Cluster algebras, quiver mutations and triangulated surfaces

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joint work with Michael Shapiro and Pavel Tumarkin

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Cluster algebras

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Hyperbolic geometry
Coxeter groups
Teichmuller theory
Triangulated surfaces
Combinatorics of polytopes
Tropical geometry

Root systems
Frieze patterns

Dilogarithm identities
Supersymmetric gauge theories
Conformal field theory
Solitons
Mirror symmetry
Poisson geometry
Integrable systems
Cluster algebras
(Fomin, Zelevinsky, 2002)

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1. Quiver mutation

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- **Quiver** is a directed graph without loops and 2-cycles.

- **Mutation** $\mu_k$ of quivers:
  - reverse all arrows incident to $k$;
  - for every oriented path through $k$ do

\[
\begin{align*}
p & \quad \quad \rightarrow \quad q \\
r & \quad \quad \quad \quad \quad \quad \rightarrow \quad \bullet
\end{align*}
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p & \quad \quad \rightarrow \quad q \\
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\[r' = pq - r\]
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q \quad p \\
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\[
q \quad p \\
\Rightarrow r' = pq - r
\]

**Example:**

![Quiver Diagram]
1. Quiver mutation

Iterated mutations $\rightarrow$ many other quivers

$Q \rightarrow$ its mutation class

Property: $\mu_k \circ \mu_k(Q) = Q$ for any quiver $Q$. 
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Property: $\mu_k \circ \mu_k(Q) = Q$ for any quiver $Q$.

Definition. A quiver is of finite mutation type if its mutation class contains finitely many quivers.

Question. Which quivers are of finite mutation type?
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Quick answer. Not many:

If $Q$ is connected, $|Q| \geq 3$ and $Q$ contains arrow $\frac{p}{\rightarrow}$ with $p > 2$, then $Q$ is mutation infinite.

Why: if $q > r > 0$, $p > 2$ then $r' = pq - r > q > r$, so the weights grow under alternating mutations $\mu_1, \mu_2$. 

\[
\begin{array}{c}
\text{p} & \frac{\rightarrow}{q} & \mu_1 \\
2 & \frac{\leftarrow}{r} & 1
\end{array}
\quad
\begin{array}{c}
\mu_1 & \frac{\rightarrow}{q} & \mu_2 \\
2 & \frac{\rightarrow}{r'} & 1
\end{array}
\quad
\begin{array}{c}
r' = pq - r
\end{array}
\]
2. Cluster algebra: seed mutation

A seed is a pair \((Q, u)\) where

- \(Q\) is a quiver with \(n := |Q|\) vertices,
- \(u = (u_1, \ldots, u_n)\) is a set of rational functions in variables \((x_1, \ldots, x_n)\).

Initial seed: \((Q_0, u_0)\), where \(u_0 = (x_1, \ldots, x_n)\).
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Seed mutation: \(\mu_k(Q, (u_1, \ldots, u_n)) = (\mu_k(Q), (u'_1, \ldots, u'_n))\)

where \(u'_k = \frac{1}{u_k} (\prod_{i \to k} u_i + \prod_{k \to j} u_j)\)

\(u'_i = u_i\) if \(i \neq k\).
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where \(u'_k = \frac{1}{u_k}(\prod_{i \to k} u_i + \prod_{k \to j} u_j)\)

- products over all incoming/outgoing arrows
- \(u'_i = u_i\) if \(i \neq k\).

**Cluster variable:** a function \(u_i\) in one of the seeds.

**Cluster algebra:** \(\mathbb{Q}\)-subalgebra of \(\mathbb{Q}(x_1, \ldots, x_n)\) generated by all cluster variables.
2. Cluster algebra: finite type

A cluster algebra is of finite type if it contains finitely many cluster variables.

Theorem. (Fomin, Zelevinsky’ 2002)

A cluster algebra $\mathcal{A}(Q)$ is of finite type iff $Q$ is mutation-equivalent to an orientation of a Dynkin diagram $A_n, D_n, E_6, E_7, E_8$.
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Note: Dynkin diagrams describe:
finite reflection groups, semisimple Lie algebras, surface singularities...
2. Cluster algebra: finite mutation type

A cluster algebra $\mathcal{A}(Q)$ is of finite mutation type if $Q$ is of finite mutation type.
3. Finite mutation type: examples

1. \( n = 2 \).

2. Quivers arising from triangulated surfaces.

3. Finitely many except that.
   (conjectured by Fomin, Shapiro, Thurston)
4. Quivers from triangulated surfaces
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Triangulated surface $\rightarrow$ Quiver
4. Quivers from triangulated surfaces

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![Diagram of a triangulated surface and its corresponding quiver]
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![Diagram of triangulated surface and quiver transformation]
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Remark. \( Q \) from a triangulation \( \Rightarrow \) weights of arrows \( \leq 2 \).

(as every arc lies at most in two triangles)
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Remark. $Q$ from a triangulation $\Rightarrow$ weights of arrows $\leq 2$.  
(as every arc lies at most in two triangles)

Theorem. (Hatcher) Every two triangulations of the same surface are connected by a sequence of flips. (Hatcher, Harer)

Corollary. (a) Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).  
(b) Quivers from triangulations are mutation-finite.
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**Remark.** $Q$ from a triangulation $\Rightarrow$ weights of arrows $\leq 2$. (as every arc lies at most in two triangles)

**Theorem.** Every two triangulations of the same surface (Hatcher, Harer) are connected by a sequence of flips.

**Corollary.** (a) Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).
(b) Quivers from triangulations are mutation-finite.

**Question.** What else is mutation-finite?
4. Quivers from triangulations: description (Fomin-Shapiro-Thurston)

Any triangulated surface can be glued of:

![Triangulated surfaces](image1)

The corresponding quiver can be glued of blocks:

![Block representations](image2)

Proposition. (Fomin-Shapiro-Thurston) 

\[
\{ Q \text{ is from triangulation} \} \iff \{ Q \text{ is block-decomposable} \}
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Question: How to find all mutation-finite but not block-decomposable quivers?
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![Triangulations]

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![Blocks]

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**Interlude:**

How to to classify discrete reflection groups in hyperbolic space?

1. They correspond to some polytopes (described by some diagrams);
2. Combinatorics of these polytopes is described by:
   a. subdiagrams corresponding to finite subgroups (classified);
   b. minimal subdiagrams corresponding to infinite subgroups.
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Idea: Classify minimal non-decomposable quivers.
Lemma 1. If $Q$ is a minimal non-decomposable quiver then $|Q| \leq 7$.

Lemma 2. If $Q$ is a minimal non-decomposable mutation-finite quiver then is mutation equivalent to one of

Now: - add vertices to these quivers (and their mutations) one by one
- check the obtained quiver is still mutation-finite.
Theorem 1. (A.F, M.Shapiro, P.Tumarkin’ 2008)

Let $Q$ be a connected quiver of finite mutation type. Then
- either $|Q| = 2$;
- or $Q$ is obtained from a triangulated surface;
- or $Q$ is mut.-equivalent to one of the following 11 quivers:
Proof:
Proof: terrible, technical .... -but follows the same steps as some classifications of reflection groups
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Example. Logic scheme for a proof of some small lemma:
Proof: **terrible, technical ....** -but follows the same steps as some classifications of reflection groups

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