Mutations of non-integer quivers: finite mutation type

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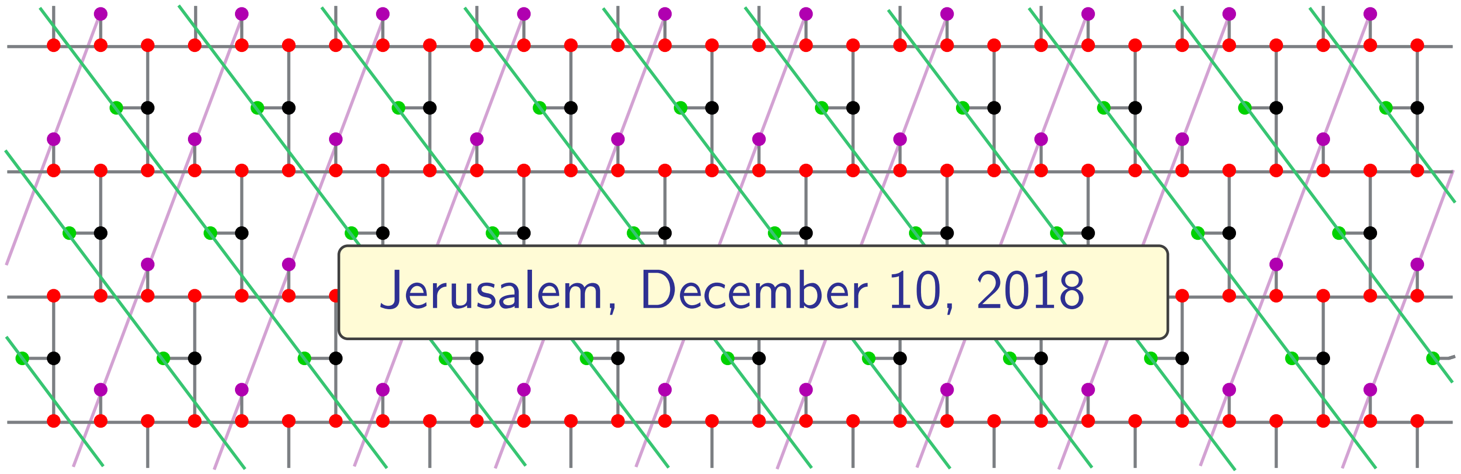
(joint works with Pavel Tumarkin and Philipp Lampe)



1. Mutations of non-integer quivers:

- $B = \{b_{ij}\}$ a skew-symmetric matrix with $b_{ij} \in \mathbb{R}$.
- Mutate B by usual mutation rule:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise} \end{cases}$$

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Why:

- Philipp Lampe, *On the approximate periodicity of sequences attached to noncrystallographic root systems*, To appear in Experimental Mathematics (2018).
- Integer finite type contains types A, B, C, D, E, F .... - but not H_3, H_4!
- Geometric realization of acyclic mutation classes by partial reflections allow non-integer values.

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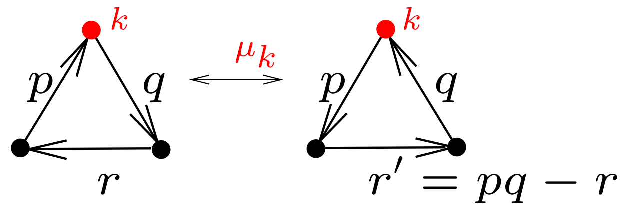
- **Mutation** μ_k of a non-integer quiver:

  1) reverse all arrows incident to k;
  2) for every path $i \xrightarrow{p} k \xrightarrow{q} j$ with $p, q > 0$ apply



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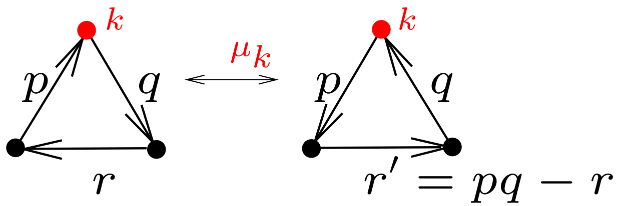
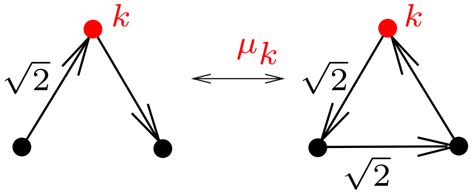


- Example: B_3



2. In Rank 3: $\left\{ \begin{array}{l} \text{acyclic} \\ \text{mutation class} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Geometric realisation} \\ \text{by partial reflections} \end{array} \right\}$

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- Let $G = \langle s_1, \ldots, s_n \rangle$ where $s_i = r_{v_i}$.

  G acts discretely in a cone $C \subset V$ with fundamental domain

$$F = \bigcap_{i=1}^{n} \Pi_i^-, \quad \text{where } \Pi_i^- = \{u \in V \mid \langle u, v_i \rangle < 0\}.$$

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Theorem. (Barot, Geiss, Zelevinsky'06; Seven'15)
For integer quivers (but also for real ones in rank 3):
The values $\langle v_i, v_j \rangle$ change under mutations
in the same way as the weights of the arrows in Q.

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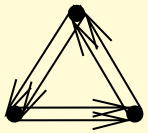
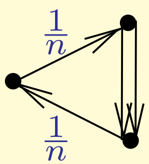
Theorem [FT'16]. Any mutation-finite rank 3 quiver is mutation-equivalent to one of

- Markov quiver:

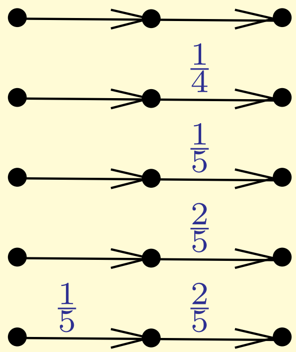


- Affine quivers:



- Finite type quivers:

A_3
B_3
H_3
H_3'
H_3''



(Here, a label $\frac{k}{m}$ stays for the weight $|b_{ij}| = 2\cos\frac{k\pi}{m}$.)

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- If Q is acyclic then the corresponding triangle is acute-angled (or has 2 obtuse angles).

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- If Q is cyclic then the corresp. triangle has 1 or 3 obtuse angles.

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- All of them geometric!

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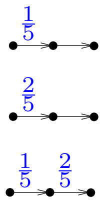
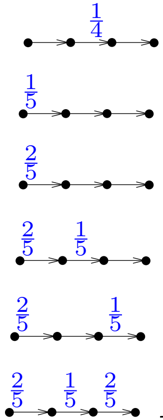
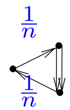
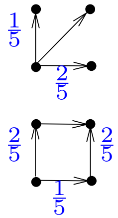
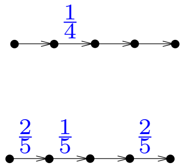
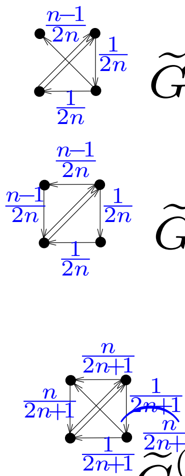
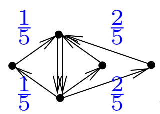
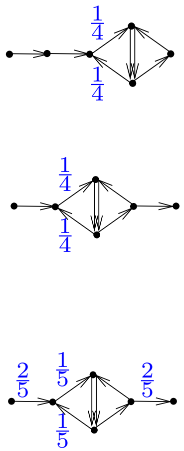
Answer:

Thm. [FT'18]
Mut.-fin. $\Leftrightarrow$
$G_{2,n}$ or
orbifold or
as in Table:

	rank 3	rank 4	rank 5	rank 6
Finite type	 H_3, H_3', H_3''	 F_4, H_4, H_4', H_4'', H_4''', H_4''''		
Affine type	 $\widetilde{G}_{2,n}$	 $\widetilde{H}_3$, $\widetilde{H}_3'$	 $\widetilde{F}_4$, $\widetilde{H}_4$	
Extended affine type		 $\widetilde{G}^{(*,+)}_{2,2n}$, $\widetilde{G}^{(*,*)}_{2,2n}$, $\widetilde{G}^{(*,*)}_{2,2n+1}$	 $H_3^{(1,1)}$	 $F_4^{(*,+)}$, $F_4^{(*,*)}$, $H_4^{(1,1)}$

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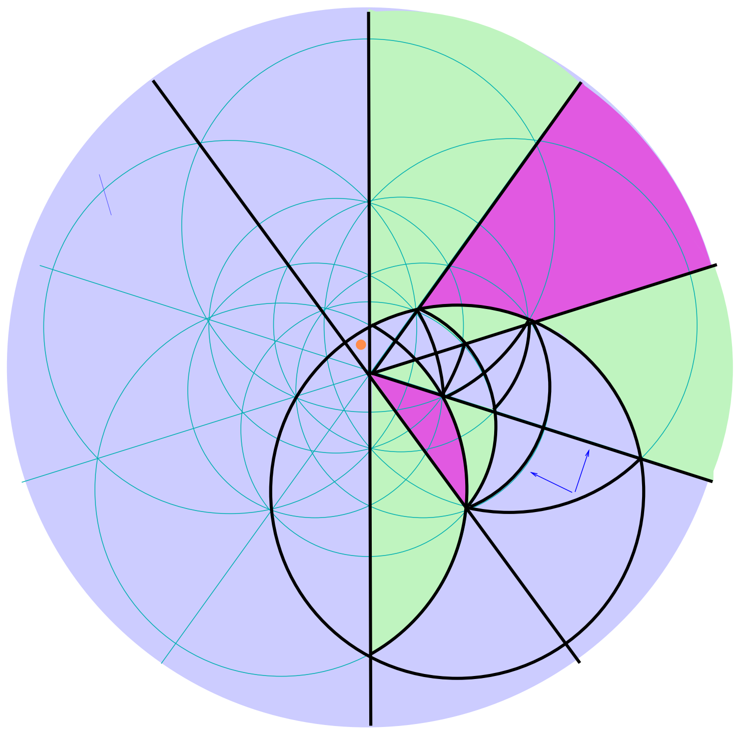
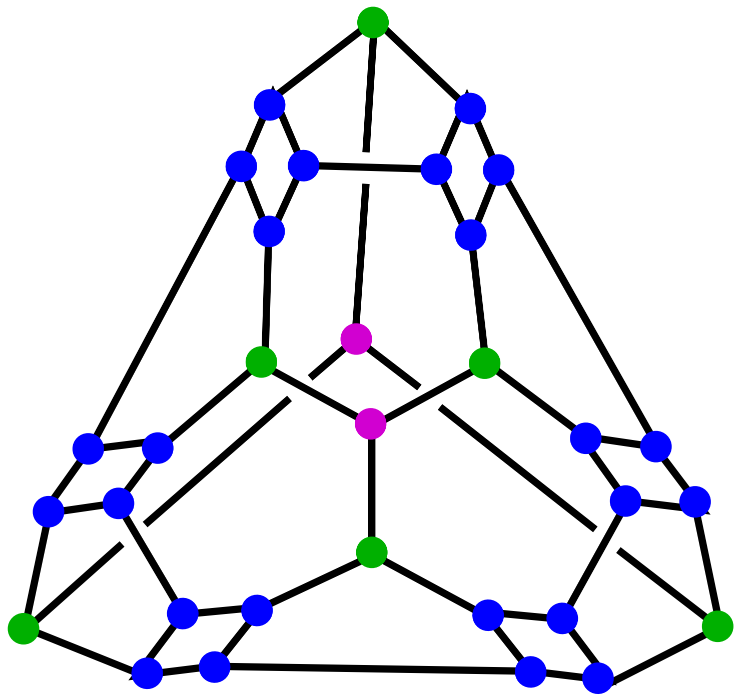
- One mutation class can have many acyclic belts
- and contain many acyclic quivers distinct up to sink/source mutations.

4. Exchange graph:

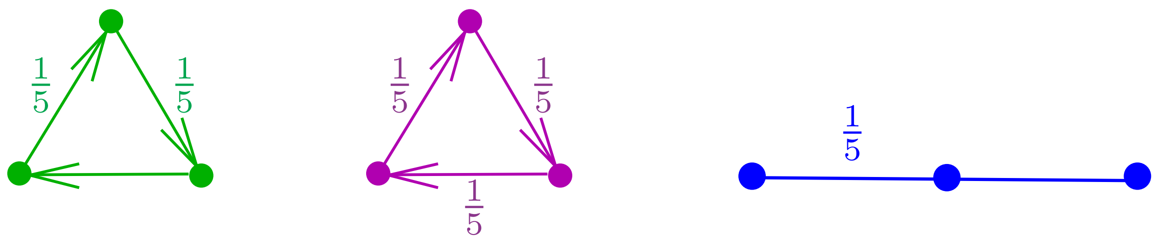
Rank 3 - finite type

- Example: H_3 (mutation class and "exchange graph")





(cf. generalised associahedron in Fomin - Reading)



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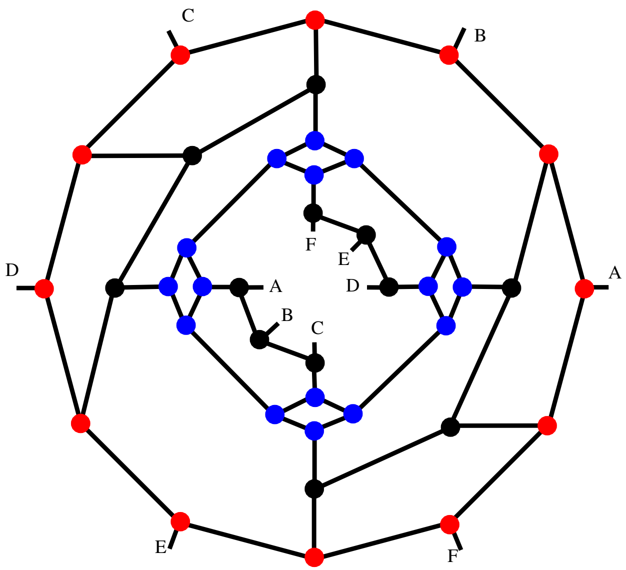
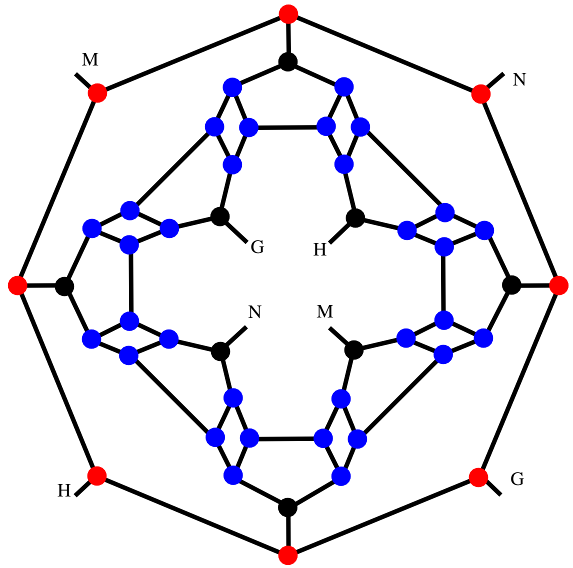
- Exchange graphs for H_3' and H_3'' are graphs on a torus (with two acyclic belts each):

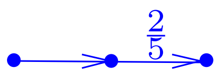


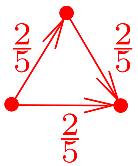
H_3'

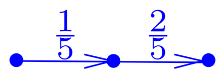


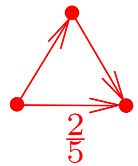
H_3''

- Two different acyclic representatives in each of H_3' and H_3'':









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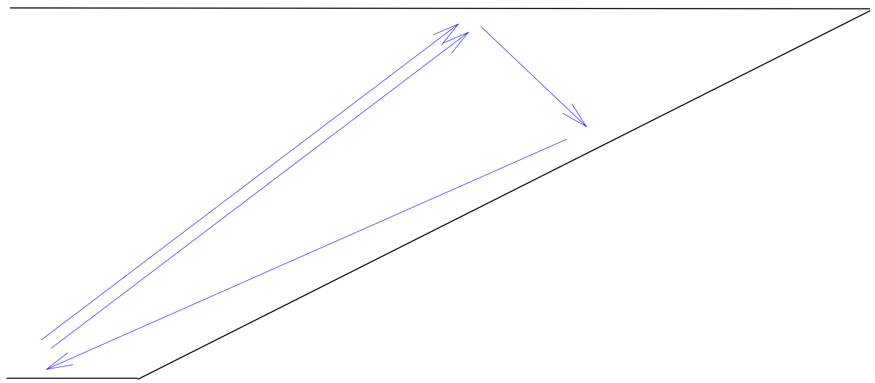
Rank 3 - affine type

from here: joint with Philipp Lampe



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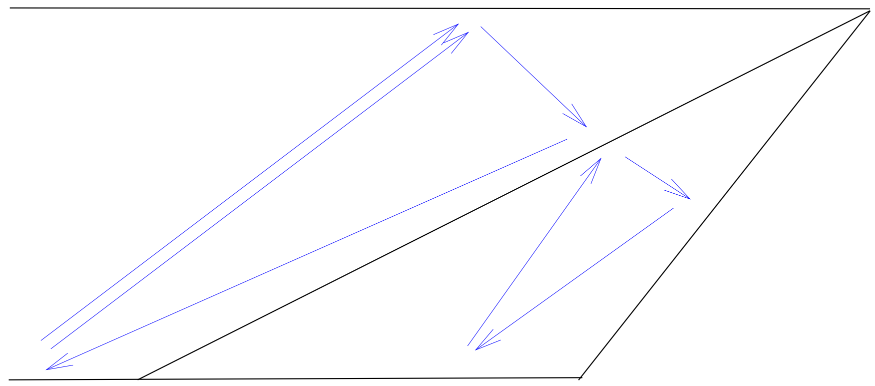

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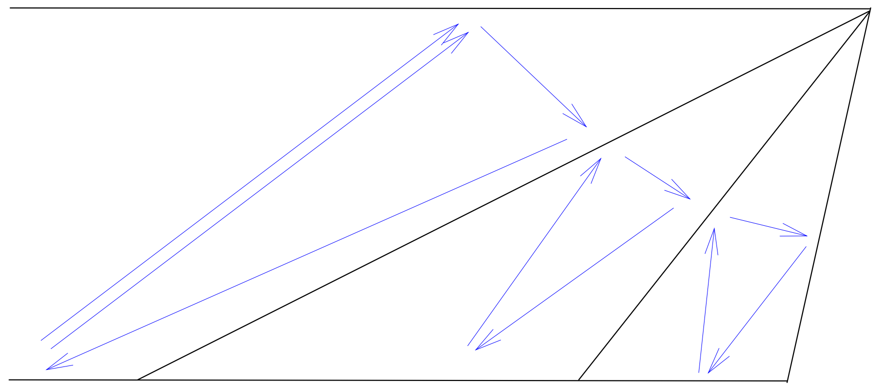

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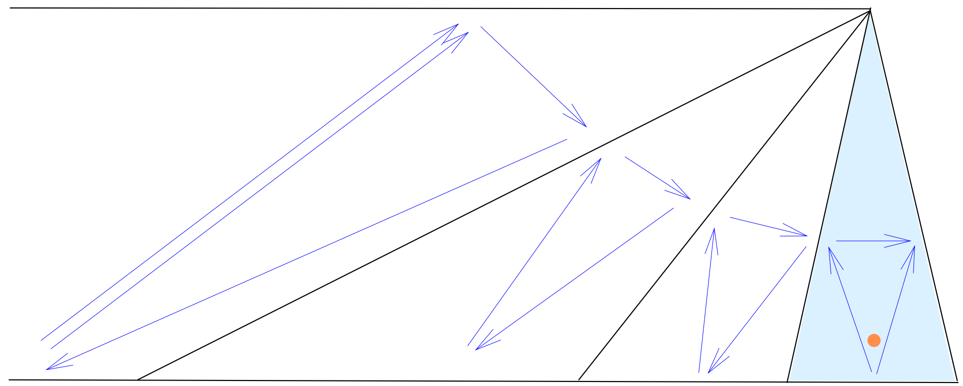

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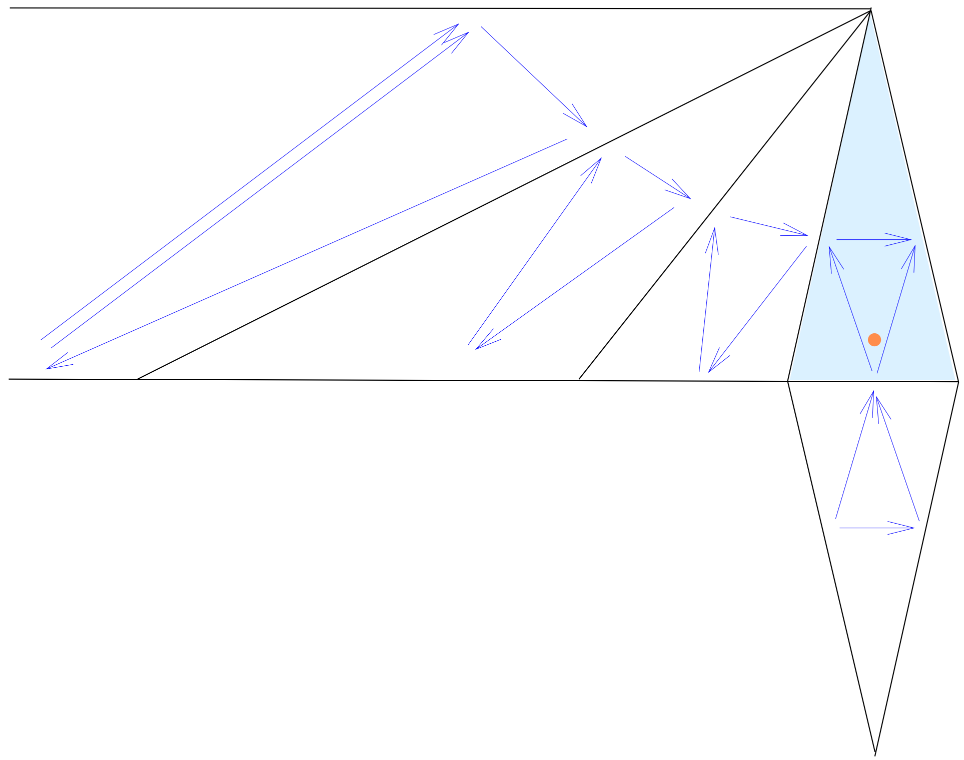

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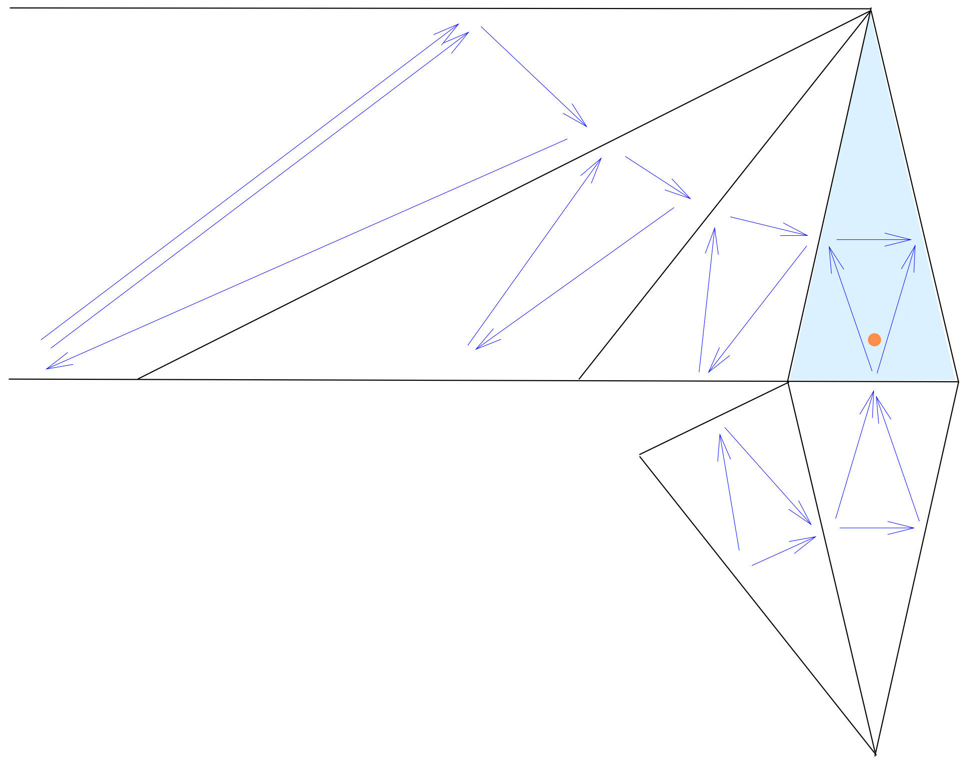

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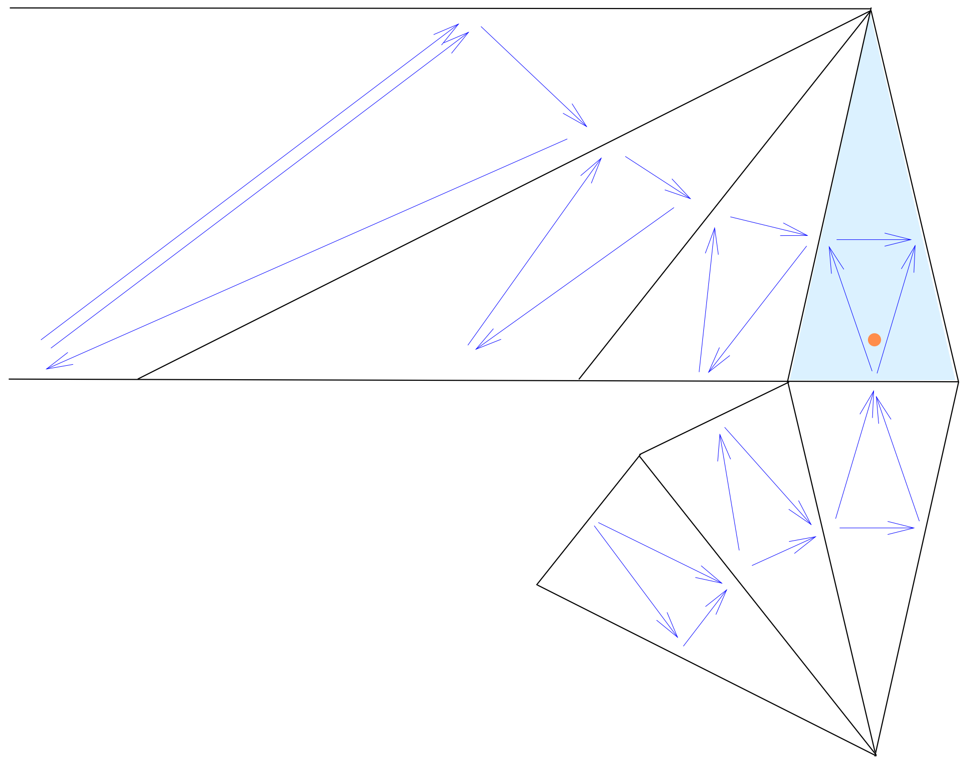

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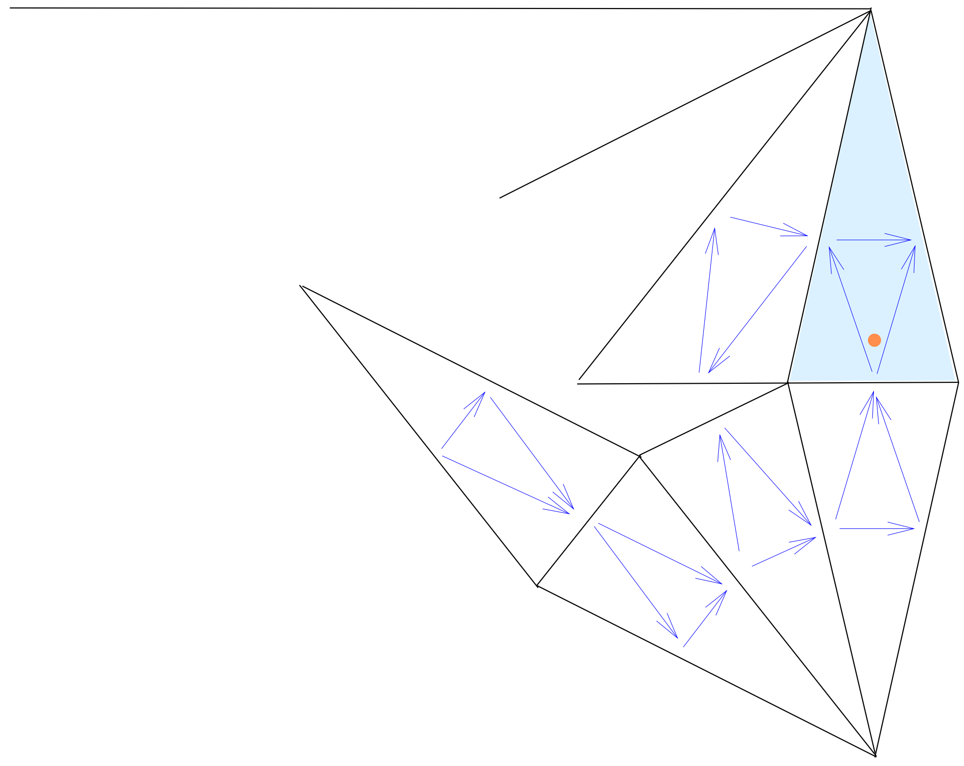

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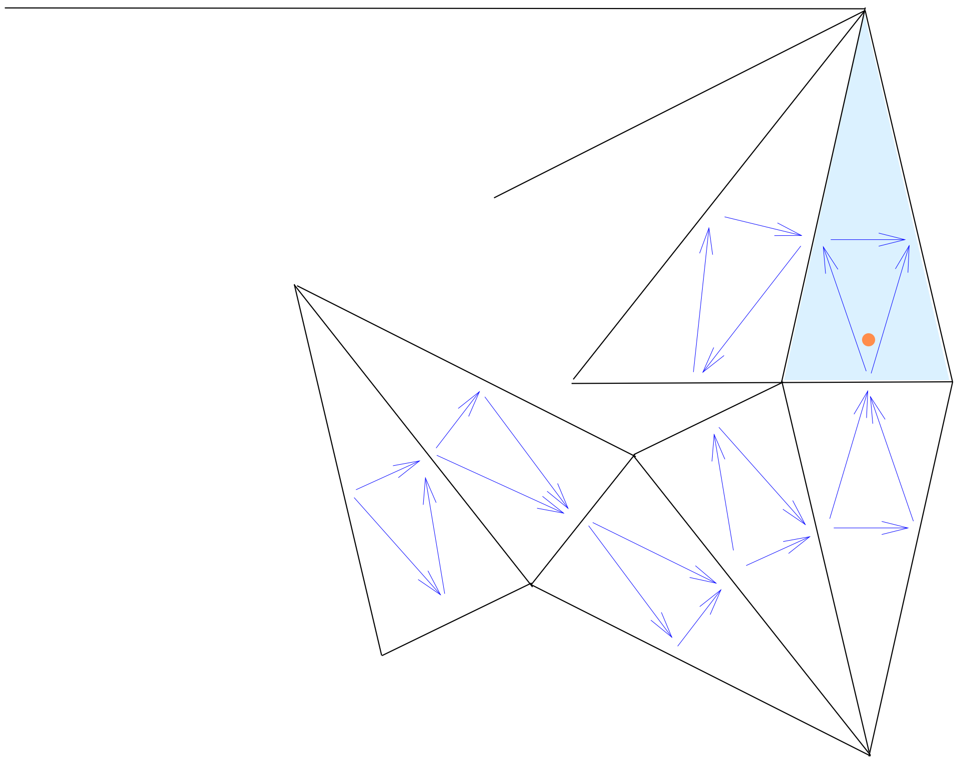

4. Exchange graph:

Rank 3 - affine type

from here: joint with Philipp Lampe

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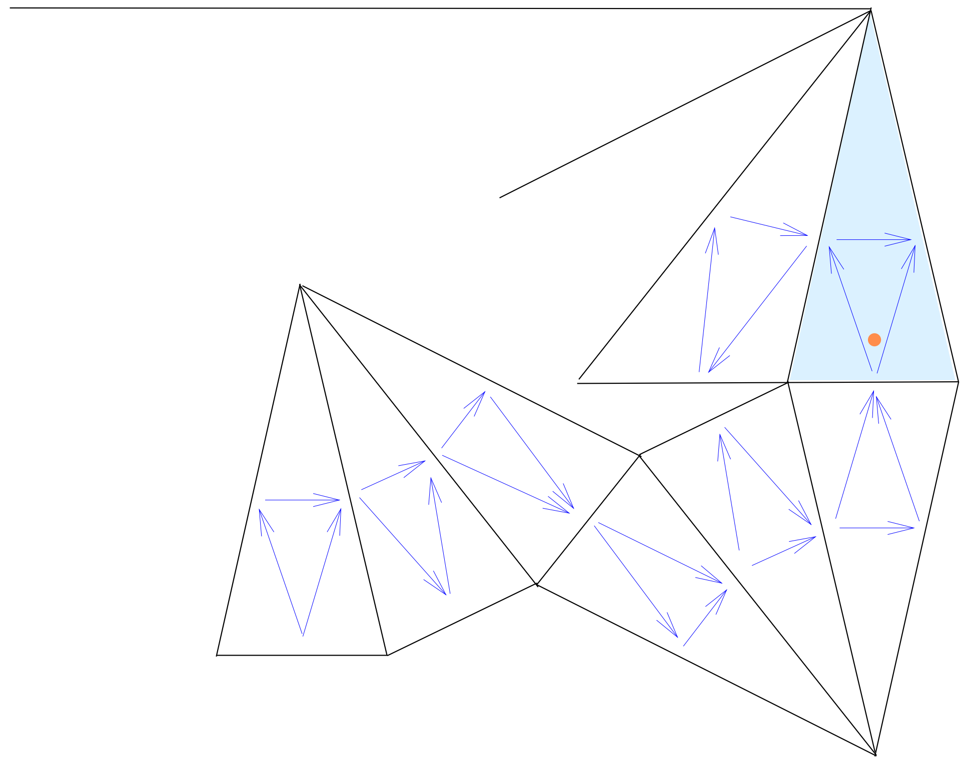

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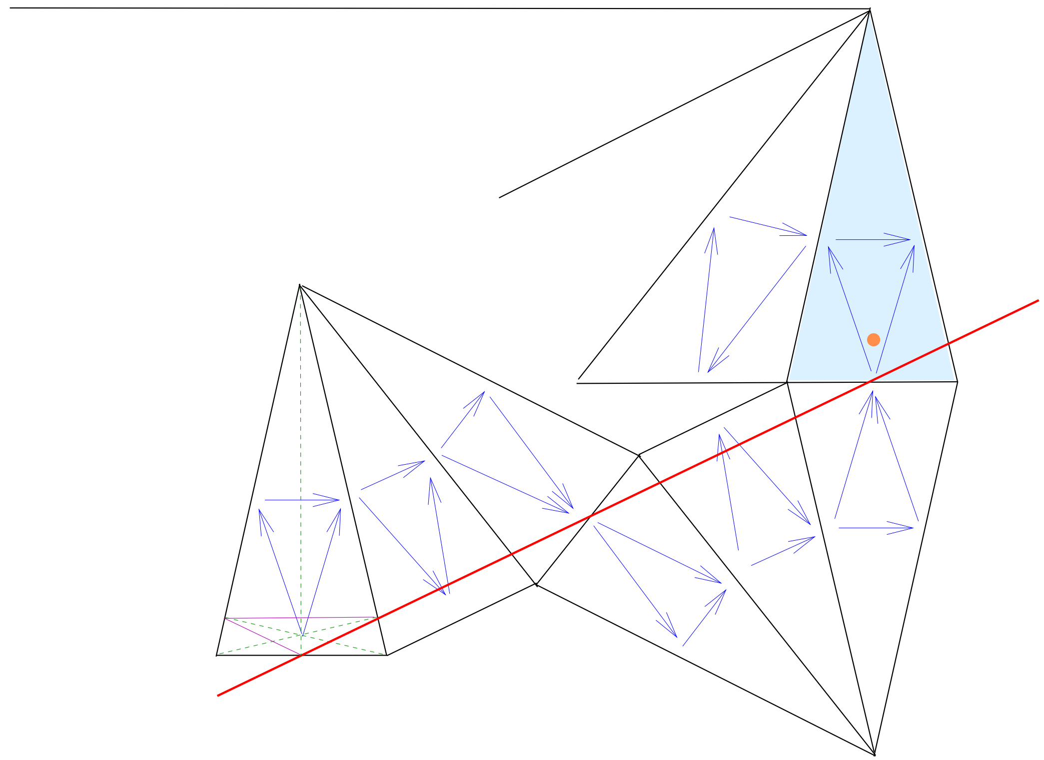
- Initial acyclic seed
- An acyclic belt
- Belt (or billiard) line

  passes through two

  feet of altitudes

  cf. Fagnano's problem:

  billiard tranjectory in triangle.



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- Away from acyclic belt:

  two feet of altitudes are
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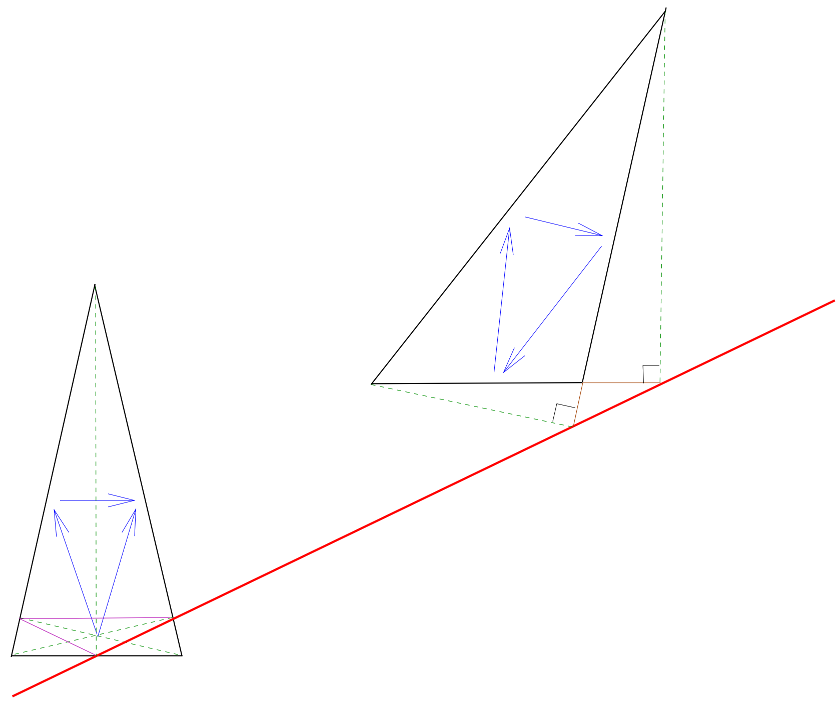

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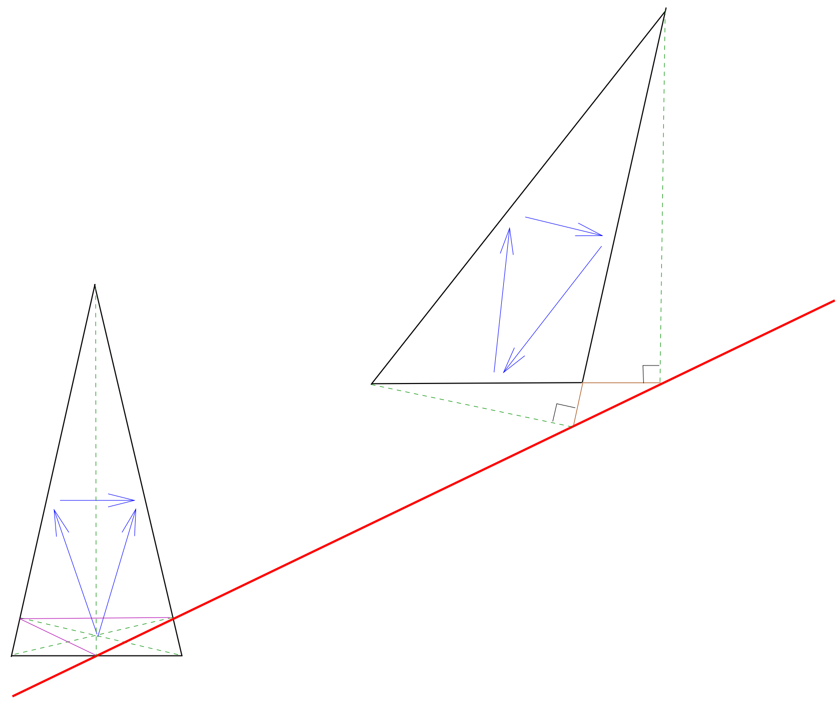
- Away from acyclic belt:

  two feet of altitudes are on the belt line.

$\Rightarrow$ If there are shifts, they are parallel to the belt line



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- Invariant under mutation:

  $T(\Delta) = a_i \sin(A_j) \sin(A_k)$

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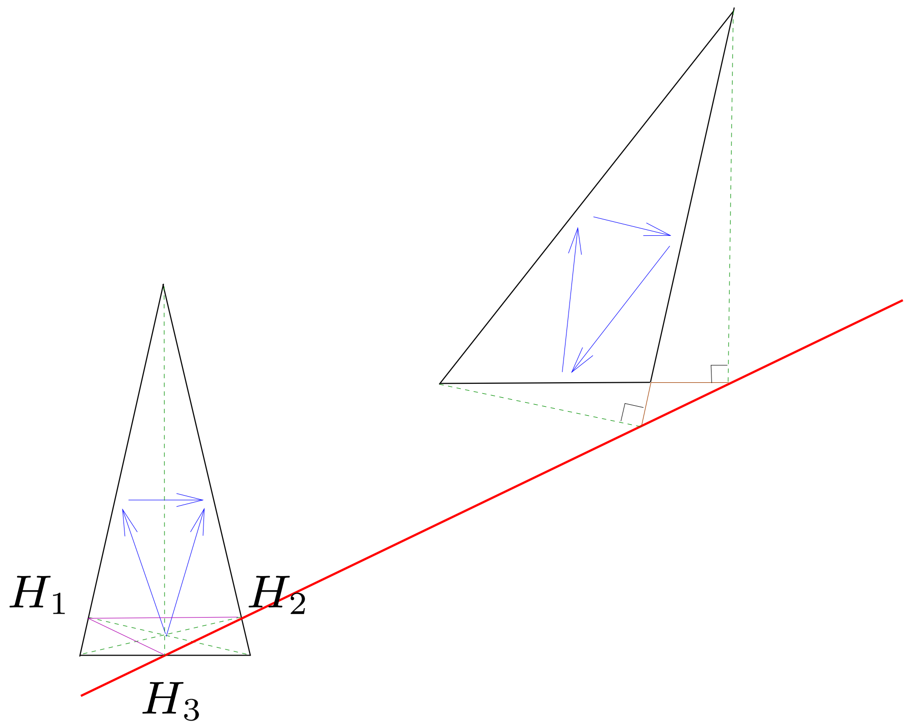

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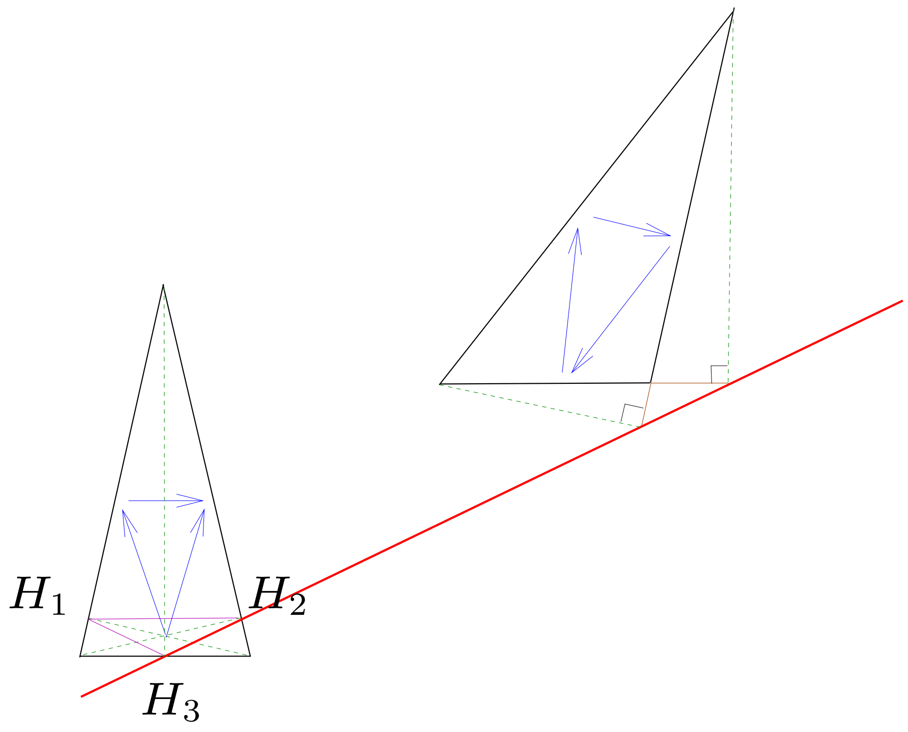

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Need: to find all shifts!

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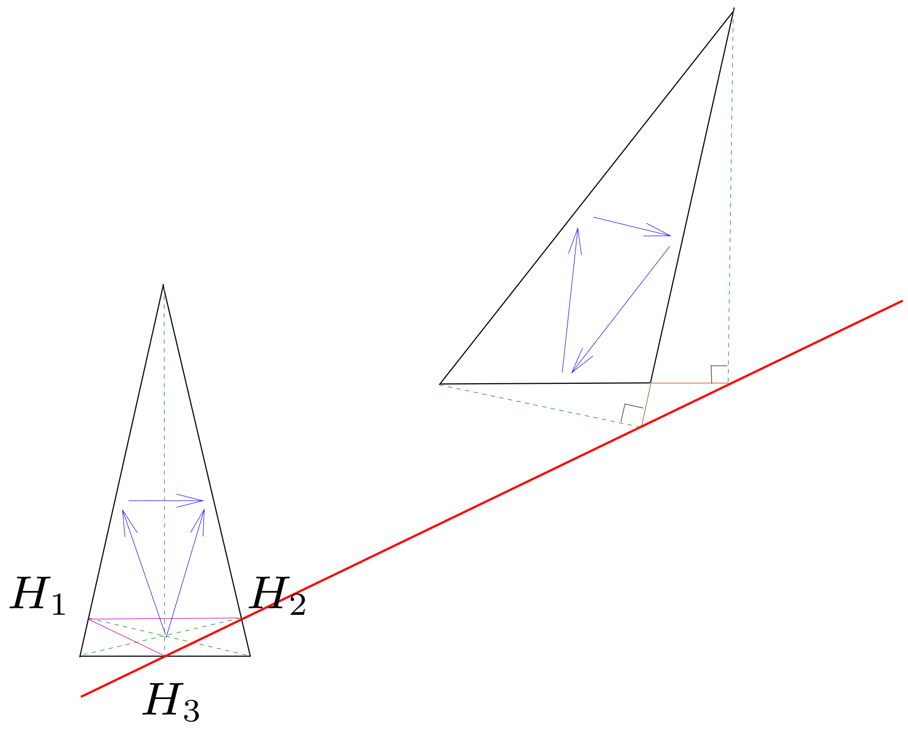

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> Triangles with same angles are congruent

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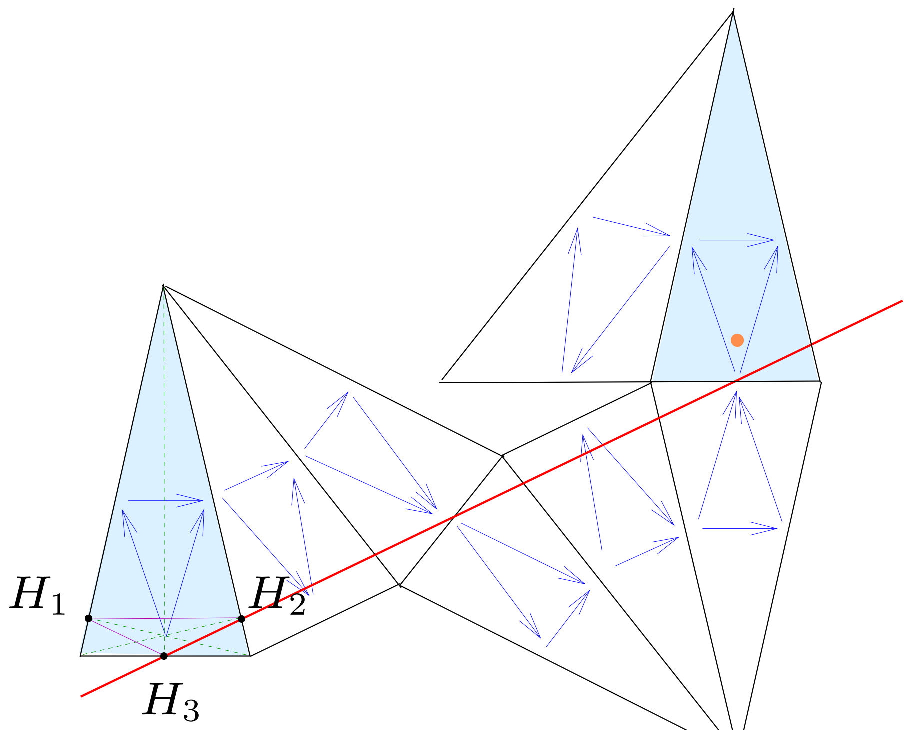
Need: to find all shifts!

- $T(\Delta) = \frac{1}{2}(|H_1H_2| + |H_2H_3| + |H_3H_1|)$

- $4T(\Delta) =$ shift along the belt line (in every acyclic belt)



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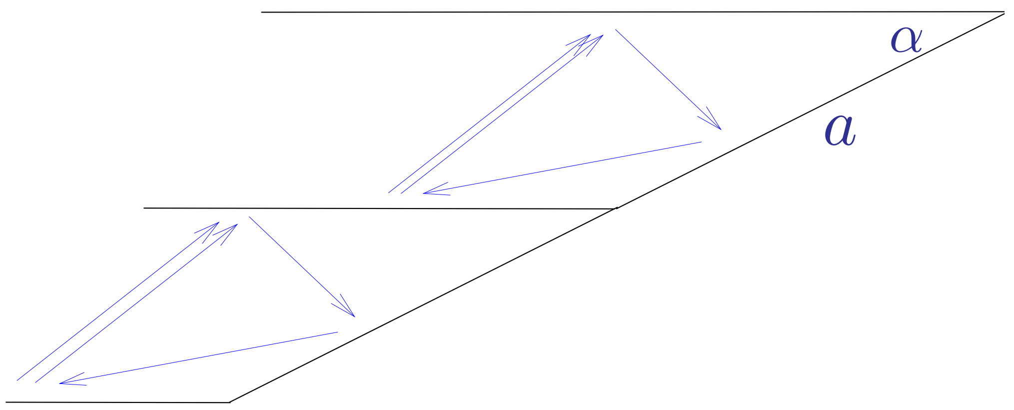
- There are more shifts:

  each infinite region induces a shift:



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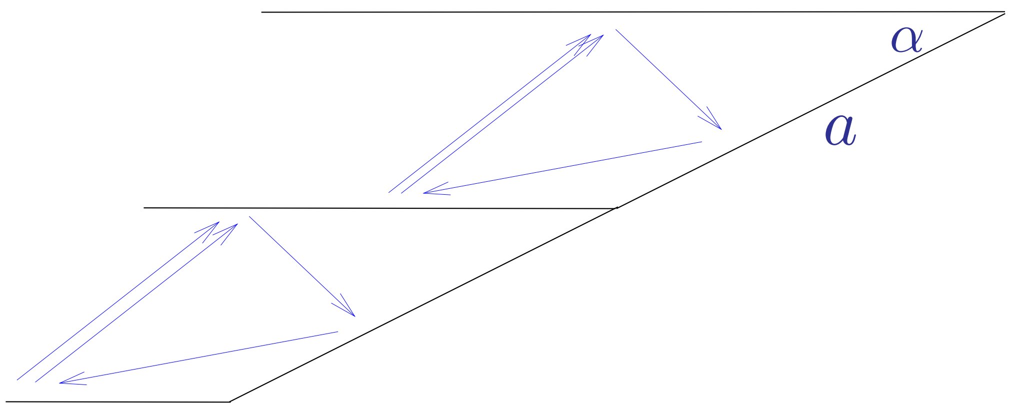
  then $T(\Delta) = a \sin^2 \alpha$

  So, $a = \frac{T(\Delta)}{\sin^2 \alpha}$



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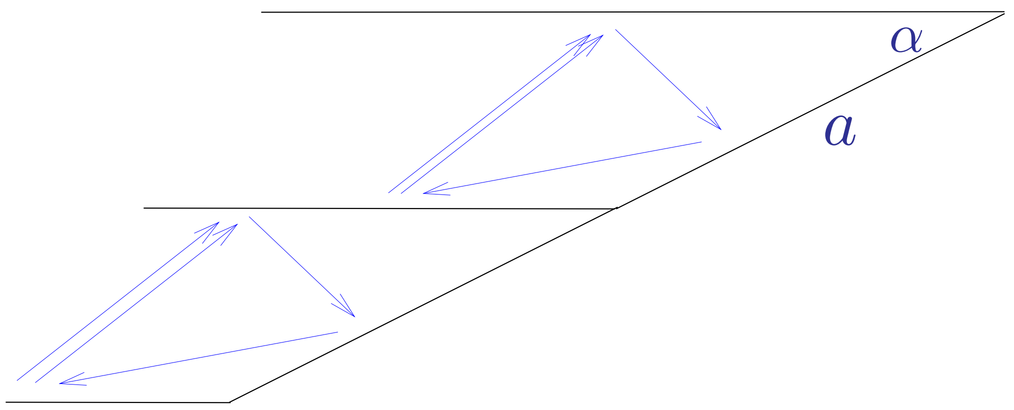

- If $\alpha = \frac{\pi}{d}$, then we have shifts:

$$4T,\ \frac{T}{\sin^2(\pi/d)},\ \frac{T}{\sin^2(2\pi/d)},\ \frac{T}{\sin^2(3\pi/d)},\ \cdots$$

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Theorem. [FL'2018]
Let Q be an affine type rank 3 mutation-finite quiver. Then the exchange graph of Q grows polynomially and is quasi-isometric to some lattice L.

$$rk_{\mathbb{Z}}(L) = \begin{cases} \varphi(d), & \text{for some } d \in 2\mathbb{Z}; \\ \frac{1}{2}\varphi(d), & \text{otherwise.} \end{cases}$$

Here, $\varphi(d) = \#\{k \in \{1, 2, \dots, d\} \mid gcd(k, d) = 1\}$

is the Euler's totient function.

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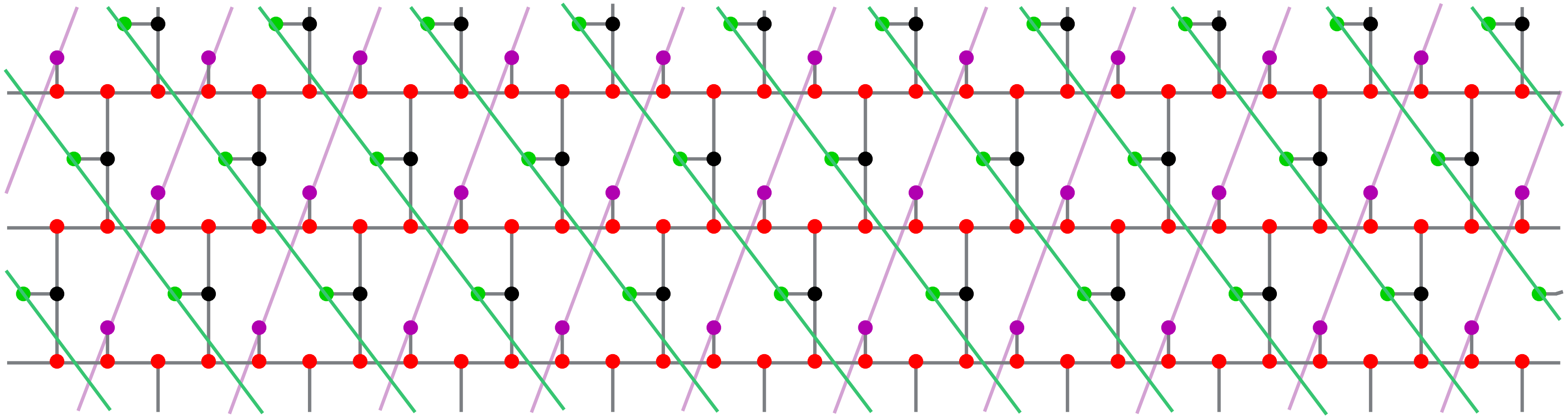
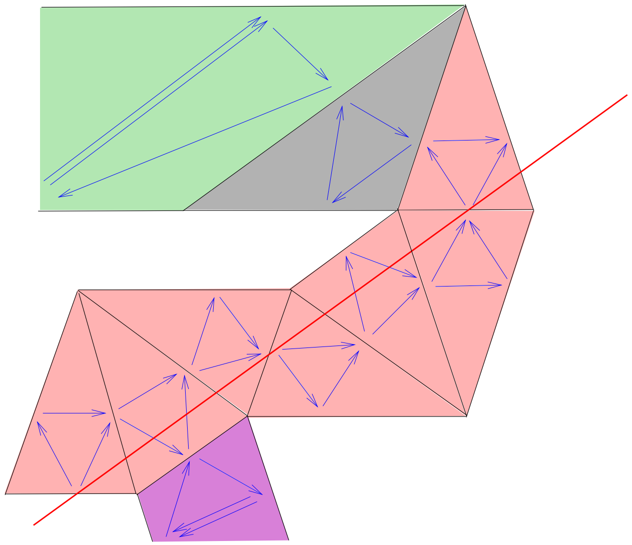


Example: exchange graph for $d = 5$



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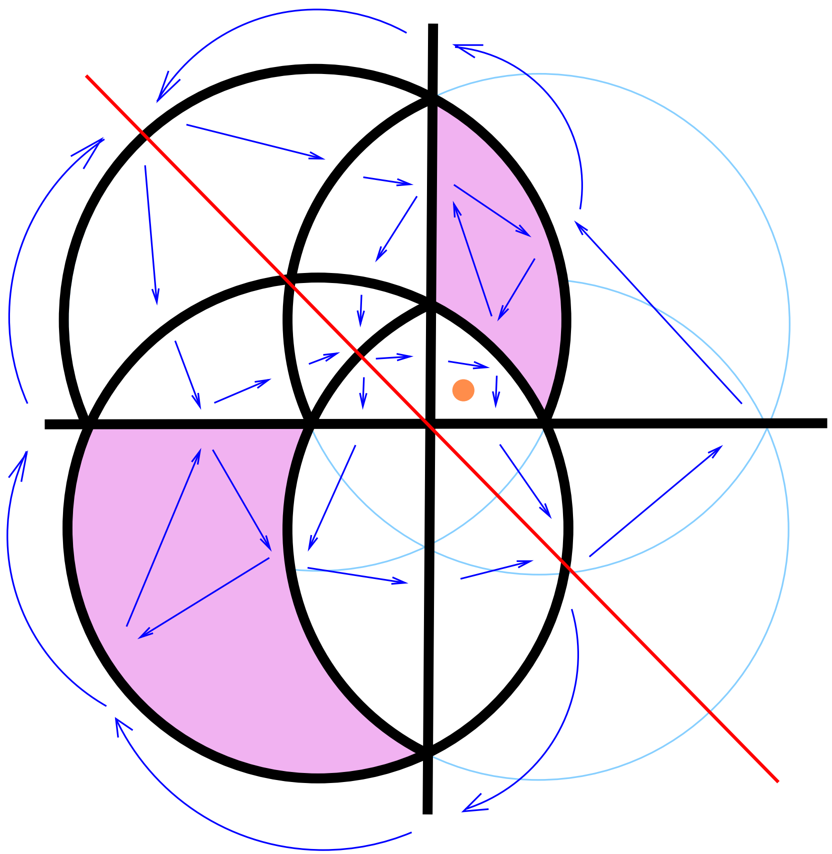
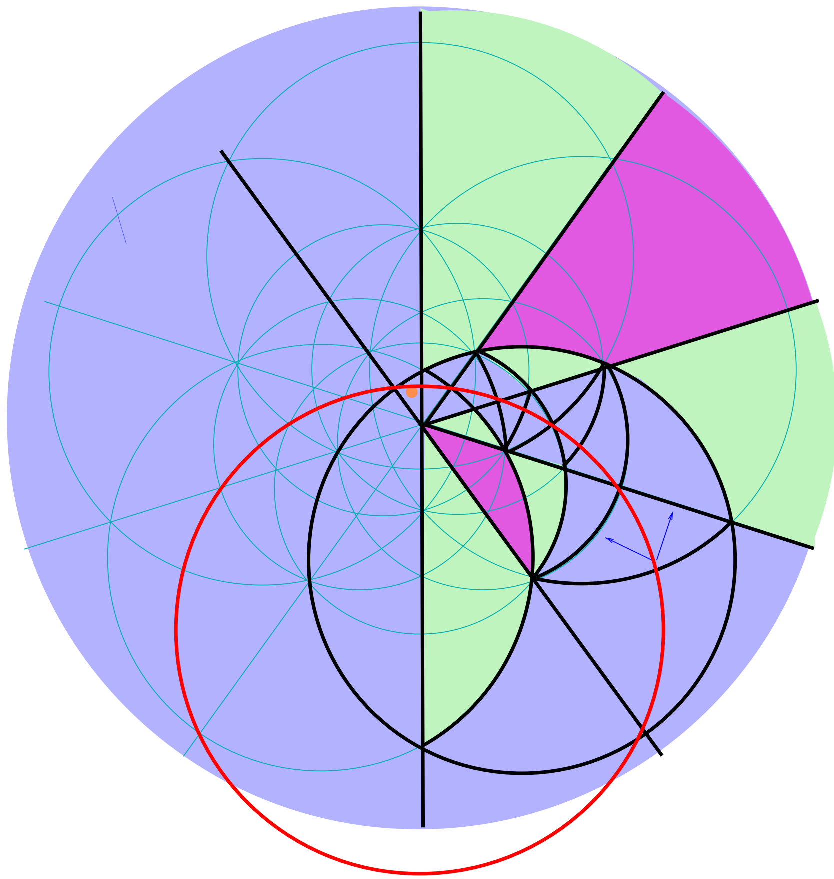
Remark: Similar belt line (and similar geometry) takes place when Q has a geometric presentation by reflections on S^2 or $\mathbb{H}^2$.





5. Remarks on definition of mutation

We defined:

Mutation $\rightsquigarrow$ Partial reflection

$$\mu_k(v_i) = \begin{cases} v_i - \langle v_i, v_k \rangle v_k, & \text{if } k \to i \text{ in } Q \\ -v_k, & \text{if } i = k \\ v_i, & \text{otherwise} \end{cases}$$

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Define:

if v_k is positive:

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if v_k is negative:

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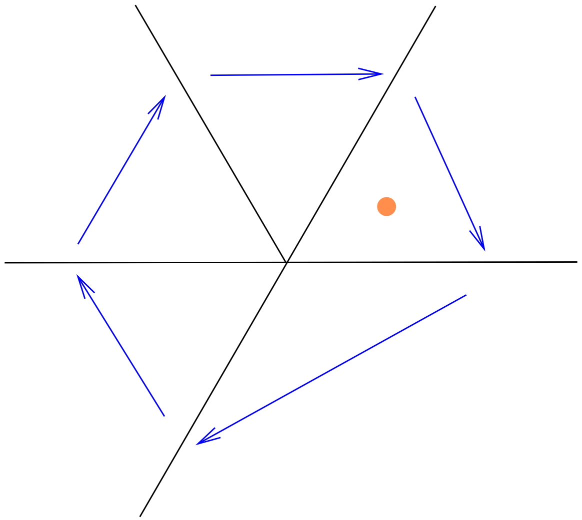
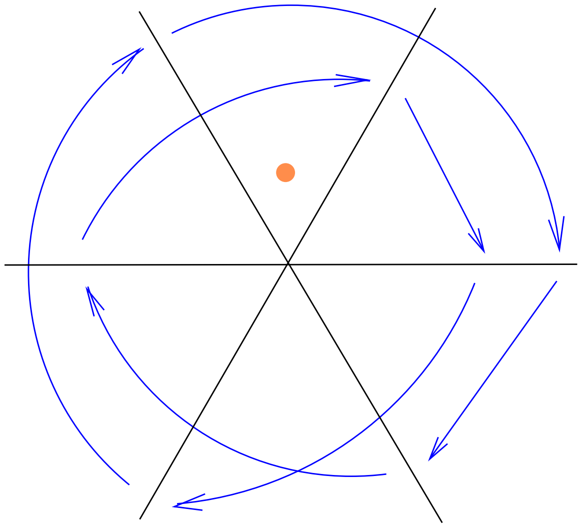
5. Remarks on definition of mutation

What to mean by positive / negative?

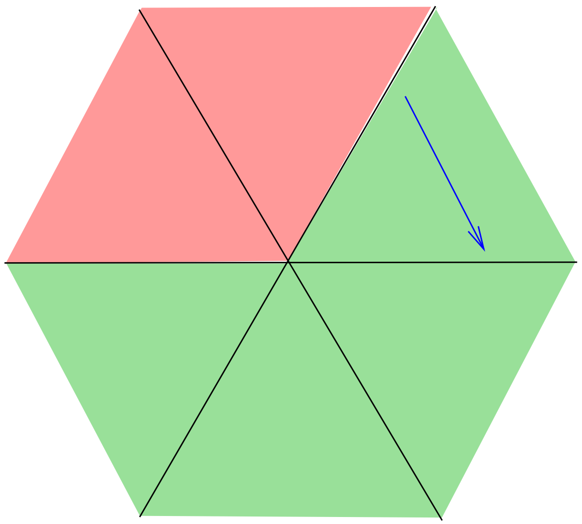
In rank 2:



Period 5



Period 7



- admissible positions
- forbidden positions

of the reference point

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What to mean by positive / negative?

In rank 3: Reference point is in admissible position,
if it is in admissible position
for every rank 2 subquiver in every cluster.

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(1) there are geometric realisations
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(2) all such realisations result in the same exchange graph;
(3) in all such realisations, the reference point belongs
to some acute-angled (i.e. acyclic) domain;
(4) every choice of reference point in an acute-angled
(i.e. acyclic) domain gives such a realisation.

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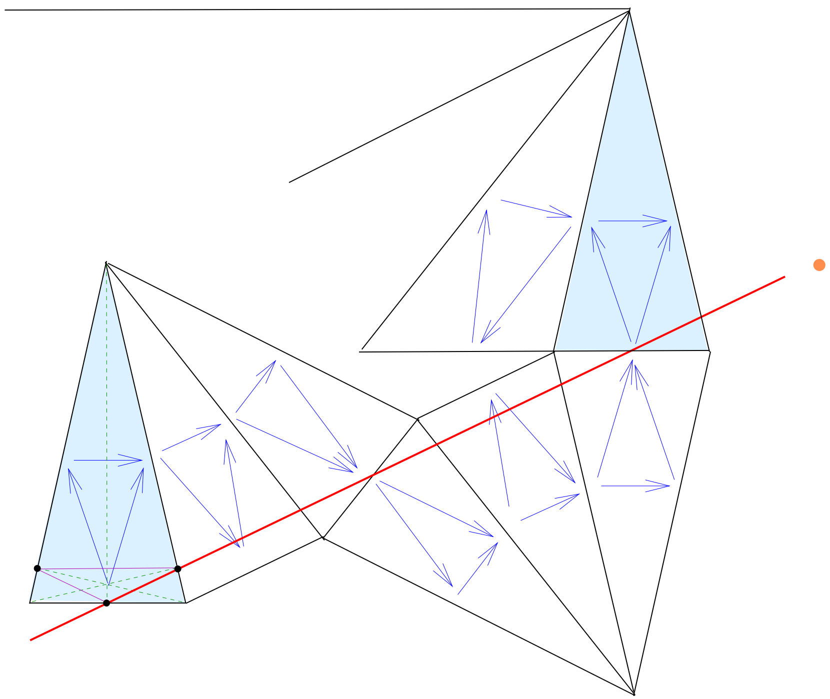
What to mean by positive / negative?

Theorem [FL'18] For every rank 3 affine quiver,

there exists a unique admissible position of the reference point:

it is the limit point at the end of the belt line.



Non-integer quivers: geometry and mutation-finiteness

Anna Felikson
Pavel Tumarkin
Durham University

Introduction

- $B = \{b_{ij}\}$ a skew-symmetric matrix with $b_{ij} \in \mathbb{R}$.
- Mutate B by usual mutation rule:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise} \end{cases}$$

- B defines a **non-integer quiver** (with arrows of real weights $b_{ij} = -b_{ji}$).

- **Mutation** μ_k of a non-integer quiver:
  1) reverse all arrows incident to k;
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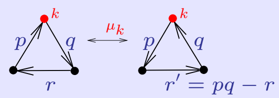
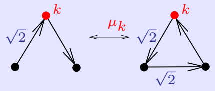


- Example: B_3



- **Question:** when a real quiver Q is mutation-finite?
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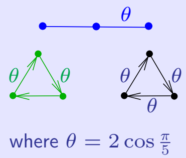
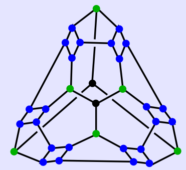




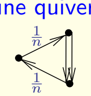
(cf. generalised associahedron in [FR])

Quivers of rank 3

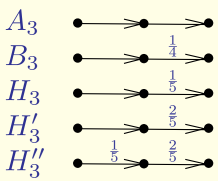
Theorem [FT1]. Any mutation-finite rank 3 quiver is mutation-equivalent to one of

- Markov quiver:

- Affine quivers:



- Finite type quivers:



Here, a label $\frac{k}{m}$ stays for the weight $|b_{ij}| = 2\cos\frac{k\pi}{m}$.

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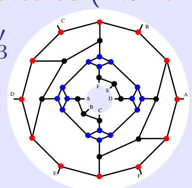
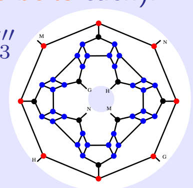
H'_3



H''_3



- Each of the mutation classes H'_3 and H''_3 has two different acyclic representatives:

Geometric realisation by reflections (GR)

- GR of a quiver of rank n: vectors $v_1, \ldots, v_n$ in a quadratic vector space V s.t. $(v_i, v_i) = 2$ and $(v_i, v_j) = -|b_{ij}|$.
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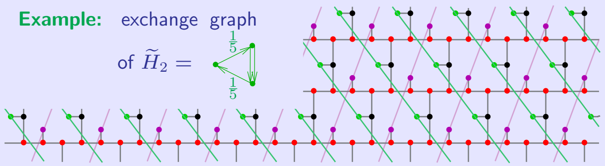
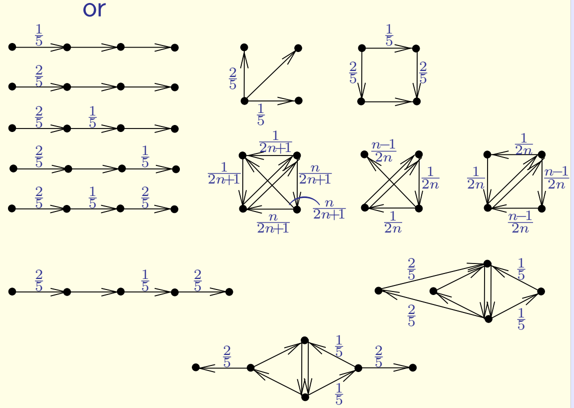
Example: exchange graph of $\widetilde{H}_2 =$



Theorem [FL]. Let Q be an affine type rank 3 mut.-fin. quiver. Then "exchange graph" of Q grows polynomially and is quasi-isometric to a lattice of some dimension.

Finite mutation type: classification

Theorem [FT3]. A mut.-fin. non-int. quiver of rank $n > 3$ is either of orbifold type, or mut.-equiv. to one of F_4, $\widetilde{F}_4$, $F_4^{(*,+)}$, $F_4^{(*,*)}$ or



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References

[FL] A. Felikson, Ph. Lampe, *Exchange graphs for non-integer affine quivers with 3 vertices*, in preparation.
[FT1] A. Felikson, P. Tumarkin, *Geometry of mutation classes of rank 3 quivers*, arXiv:1609.08828.
[FT2] A. Felikson, P. Tumarkin, *Acyclic cluster algebras, reflection groups and curves on a punctured disc*, arXiv:1709.10360.
[FT3] A. Felikson, P. Tumarkin, *Non-integer quivers of finite mutation type*, in preparation.
[FR] S. Fomin, N. Reading, *Root systems and generalized associahedra*, Geometric combinatorics, 63-131, IAS/Park City Math. Ser., 13, Amer. Math. Soc., Providence, RI, 2007.
[L] Ph. Lampe, *On the approximate periodicity of sequences attached to noncrystallographic root systems*, To appear in Experimental Mathematics (2018), arXiv:1607.04223.

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THANKS!

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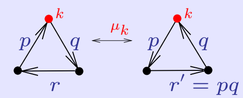

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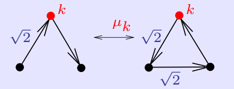

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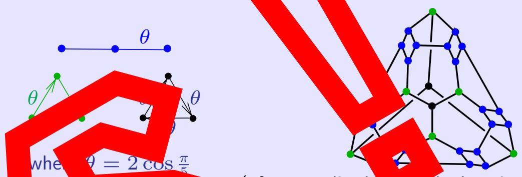

where $\theta = 2\cos\frac{\pi}{5}$ (cf. generalised associahedron in [FR])

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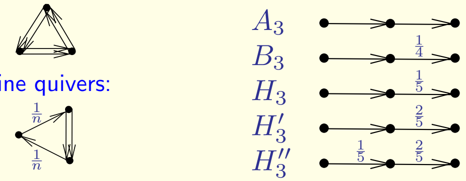

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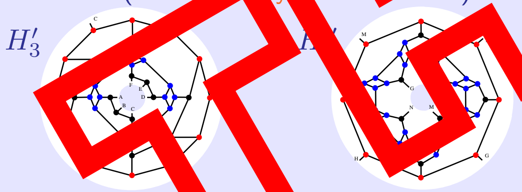

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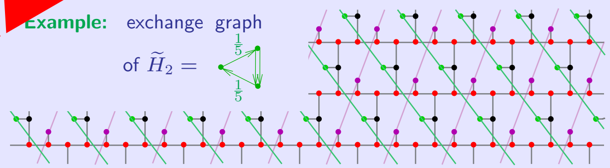
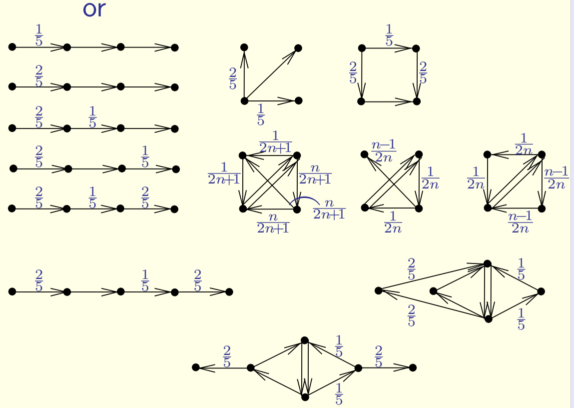


Theorem [FL]. Let Q be an affine type rank 3 mut.-fin. quiver. Then "exchange graph" of Q grows polynomially and is quasi-isometric to a lattice of some dimension.

Finite mutation type: classification

Theorem [FT3]. A mut.-fin. non-int. quiver of rank $n > 3$ is either of orbifold type, or mut.-equiv. to one of F_4, $\widetilde{F}_4$, $F_4^{(*,+)}$, $F_4^{(*,*)}$ or



Here, a label $\frac{k}{m}$ stays for the weight $|b_{ij}| = 2\cos\frac{k\pi}{m}$.

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