Problems 1 - Piecewise Polynomial Interpolation

Approximation Theory (MATH3081/4221) — Epiphany 2015 — anthony.yeates@dur.ac.uk

The problem marked $\star$ should be handed in for marking at the lecture on **Thursday 5th February**.
There will be a problem class on this chapter on Monday 2nd February.
I use † to indicate (what I consider to be) trickier problems.

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5. *General splines.* An interpolating spline of degree N is required to have continuous derivatives up to and including order $N-1$. How many additional conditions are required to specify the spline uniquely?

Solution: A degree N spline has $N+1$ unknowns on each of the n intervals (assuming there are $n+1$ knots), so $n(N+1)$ unkowns in total.

We have two interpolation conditions for each interval [$2n$ equations]. We have a matching condition for each derivative from 1 to $N-1$, at each of the $(n-1)$ interior knots [$(N-1)(n-1)$ equations]. So in total we have $2n+(N-1)(n-1)$ equations.

The number of additional conditions required is therefore

$$n(N+1) - 2n - (N-1)(n-1) = N-1.$$

[So for linear splines the system is fully determined, while for cubic splines there are two additional conditions required. This agrees with what we know already.]

6. *Flatness of linear splines.* Let $f(x) = x^3$.

(a) Compute the linear spline s which interpolates f at the knots 0, 1, and 2.

(b) Compute the quadratic polynomial p_2 which interpolates f at the same knots.

(c) Verify that $\|s'\|_2 \leq \|f'\|_2$ and $\|s'\|_2 \leq \|p_2'\|_2$, where the norm is defined on the interval $[0,2]$.

Solution: (a) We have $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $f_0 = 0$, $f_1 = 1$, $f_2 = 8$, so

$$s(x) = \begin{cases} s_0(x), & 0 \leq x \leq 1, \\ s_1(x), & 1 \leq x \leq 2, \end{cases}$$

where

$$s_0(x) = 0 + \left(\frac{1-0}{1-0}\right)(x-0) = x, \qquad s_1(x) = 1 + \left(\frac{8-1}{2-1}\right)(x-1) = 7x-6.$$

(b) The three Lagrange polynomials for these nodes are

$$l_0(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \tfrac{1}{2}(x-1)(x-2),$$
$$l_1(x) = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2),$$
$$l_2(x) = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \tfrac{1}{2}x(x-1).$$

The quadratic interpolant is then

$$p_2(x) = \sum_{i=0}^{2} f_i l_i(x) = l_1(x) + 8l_2(x) = -x(x-2) + 4x(x-1) = x(3x-2).$$

(c) Since

$$s' = \begin{cases} 1, & 0 \leq x \leq 1, \\ 7, & 1 \leq x \leq 2, \end{cases}$$

we have

$$\|s'\|_2^2 = \int_0^2 |s'|^2\, dx = \int_0^1 dx + \int_1^2 7^2\, dx = 50.$$

On the other hand,

$$\|f'\|_2^2 = \int_0^2 9x^4\,dx = \tfrac{9}{5}2^5 = 57.6$$

and

$$\|p_2'\|_2^2 = \int_0^2 (6x-2)^2\,dx = \int_0^2 (36x^2 - 24x + 4)\,dx = 12(2)^3 - 12(2)^2 + 4(2) = 56.$$

7. *Quadratic splines.* Let $a = x_0 < x_1 < \ldots < x_n = b$ be a sequence of equally spaced knots on an interval $[a,b]$, and let $s \in C^1[a,b]$ be a quadratic spline that interpolates a function f at the knots.

   (a) Define the moments $M_i = s'(x_i)$ for $i = 0, \ldots, n$. Construct the spline s in terms of these M_i and derive a system of linear equations for the M_i. For what values of i must they hold? How many extra conditions are necessary?

   † (b) If the function to be interpolated is periodic, we might try to introduce the extra condition $s'(a) = s'(b)$. Show, by considering the resulting system of equations, that this "periodic" quadratic spline exists for certain conditions on the knots.

   (c) Use the above to approximate $f(x) = \sin(2\pi x)$ on the interval $[0,1]$ with four knots.

Solution: (a) Follow a similar procedure as we used in lectures for the cubic spline. Since each piece s_i is quadratic, s_i' must be linear, so in $[x_i, x_{i+1}]$ we can write

$$s_i'(x) = \left(\frac{x_{i+1} - x}{h}\right) M_i + \left(\frac{x - x_i}{h}\right) M_{i+1},$$

where h is the (uniform) grid spacing. This construction guarantees that s' is continuous at the interior knots. Now integrate to find

$$s_i(x) = \left(\frac{-(x_{i+1} - x)^2}{2h}\right) M_i + \left(\frac{(x - x_i)^2}{2h}\right) M_{i+1} + \alpha_i,$$

We must then apply the interpolation conditions $s_i(x_i) = f_i$ and $s_i(x_{i+1}) = f_{i+1}$, for each interval $i = 0, \ldots, n-1$. The first gives the equations

$$\left(\frac{-(x_{i+1} - x_i)^2}{2h}\right) M_i + \alpha_i = f_i \quad\iff\quad -\tfrac{h}{2} M_i + \alpha_i = f_i, \quad \text{for } i = 0, \ldots, n-1.$$

The second gives the equations

$$\left(\frac{(x_{i+1} - x_i)^2}{2h}\right) M_{i+1} + \alpha_i = f_{i+1} \quad\iff\quad \tfrac{h}{2} M_{i+1} + \alpha_i = f_{i+1}, \quad \text{for } i = 0, \ldots, n-1.$$

We can eliminate the α_i to give a system of n equations

$$\tfrac{h}{2} M_i + \tfrac{h}{2} M_{i+1} = f_{i+1} - f_i, \quad \text{for } i = 0, \ldots, n-1.$$

Since there are $n+1$ unknowns (the M_i), one additional condition is necessary [cf. Problem 5].

(b) The extra condition gives the extra equation $s_0'(x_0) = s_{n-1}'(x_n)$, i.e.,

$$\left(\frac{x_1 - x_0}{h}\right) M_0 = \left(\frac{x_n - x_{n-1}}{h}\right) M_n \quad\iff\quad M_0 - M_n = 0.$$

Thus the system is

$$\begin{pmatrix} \frac{h}{2} & \frac{h}{2} & 0 & \cdots & & 0 \\ 0 & \frac{h}{2} & \frac{h}{2} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & & \frac{h}{2} & \frac{h}{2} \\ 1 & 0 & \cdots & & 0 & -1 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n-1} \\ M_n \end{pmatrix} = \begin{pmatrix} f_1 - f_0 \\ f_2 - f_1 \\ \vdots \\ f_n - f_{n-1} \\ 0 \end{pmatrix}$$

For the quadratic spline to exist, the determinant of the above matrix (call it A) must be non-zero. We have

$$\det A = (\tfrac{h}{2})^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & & -1 \end{vmatrix} = (\tfrac{h}{2})^n \begin{vmatrix} 1 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 1 & 1 \\ 0 & \cdots & & -1 \end{vmatrix} + (\tfrac{h}{2})^n(-1)^n \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 1 & 1 \end{vmatrix}$$
$$= (\tfrac{h}{2})^n\Big(-1 + (-1)^n\Big).$$

which is non-zero iff $n+1$ is even (i.e., there are an even number of knots).

(c) We have $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$, and $f_0 = 0$, $f_1 = \frac{\sqrt{3}}{2}$, $f_2 = -\frac{\sqrt{3}}{2}$, $f_3 = 0$. Since $h = \frac{1}{3}$ the system is

$$\frac{1}{6}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 6 & 0 & 0 & -6 \end{pmatrix}\begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ -\sqrt{3} \\ \sqrt{3}/2 \\ 0 \end{pmatrix}$$

The solution to this system (by e.g. Gaussian elimination) is

$$M_0 = 6\sqrt{3}, \quad M_1 = -3\sqrt{3}, \quad M_2 = -3\sqrt{3}, \quad M_3 = 6\sqrt{3}.$$

Computing the $\alpha_i = f_i + \frac{1}{6}M_i$ we find

$$\alpha_0 = \sqrt{3}, \quad \alpha_1 = 0, \quad \alpha_2 = -\sqrt{3}.$$

Hence the pieces of the quadratic spline are

$$s_0(x) = \frac{-(\frac{1}{3}-x)^2}{\frac{2}{3}}(6\sqrt{3}) + \frac{x^2}{\frac{2}{3}}(-3\sqrt{3}) + \sqrt{3},$$
$$s_1(x) = \frac{-(\frac{2}{3}-x)^2}{\frac{2}{3}}(-3\sqrt{3}) + \frac{(\frac{1}{3}-x)^2}{\frac{2}{3}}(-3\sqrt{3}),$$
$$s_2(x) = \frac{-(1-x)^2}{\frac{2}{3}}(-3\sqrt{3}) + \frac{(\frac{2}{3}-x)^2}{\frac{2}{3}}(6\sqrt{3}) - \sqrt{3}.$$

8. *Is it a spline?* Is the following function a cubic spline? Why or why not?

$$s(x) = \begin{cases} 0, & x < 0, \\ x^3, & 0 \le x < 1, \\ x^3 + (x-1)^3, & 1 \le x < 2, \\ -(x-3)^3 - (x-4)^3, & 2 \le x < 3, \\ -(x-4)^3, & 3 \le x < 4, \\ 0, & 4 \le x. \end{cases}$$

Solution: To decide this, we must check the continuity (interpolation) and derivative conditions. Firstly, s is continuous at each node. Next we have

$$s'(x) = \begin{cases} 0, & x < 0, \\ 3x^2, & 0 \le x < 1, \\ 3x^2 + 3(x-1)^2, & 1 \le x < 2, \\ -3(x-3)^2 - 3(x-4)^2, & 2 \le x < 3, \\ -3(x-4)^2, & 3 \le x < 4, \\ 0, & 4 \le x. \end{cases}$$

At $x = 2$, we find that s' is discontinuous (15 on one side and -15 on the other), so this cannot be a cubic spline.

9. *Natural cubic spline.* Compute the natural cubic spline interpolating $(0,0)$, $(1,\frac{1}{2})$, $(2,0)$.

Solution: Each piece of the cubic spline $i = 0, 1$ takes the form

$$s_i(x) = \frac{(x_{i+1}-x)^3}{6h}M_i + \frac{(x-x_i)^3}{6h}M_{i+1} + \alpha_i(x-x_i) + \beta_i(x_{i+1}-x),$$

so using the natural conditions $M_0 = M_2 = 0$ we have

$$s_0(x) = \frac{(1-x)^3}{6}M_0 + \frac{x^3}{6}M_1 + \alpha_0 x + \beta_0(1-x) = \tfrac{1}{6}x^3 M_1 + \alpha_0 x + \beta_0(1-x),$$
$$s_1(x) = \frac{(2-x)^3}{6}M_1 + \frac{(x-1)^3}{6}M_2 + \alpha_1(x-1) + \beta_1(2-x) = \tfrac{1}{6}(2-x)^3 M_1 + \alpha_1(x-1) + \beta_1(2-x).$$

The interpolation conditions give

$$s_0(0) = 0 \implies \beta_0 = 0, \qquad s_0(1) = \tfrac{1}{2} \implies \alpha_0 = \tfrac{1}{2} - \tfrac{1}{6}M_1,$$
$$s_1(1) = \tfrac{1}{2} \implies \beta_1 = \tfrac{1}{2} - \tfrac{1}{6}M_1, \qquad s_1(2) = 0 \implies \alpha_1 = 0.$$

Hence

$$s_0(x) = (\tfrac{1}{6}x^3 - \tfrac{1}{6}x)M_1 + \tfrac{1}{2}x, \qquad s_1(x) = (\tfrac{1}{6}(2-x)^3 - \tfrac{1}{6}(2-x))M_1 + \tfrac{1}{2}(2-x).$$

Finally we determine M_1 from the first derivative condition at $x = 1$. Matching $s_0'(1) = s_1'(1)$ gives

$$(\tfrac{1}{2}(1)^2 - \tfrac{1}{6})M_1 + \tfrac{1}{2} = (-\tfrac{1}{2}(1)^2 + \tfrac{1}{6})M_1 - \tfrac{1}{2} \quad \implies \quad M_1 = -\tfrac{3}{2}.$$

So the two pieces of the cubic spline are

$$s_0(x) = -\tfrac{1}{4}x^3 + \tfrac{3}{4}x, \qquad s_1(x) = -\tfrac{1}{4}(2-x)^3 + \tfrac{3}{4}(2-x).$$

⋆ **10.** *Not-a-knot cubic spline.* Let s be a cubic spline interpolating a function f at the evenly-spaced knots $a = x_0 < x_1 < \ldots < x_n = b$, with spacing h, and suppose that s satisfies the so-called "not-a-knot" conditions that s''' is continuous at the two knots x_1 and x_{n-1}.

(a) Derive the system of linear equations satisfied by the moments $M_i := s''(x_i)$ for $n+1$ knots.

(b) Suppose we try to find the not-a-knot cubic spline through the data $(0,0)$, $(1,1)$ and $(2,8)$. Write down the system of linear equations in this case, show that the solution is not unique, and write the resulting spline in terms of M_0 as a free parameter.

(c) Explain why you expected the solution in (b) to be non-unique.

(d) By considering the two pieces of the cubic spline derived in (b), or otherwise, explain why this type of cubic spline is called "not-a-knot".

Solution: (a) As for the natural cubic spline, the general form is

$$s_i(x) = \frac{(x_{i+1}-x)^3}{6h}M_i + \frac{(x-x_i)^3}{6h}M_{i+1} + \alpha_i(x-x_i) + \beta_i(x_{i+1}-x),$$

where the moments M_i and the constants α_i and β_i are to be determined. First apply the interpolation conditions:

$$f_i = s_i(x_i) = \tfrac{1}{6}h^2 M_i + h\beta_i \quad \implies \quad \beta_i = (f_i - \tfrac{1}{6}h^2 M_i)/h,$$
$$f_{i+1} = s_i(x_{i+1}) = \tfrac{1}{6}h^2 M_{i+1} + h\alpha_i \quad \implies \quad \alpha_i = (f_{i+1} - \tfrac{1}{6}h^2 M_{i+1})/h.$$

We then have to apply the first derivative condition. For each piece, we have

$$s_i'(x) = \frac{-(x_{i+1}-x)^2}{2h}M_i + \frac{(x-x_i)^2}{2h}M_{i+1} + \alpha_i - \beta_i,$$
$$= \frac{-(x_{i+1}-x)^2}{2h}M_i + \frac{(x-x_i)^2}{2h}M_{i+1} + \frac{f_{i+1}-f_i}{h} + \frac{M_i - M_{i+1}}{6}h.$$

Equating $s'_{i-1}(x_i)$ and $s'_i(x_i)$ and rearranging gives the equations

$$hM_{i-1} + 4hM_{i-1} + hM_{i+1} = 6\left(\frac{f_{i-1} - 2f_i + f_{i+1}}{h}\right), \quad \text{for } i = 1, \ldots, n = 1.$$

Up to here, everything is the same as for the natural cubic spline. The new things here are the two not-a-knot conditions. We have

$$s_i'''(x) = \frac{M_{i+1} - M_i}{h}$$

so the two equations that close the system are

$$s_0'''(x_1) = s_1'''(x_1) \iff \frac{M_1 - M_0}{h} = \frac{M_2 - M_1}{h} \iff M_0 - 2M_1 + M_2 = 0,$$
$$s_{n-2}'''(x_{n-1}) = s_{n-1}'''(x_{n-1}) \iff \frac{M_{n-1} - M_{n-2}}{h} = \frac{M_n - M_{n-1}}{h} \iff M_{n-2} - 2M_{n-1} + M_n = 0.$$

In summary, the system of equations has the form

$$\begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & \\ h & 4h & h & 0 & & \vdots & \\ 0 & h & 4h & h & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & & h & 4h & h \\ 0 & \cdots & & & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{pmatrix} = \frac{6}{h} \begin{pmatrix} 0 \\ f_0 - 2f_1 + f_2 \\ f_1 - 2f_2 + f_3 \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \\ 0 \end{pmatrix} \tag{1}$$

(b) We can use the general system derived in part (a) with $h = 1$, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $f_0 = 0$, $f_1 = 1$, $f_2 = 8$, i.e.,

$$\begin{pmatrix} 1 & -2 & 1 \\ 1 & 4 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 36 \\ 0 \end{pmatrix}.$$

This system clearly has multiple solutions, since there are only two distinct equations. Solving in terms of M_0 leads to

$$M_1 = 6, \qquad M_2 = 12 - M_0,$$

which gives

$$\alpha_0 = 1 - \tfrac{1}{6}M_1 = 0, \quad \beta_0 = -\tfrac{1}{6}M_0, \quad \alpha_1 = 8 - \tfrac{1}{6}(12 - M_0) = 6 + \tfrac{1}{6}M_0, \quad \beta_1 = 0.$$

Thus the components of the spline are

$$s_0(x) = \tfrac{1}{6}(1-x)^3M_0 + x^3 - \tfrac{1}{6}M_0(1-x), \qquad s_1(x) = (2-x)^3 + \tfrac{1}{6}(x-1)^3(12-M_0) + (6 + \tfrac{1}{6}M_0)(x-1). \tag{2}$$

(c) I expected the solution to be non-unique because we are only really imposing one additional condition: for three knots, we have that $x_1 = x_{n-1}$.

(d) If you multiply out the brackets in (2), you will see that $s_0 \equiv s_1$. So $x = 1$ is "not a knot", in the sense that s is infinitely differentiable at $x = 1$. For larger n, this will be true at the knots x_1 and x_{n-1} (because two cubics which match first, second and third derivatives must be the same), meaning that these two knots are removable! At the other knots in between, the cubic pieces will of course be different on either side.

† 11. By reducing the linear system found in Problem 10(a) to a strictly diagonally dominant form, or otherwise, show that there is a unique not-a-knot cubic spline for $n \geq 3$.

Solution: One possible way is to eliminate M_0 and M_n from the second and $(n-1)$th rows of the matrix in (1) respectively. To do this, subtract h times row 1 from row 2, and h times row n from row $n-1$. This leaves an $(n-1) \times (n-1)$ linear system

$$\begin{pmatrix} 6h & 0 & 0 & & 0 \\ h & 4h & h & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ & & h & 4h & h \\ 0 & \cdots & & 0 & 6h \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ \vdots \\ M_{n-1} \end{pmatrix} = \frac{6}{h} \begin{pmatrix} f_0 - 2f_1 + f_2 \\ f_1 - 2f_2 + f_3 \\ \vdots \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \end{pmatrix},$$

which is strictly diagonally dominant and therefore invertible. One can then find M_0 and M_n from the original rows 1 and n.

12. *Holladay's Theorem for complete cubic splines.* The cubic spline interpolating the points (x_0, f_0), (x_1, f_1), $\ldots (x_n, f_n)$ and satisfying the end conditions $s'(x_0) = c$, $s'(x_n) = d$ for fixed constants c, d is known as the *complete* cubic spline. Adapt the proof of Theorem 1.3 to show that the *complete* cubic spline minimises $\|f''\|_2$ among all functions $f \in C^2[x_0, x_n]$ that satisfy $f'(x_0) = c$ and $f'(x_n) = d$.

Solution: As for the natural cubic spline we have

$$\|f'' - s''\|_2^2 = \|f''\|_2^2 - \|s''\|_2^2 - 2\int_a^b (f'' - s'')s''\,dx,$$

and we need to show that the last term vanishes so that $\|s''\|_2 \leq \|f''\|_2$. As for the natural spline, the last term reduces to

$$\begin{aligned}\int_a^b (f'' - s'')s''\,dx &= \Big(f'(x_n) - s'_{n-1}(x_n)\Big)s''_{n-1}(x_n) - \Big(f'(x_0) - s'_0(x_0)\Big)s''_0(x_0),\\ &= \Big(f'(x_n) - d\Big)s''_{n-1}(x_n) - \Big(f'(x_0) - c\Big)s''_0(x_0)\end{aligned}$$

So provided that we consider only those f for which $f'(x_0) = c$ and $f'(x_n) = d$, this term vanishes.

13. *General B-splines.* The B-spline of degree N for equally-spaced knots is defined as

$$B_N(x) = \sum_{k=0}^{N+1} (-1)^k \binom{N+1}{k} (x - kh)_+^N,$$

where h is the knot spacing, the subscript $+$ means the *positive part*

$$f(x)_+ = \begin{cases} f(x), & f(x) > 0,\\ 0, & f(x) \leq 0,\end{cases}$$

and

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is the usual binomial coefficient.

(a) Find and sketch the linear B-spline $B_1(x)$ and verify that it is indeed a linear spline function.

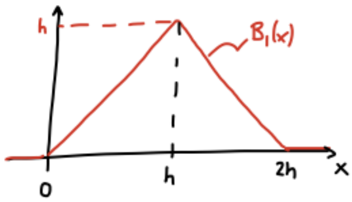
(b) Find the cubic B-spline $B_3(x)$ and verify that it is the same function as given in the lecture.

† (c) Show that the function $B_N(x)$ is always a spline of degree N.

Solution: (a) For $N = 1$ we have

$$\begin{aligned}B_1(x) &= (-1)^0\binom{2}{0}x_+ + (-1)^1\binom{2}{1}(x-h)_+ + (-1)^2\binom{2}{2}(x-2h)_+,\\ &= \begin{cases} 0, & x < 0,\\ x, & 0 \leq x \leq h,\\ x - 2(x-h) = 2h - x, & h \leq x \leq 2h,\\ x - 2(x-h) + x - 2h = 0, & x > 2h.\end{cases}\end{aligned}$$

This function is piecewise-linear and continuous, so is a linear spline.



(b) For $N = 3$ we have

$$\begin{aligned} B_3(x) &= (-1)^0 \binom{4}{0} x_+^3 + (-1)^1 \binom{4}{1} (x-h)_+^3 + (-1)^2 \binom{4}{2} (x-2h)_+^3 \\ &\quad + (-1)^3 \binom{4}{3} (x-3h)_+^3 + (-1)^4 \binom{4}{4} (x-4h)_+^3, \\ &= x_+^3 - 4(x-h)_+^3 + 6(x-2h)_+^3 - 4(x-3h)_+^3 + (x-4h)_+^3, \\ &= \begin{cases} 0, & x < 0, \\ x^3, & 0 \le x \le h, \\ x^3 - 4(x-h)^3, & h \le x \le 2h, \\ x^3 - 4(x-h)^3 + 6(x-2h)^3, & 2h \le x \le 3h, \\ x^3 - 4(x-h)^3 + 6(x-2h)^3 - 4(x-3h)^3, & 3h \le x \le 4h, \\ x^3 - 4(x-h)^3 + 6(x-2h)^3 - 4(x-3h)^3 + (x-4h)^3, & x > 4h. \end{cases} \end{aligned}$$

To see that this is equal to the cubic B-spline given in the lecture notes, multiply out the penultimate part and show that it equals $(4h - x)^3$, and multiply out the last part and show that it vanishes.

(c) To see that $B_N(x)$ is a spline of degree N, we need to show that it is continuous at the knots $x = 0, h, 2h, \ldots, (N+1)h$ and has continuous first and second derivatives there. To see why this holds, look at what we found above for $B_3(x)$. You see that in general the neighbouring pieces for $(k-1)h \le x \le kh$ and $kh \le x \le (k+1)h$ differ by a term

$$(-1)^k \binom{N+1}{k} (x - kh)^N$$

which vanishes at $x = kh$ along with its first $N-1$ derivatives. Therefore this pair of neighbouring pieces match at $x = kh$ along with their first $N-1$ derivatives. The same is true at every knot, so $B_N(x)$ is a spline of degree N.

14. *Applying linear B-splines.* Any linear spline may be expressed as

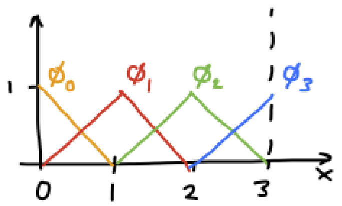
$$s(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x)$$

where the basis functions are linear B-splines $\phi_i(x) = h^{-1} B_1(x - x_{i-1})$, with B_1 as defined in Problem 13. For the data $(0,1)$, $(1,2)$, $(2,3)$ and $(3,4)$, write down and sketch the linear spline basis functions $\phi_k(x)$, and hence form the linear spline $s(x)$. Use this to evaluate the spline at $x = \frac{3}{2}$.

Solution: The four required basis functions are

$$\phi_0(x) = \begin{cases} 2-(x+1), & 0 \le x \le 1, \\ 0, & 1 \le x \le 2, \\ 0, & 2 \le x \le 3, \end{cases}, \qquad \phi_1(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2, \\ 0, & 2 \le x \le 3, \end{cases},$$

$$\phi_2(x) = \begin{cases} 0, & 0 \le x \le 1, \\ x-1, & 1 \le x \le 2, \\ 2-(x-1), & 2 \le x \le 3, \end{cases}, \qquad \phi_3(x) = \begin{cases} 0, & 0 \le x \le 1, \\ 0, & 1 \le x \le 2, \\ x-2, & 2 \le x \le 3. \end{cases}$$



The interpolation conditions at the four nodes then give

$$\begin{cases} \alpha_0\phi_0(0)+\alpha_1\phi_1(0)+\alpha_2\phi_2(0)+\alpha_3\phi_3(0) &= 1,\\ \alpha_0\phi_0(1)+\alpha_1\phi_1(1)+\alpha_2\phi_2(1)+\alpha_3\phi_3(1) &= 2,\\ \alpha_0\phi_0(2)+\alpha_1\phi_1(2)+\alpha_2\phi_2(2)+\alpha_3\phi_3(2) &= 3,\\ \alpha_0\phi_0(3)+\alpha_1\phi_1(3)+\alpha_2\phi_2(3)+\alpha_3\phi_3(3) &= 4. \end{cases} \implies \begin{cases} \alpha_0 &= 1,\\ \alpha_1 &= 2,\\ \alpha_2 &= 3,\\ \alpha_3 &= 4. \end{cases}$$

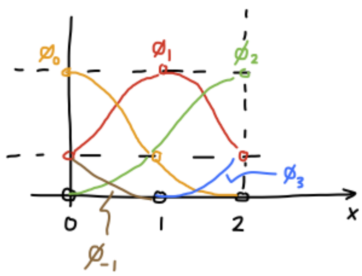
Hence the linear spline is

$$s(x) = \begin{cases} 1-x+2x, & 0\le x\le 1,\\ 2(2-x)+3(x-1), & 1\le x\le 2,\\ 3(3-x)+4(x-2), & 2\le x\le 3. \end{cases}$$

We have $s(\frac{3}{2}) = 2(2-\frac{3}{2}) + 3(\frac{3}{2}-1) = \frac{5}{2}$.

15. *Applying cubic B-splines.* If cubic B-splines are used to compute the *complete* cubic spline that interpolates the function $f(x) = \sin(\pi x/2)$ at the knots $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$, find the linear system that has to be solved.

Solution: Here $h = 1$ and $n = 2$, so we need ϕ_{-1}, ϕ_0, ϕ_1, ϕ_2, ϕ_3. We have simply $x_j = j$. It is dangerous not to first sketch the basis functions that are involved:



Using the formula for cubic B-splines given in the lecture, and taking only the parts in $[0,1]$ and $[1,2]$, we get

$$\phi_{-1}(x) = B_3(x-x_{-3}) = \begin{cases} (4-x-3)^3, & 0\le x\le 1,\\ 0, & 1\le x\le 2, \end{cases}$$

$$\phi_0(x) = B_3(x-x_{-2}) = \begin{cases} (x+2)^3-4(x+1)^3+6x^3, & 0\le x\le 1,\\ (2-x)^3, & 1\le x\le 2, \end{cases}$$

$$\phi_1(x) = B_3(x-x_{-1}) = \begin{cases} (x+1)^3-4x^3, & 0\le x\le 1,\\ (x+1)^3-4x^3+6(x-1)^3, & 1\le x\le 2, \end{cases}$$

$$\phi_2(x) = B_3(x-x_0) = \begin{cases} x^3, & 0\le x\le 1,\\ x^3-4(x-1)^3, & 1\le x\le 2, \end{cases}$$

$$\phi_3(x) = B_3(x-x_1) = \begin{cases} 0, & 0\le x\le 1,\\ (x-1)^3, & 1\le x\le 2. \end{cases}$$

The interpolation conditions give three equations

$$\begin{aligned} \alpha_{-1}\phi_{-1}(0)+\alpha_0\phi_0(0)+\alpha_1\phi_1(0)+\alpha_2\phi_2(0)+\alpha_3\phi_3(0) &= \sin(0) = 0,\\ \alpha_{-1}\phi_{-1}(1)+\alpha_0\phi_0(1)+\alpha_1\phi_1(1)+\alpha_2\phi_2(1)+\alpha_3\phi_3(1) &= \sin(\pi/2) = 1,\\ \alpha_{-1}\phi_{-1}(2)+\alpha_0\phi_0(2)+\alpha_1\phi_1(2)+\alpha_2\phi_2(2)+\alpha_3\phi_3(2) &= \sin(\pi) = 0. \end{aligned}$$

You can quickly get the values of the basis functions at the knots by referring to the sketch and knowing that they all have the same shape, with knot values 0, 1, 4, 1, 0. This gives

$$\begin{aligned} \alpha_{-1} + 4\alpha_0 + \alpha_1 &= 0, \\ \alpha_0 + 4\alpha_1 + \alpha_2 &= 1, \\ \alpha_1 + 4\alpha_2 + \alpha_3 &= 0. \end{aligned}$$

We get a further two equations from the complete conditions that $s_0'(0) = f'(0) = \frac{\pi}{2}$ and $s_1'(2) = f'(2) = -\frac{\pi}{2}$. Using the fact that the derivatives of the basis functions are 3, 0, -3 at the knots, we have

$$\begin{aligned} \alpha_{-1}\phi_{-1}'(0) + \alpha_0\phi_0'(0) + \alpha_1\phi_1'(0) = \tfrac{\pi}{2} \quad &\iff \quad -3\alpha_{-1} + 3\alpha_1 = \tfrac{\pi}{2}, \\ \alpha_1\phi_1'(2) + \alpha_2\phi_2'(2) + \alpha_3\phi_3'(2) = -\tfrac{\pi}{2} \quad &\iff \quad -3\alpha_1 + 3\alpha_3 = -\tfrac{\pi}{2}. \end{aligned}$$

So the linear system we have to solve is

$$\begin{pmatrix} -3 & 0 & 3 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \\ 0 \\ 1 \\ 0 \\ -\frac{\pi}{2} \end{pmatrix}$$

Here is a plot of the spline obtained from solving the linear system, and the original function f (black dashed line) - you can see the approximation is pretty good: