$ the viscous term in (6.65) becomes smaller and smaller. However, this is a singular limit, meaning that the solution of the Navier-Stokes equation does *not* (in general) tend to the solution of the Euler equation. This is because the viscous term has the highest derivative.

Example → The ODE $\varepsilon u'' + u' = 1$, $\varepsilon \ll 1$ with boundary conditions $u(0) = 0$, $u(1) = 2$.
This is a simpler differential equation to illustrate what happens when the highest derivative term is multiplied by a small parameter.

If we set $\varepsilon = 0$ and neglect the first term, then we have

$$u' = 1 \qquad \Longrightarrow u(x) = x + C.$$

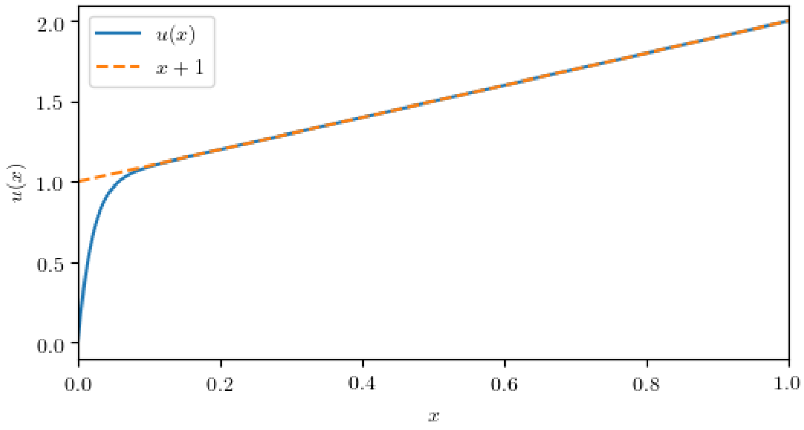
It is not possible to satisfy both boundary conditions.

To find the solution of the full equation, use an integrating factor to write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x/\varepsilon}u'\right) = \frac{1}{\varepsilon}\mathrm{e}^{x/\varepsilon} \quad \Longrightarrow \quad u' = 1 + C_1\mathrm{e}^{-x/\varepsilon} \quad \Longrightarrow \quad u(x) = x + C_2 - C_1\varepsilon\mathrm{e}^{-x/\varepsilon}.$$

Now imposing the condition at $x = 0$ gives $C_2 = \varepsilon C_1$ and at $x = 1$ gives $C_1 = \varepsilon^{-1}(1 - \mathrm{e}^{-1/\varepsilon})^{-1}$. Thus

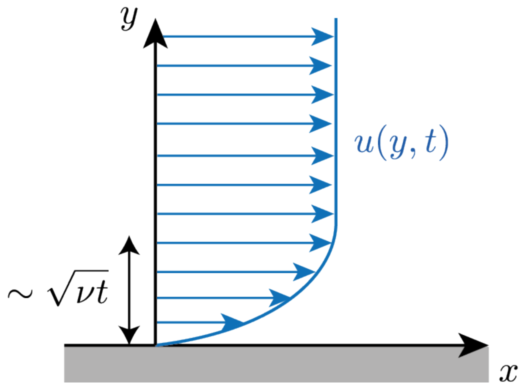
$$u(x) = x + \frac{1 - \mathrm{e}^{-x/\varepsilon}}{1 - \mathrm{e}^{-1/\varepsilon}}.$$



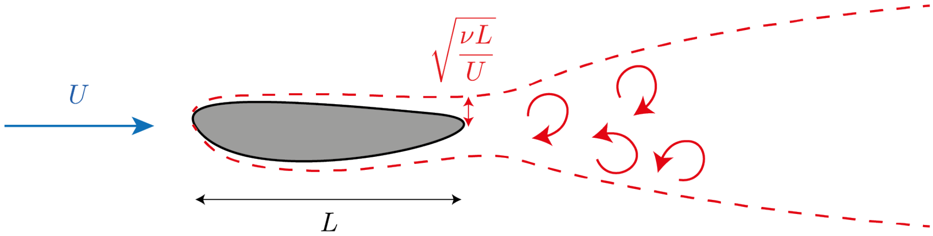
The solution to the second-order equation looks like the first-order solution except in a thin layer near the boundary where it changes to satisfy the second boundary condition. In this layer, u'' is large enough that it compensates for the smallness of ε. This layer will get thinner for smaller ε, but will continue to exist for any $\varepsilon > 0$.

In the fluid case, there is an analogous layer above a solid boundary, where $\boldsymbol{u}$ has to depart from an inviscid flow in order to satisfy the no-slip condition. This is called a *boundary layer*.

▶ We already saw an example with the impulsively moved boundary. In that example, we saw that the boundary layer thickness grew like $\sqrt{\nu t}$.



Example → Flow past an obstacle.
A smooth solid object moving through an otherwise inviscid flow with high Re will generate a thin boundary layer of vorticity. If the object has length L and the flow speed is U, the thickness of the boundary layer will reach $\sqrt{\nu t} = \sqrt{\nu L/U}$ by the time it reaches the rear.



This vorticity will then be advected beyond the object as a *wake*, which will continue to widen due to viscous diffusion. The flow takes time x/U to move a distance x, so the wake will spread a distance $\sqrt{\nu t} = \sqrt{\nu x/U}$.

▶ In fact, secondary flows can cause the boundary layer to *separate* before it reaches the rear of the obstacle. Aerofoils are shaped to prevent this premature separation compared to (for instance) a cylinder.

6.6.2 Low Reynolds-number limit

If $\mathrm{Re} \ll 1$ then viscosity dominates. This applies to flows with extremely small length scales or very viscous liquids like honey.

When $\mathrm{Re} \ll 1$, we neglect $\dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$ and obtain the *Stokes equations*

$$\nabla \cdot \boldsymbol{u} = 0, \tag{6.67}$$

$$\mu \Delta \boldsymbol{u} = \nabla p. \tag{6.68}$$

We still have the no-slip boundary condition.

▶ Equation (6.65) suggests that we can also neglect the pressure. But this would not account for all of the possible Stokes flows. We could instead have defined the alternative dimensionless pressure $p' = Lp/(\mu U)$ so that

$$\frac{\partial \boldsymbol{u}'}{\partial t'} + (\boldsymbol{u}' \cdot \nabla')\boldsymbol{u}' = -\frac{1}{\mathrm{Re}}\nabla' p' + \frac{1}{\mathrm{Re}}\Delta' \boldsymbol{u}'.$$

The Stokes equations are linear, so are much easier to deal with.

Proposition 6.4. *The Stokes equations have at most one solution $\boldsymbol{u}(\boldsymbol{x})$, $p(\boldsymbol{x})$ in a domain V matching a given boundary condition $\boldsymbol{u} = \boldsymbol{u}_S(\boldsymbol{x})$ on ∂V. The pressure p is unique up to an additive constant.*

▶ To get a non-zero flow, we therefore need $\boldsymbol{u}_S \neq 0$, which implies that the boundary itself is moving.

Proof. Suppose there exist two solutions $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and consider the difference $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ with corresponding pressure difference $q = p_1 - p_2$. By linearity, we have

$$\mu \Delta \boldsymbol{v} = \nabla q, \qquad \nabla \cdot \boldsymbol{v} = 0, \tag{6.69}$$

with $\boldsymbol{v} = \boldsymbol{0}$ on ∂V. It follows that

$$\int_V \boldsymbol{v} \cdot \Big(\mu \Delta \boldsymbol{v} - \nabla q\Big)\, \mathrm{d}V = 0 \tag{6.70}$$

$$\implies \int_V v_i \Big(\mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \frac{\partial q}{\partial x_i}\Big)\, \mathrm{d}V = 0 \tag{6.71}$$

$$\implies \mu \int_V \frac{\partial}{\partial x_j}\left(v_i \frac{\partial v_i}{\partial x_j}\right) \mathrm{d}V - \mu \int_V \left(\frac{\partial v_i}{\partial x_j}\right)^2 \mathrm{d}V - \int_V \frac{\partial}{\partial x_i}\Big(v_i q\Big)\, \mathrm{d}V = 0. \tag{6.72}$$

Applying the divergence theorem to the first and third terms gives

$$\mu \oint_{\partial V} v_i \frac{\partial v_i}{\partial x_j} n_j \, \mathrm{d}S - \mu \int_V \left(\frac{\partial v_i}{\partial x_j}\right)^2 \mathrm{d}V - \oint_{\partial V} q v_i n_i \, \mathrm{d}S = 0 \tag{6.73}$$

$$\implies \mu \int_V \left(\frac{\partial v_i}{\partial x_j}\right)^2 \mathrm{d}V = 0, \tag{6.74}$$

where we used the boundary condition $\boldsymbol{v} = \boldsymbol{0}$. The remaining integrand is

$$\left(\frac{\partial v_i}{\partial x_j}\right)^2 = \left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial y}\right)^2 + \left(\frac{\partial v_x}{\partial z}\right)^2 + \left(\frac{\partial v_y}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_y}{\partial z}\right)^2 + \left(\frac{\partial v_z}{\partial x}\right)^2 + \left(\frac{\partial v_z}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2. \tag{6.75}$$

So the only way for the integral to vanish is if $\boldsymbol{v} = \boldsymbol{0}$. It follows that $\boldsymbol{u}_1 = \boldsymbol{u}_2$, hence $\nabla q = 0$ and $p_1 = p_2 + c$. □

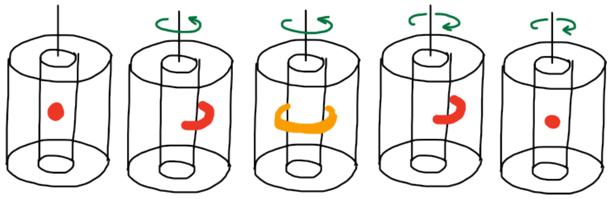
Example → Time-reversibility.
It follows from Proposition 6.4 that reversing the boundary conditions will lead to reversal of the flow. To see this, let $\boldsymbol{u}_S(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x})$ for some particular function $\boldsymbol{F}$ on ∂V, and suppose that $\boldsymbol{u}_1$, p_1 is the corresponding solution of the Stokes equations (with p_1 unique up to a constant).

Now change the boundary conditions to $\boldsymbol{u}_S(\boldsymbol{x}) = -\boldsymbol{F}(\boldsymbol{x})$ on ∂V. The solution

$$\boldsymbol{u}_2(\boldsymbol{x}) = -\boldsymbol{u}_1(\boldsymbol{x}), \qquad p_2 = C - p_1$$

matches the new boundary conditions, so by Proposition 6.4, it is the only solution.

We can illustrate this by dyeing a blob of fluid between two rotating cylinders:



6.7 The million-dollar question

You may know that the Navier-Stokes equations constitute one of the Clay Millennium problems:

Clay Prize Problem #6 - incompressible Navier-Stokes. *Either prove that initially smooth solutions with periodic boundary conditions (or in $\mathbb{R}^3$ with strong decay conditions toward infinity) remain smooth for all time, or find at least one solution which blows up in finite time.*

▶ This is a question about whether the PDEs are *well-posed*, meaning that, for given initial conditions, a unique solution exists for all time and it depends continuously on the initial conditions. The latter means that the solution at a later time can be determined to arbitrary precision by knowing the initial conditions to sufficient precision.

▶ The same question remains open for the Euler equations, although with no prize associated.

Firstly, we show that no solution to the *two-dimensional* Euler equations will blow up. This is because the derivatives of such a flow $\boldsymbol{u}(x, y, t)$ are constrained.

Proposition 6.5. *Under the 2D incompressible Euler equations,* $\sup_{\boldsymbol{x}} |\boldsymbol{\omega}(\boldsymbol{x}, t)| \le \sup_{\boldsymbol{x}} |\boldsymbol{\omega}(\boldsymbol{x}, 0)|$.

Proof. In 2D we have $\boldsymbol{u} = \boldsymbol{u}(x, y, t)$ so the vorticity has only one non-zero component, $\boldsymbol{\omega} = \omega(x, y, t)\boldsymbol{e}_z$. We saw last term that the vorticity equation in this case reduces to

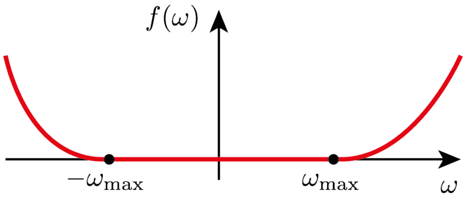
$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0, \tag{6.76}$$

so that ω is a material scalar.

Now consider the evolution of an arbitrary function $f(\omega)$. Taking a fixed region $D \subset \mathbb{R}^2$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_D f(\omega)\,\mathrm{d}S &= \int_D \frac{\partial f}{\partial t}\,\mathrm{d}S && (6.77)\\
&= \int_D f'(\omega)\frac{\partial \omega}{\partial t}\,\mathrm{d}S && (6.78)\\
&= -\int_D f'(\omega)\boldsymbol{u}\cdot\nabla\omega\,\mathrm{d}S \qquad \text{[using (6.76)]} && (6.79)\\
&= -\int_D \boldsymbol{u}\cdot\nabla f\,\mathrm{d}S && (6.80)\\
&= -\oint_{\partial D} f\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}l \qquad [\text{using } \nabla\cdot\boldsymbol{u} = 0] && (6.81)\\
&= 0. && (6.82)
\end{aligned}$$

Now let $\omega_{\max} = \sup_{\boldsymbol{x}} \left|\boldsymbol{\omega}(\boldsymbol{x}, 0)\right|$ and choose $f(\omega)$ to be any function with the properties that $f(\omega) = 0$ for $|\omega| < \omega_{\max}$ and $f(\omega) > 0$ otherwise. For example:



Then

$$\int_D f\big(\omega(\boldsymbol{x}, t)\big)\,\mathrm{d}S = \int_D f\big(\omega(\boldsymbol{x}, 0)\big)\,\mathrm{d}S = 0. \tag{6.83}$$

But this means that there cannot be any points where $\omega(\boldsymbol{x}, t) > \omega_{\max}$. □

► This proof fails in 3D because of the vortex stretching term in the vorticity equation. Indeed, the most promising candidate flows for blow up are designed to maximise the local amplification of vorticity. For the (3D) Euler equations, one can show the *Beale-Kato-Majda criterion*: if there is a solution $\boldsymbol{u}$ with finite blow-up time T, then its vorticity must blow up with $\displaystyle\int_0^T \sup_{\boldsymbol{x}} \left|\boldsymbol{\omega}(\boldsymbol{x}, t)\right| \mathrm{d}t = \infty$.

To analyse the (incompressible) Navier-Stokes equations, recall Proposition 6.3, which says that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V |\boldsymbol{u}|^2\,\mathrm{d}V = -\frac{2\mu}{\rho_0} \int_V |\boldsymbol{\omega}|^2\,\mathrm{d}V. \tag{6.84}$$

It follows that a smooth solution will remain bounded. However, to show that the solution remains smooth, we need to bound the derivatives, and in particular $|\nabla \boldsymbol{u}|^2 := \left|\dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_i}{\partial x_j}\right|$.

Proposition 6.6. *A solution of the incompressible Navier-Stokes equations with conservative body force and periodic boundary conditions satisfies*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_V |\nabla \boldsymbol{u}|^2\,\mathrm{d}V = -\int_V \frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k}\,\mathrm{d}V - \nu \int_V |\Delta \boldsymbol{u}|^2\,\mathrm{d}V.$$

► Taking periodic boundary conditions allows us to cancel some boundary terms.

Proof. To derive this equation, dot the momentum equation with $\Delta \boldsymbol{u}$ and integrate:

$$\int_V \Delta \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}V = \int_V \Delta \boldsymbol{u} \cdot \left(-(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla U - \frac{1}{\rho_0}\nabla p + \nu \Delta \boldsymbol{u}\right)\mathrm{d}V. \tag{6.85}$$

Integrating the left-hand side by parts gives

$$\int_V \Delta \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}V = \int_V \frac{\partial^2 u_i}{\partial x_k \partial x_k}\frac{\partial u_i}{\partial t}\,\mathrm{d}V \tag{6.86}$$

$$= \int_V \frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial t}\right)\mathrm{d}V - \int_V \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}V \tag{6.87}$$

$$= \oint_{\partial V} \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial t} n_k\,\mathrm{d}S - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(\frac{\partial u_i}{\partial x_k}\right)^2\mathrm{d}V \tag{6.88}$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V |\nabla \boldsymbol{u}|^2\,\mathrm{d}V, \tag{6.89}$$

where we used the fact that $\boldsymbol{u}$ is periodic. So

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V |\nabla \boldsymbol{u}|^2\,\mathrm{d}V = \int_V \Delta \boldsymbol{u} \cdot \left[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla\left(U + \frac{p}{\rho_0}\right)\right]\mathrm{d}V - \nu \int_V |\Delta \boldsymbol{u}|^2\,\mathrm{d}V. \tag{6.90}$$

Dealing with the remaining term is left to the problem sheet. □

Notice that the first term on the right-hand side of Proposition 6.6 is the only one that could blow up, but in 3D this remains an open question.

▶ It can be shown (by some more advanced functional analysis) that

$$\left| \int_V \frac{\partial u_j}{\partial x_i} \frac{\partial u_k}{\partial x_i} \frac{\partial u_j}{\partial x_k} \, \mathrm{d}V \right| \leq \nu \int_V \left| \Delta \boldsymbol{u} \right|^2 \, \mathrm{d}V + C(\nu) \left(\int_V \left| \nabla \boldsymbol{u} \right|^2 \, \mathrm{d}V \right)^d,$$

where d is the number of dimensions and C is a constant. If $d = 2$ (2D) then the second term can be bounded, but if $d = 3$ (3D) then this bound blows up. But this could just be because the estimates that went into the inequality are not sharp enough. In particular, the three-dimensional geometry of the flow was thrown out.