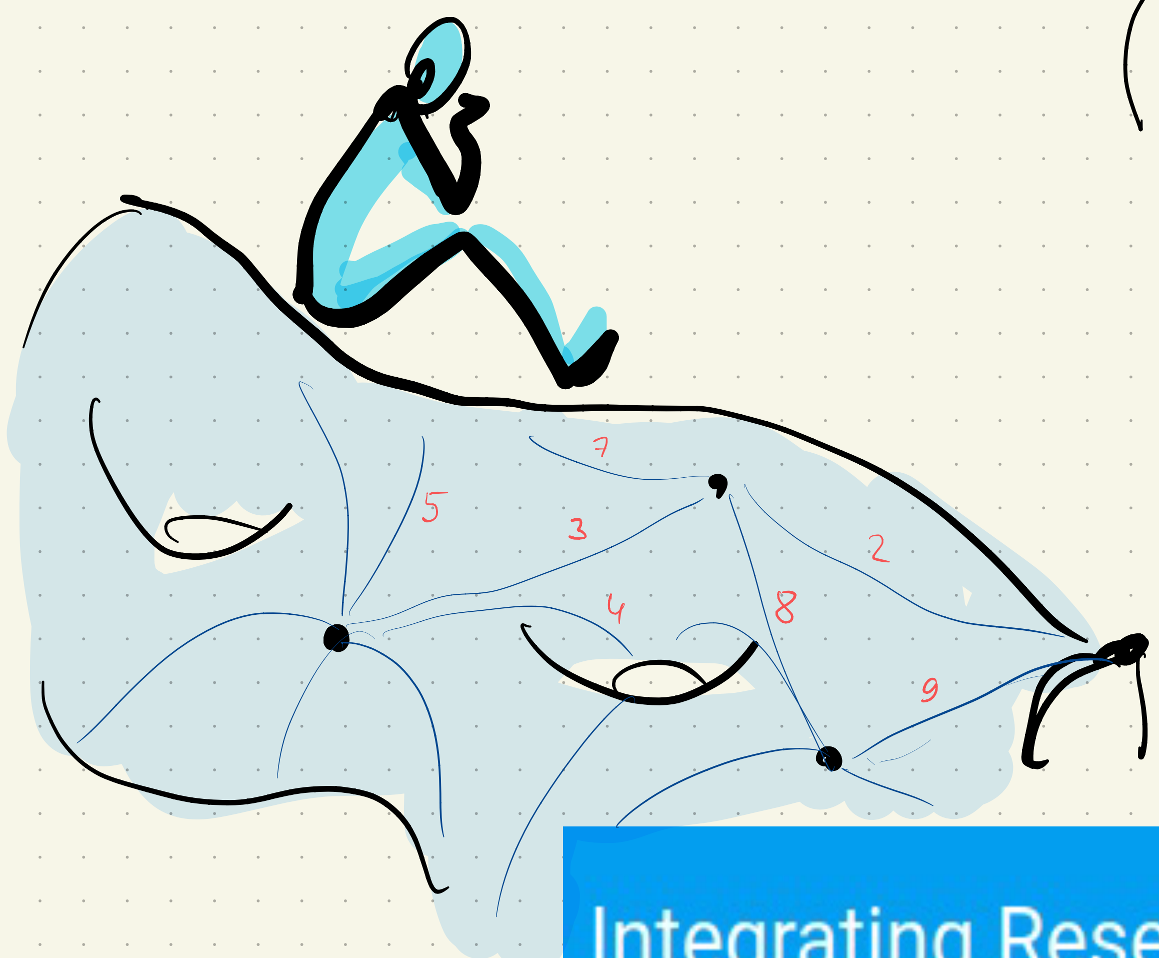
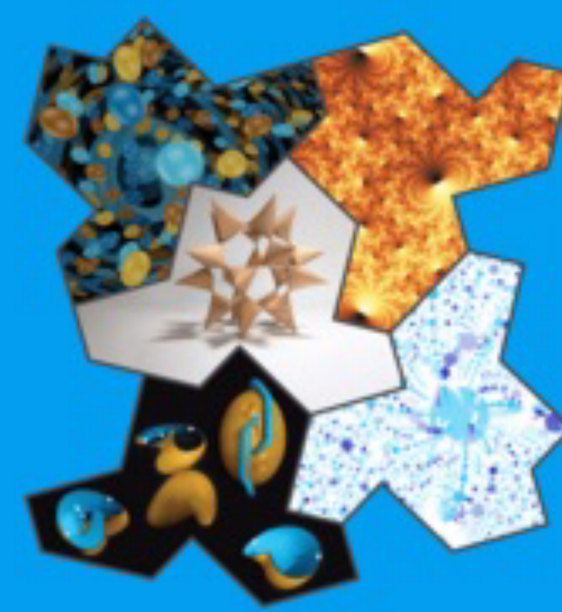
Unintentional illustration

Anna Felikson

(Durham University)



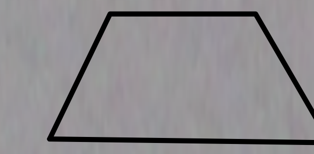


Integrating Research and Illustration in Number Theory

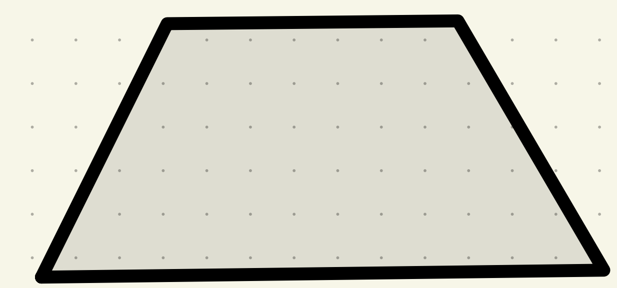
23–27 Mar 2026
Institut Henri Poincaré
Europe/Paris timezone

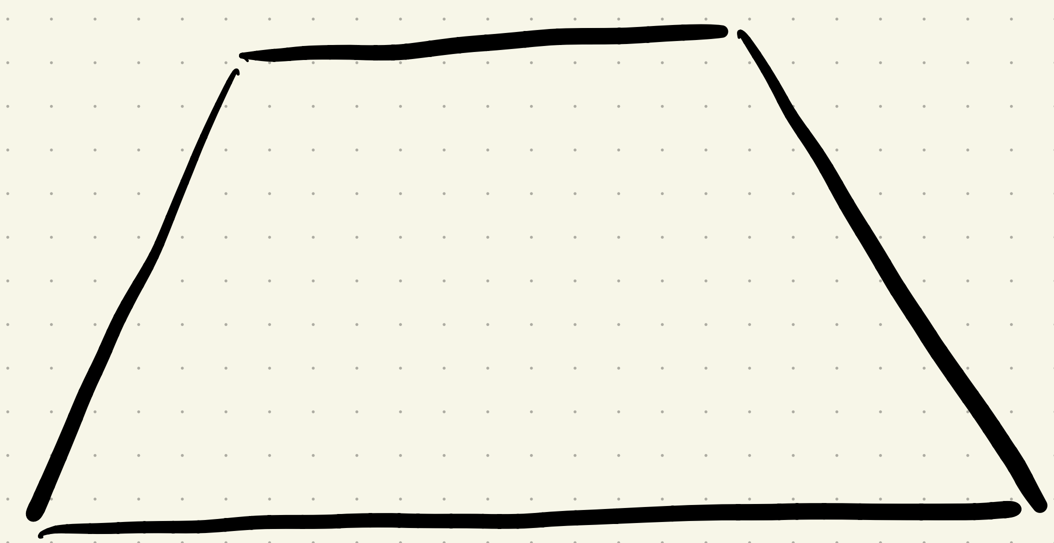
3. РАЗРЕЖЬТЕ ПОЛОВИНКУ ПРАВИЛЬНОГО ШЕСТИУГОЛЬНИКА (СМ.РИС
НА 4 ОДИНАКОВЫЕ ЧАСТИ. ПОСТАРАЙТЕСЬ ПРИДУМАТЬ НЕ ОДНО
РЕШЕНИЕ.

(4 ОЧКА)

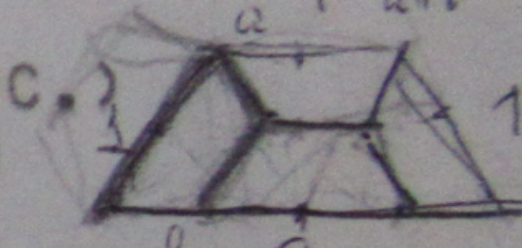


Cut the half of a regular hexagon into 4 congruent pieces.
Find more than one solution.





(3. РАЗРЕЖЬТЕ ПОЛОВИНКУ ПРАВИЛЬНОГО ШЕСТИУГОЛЬНИКА (СМ.РИС.) НА 4 ОДИНАКОВЫЕ ЧАСТИ. ПОСТАРАЙТЕСЬ ПРИДУМАТЬ НЕ ОДНО РЕШЕНИЕ.



(4 ОЧКА)

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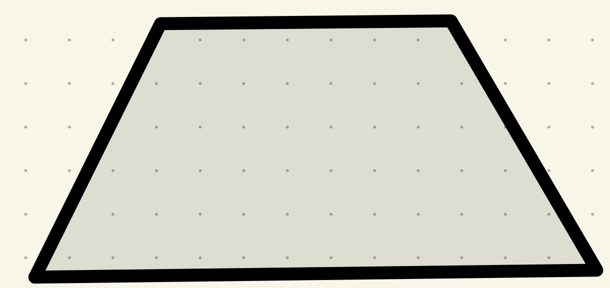
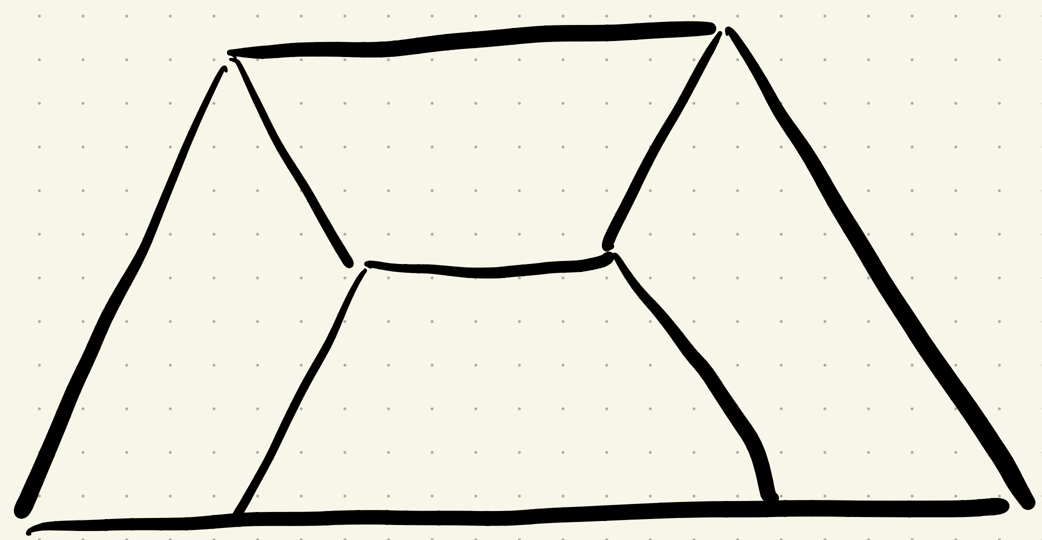
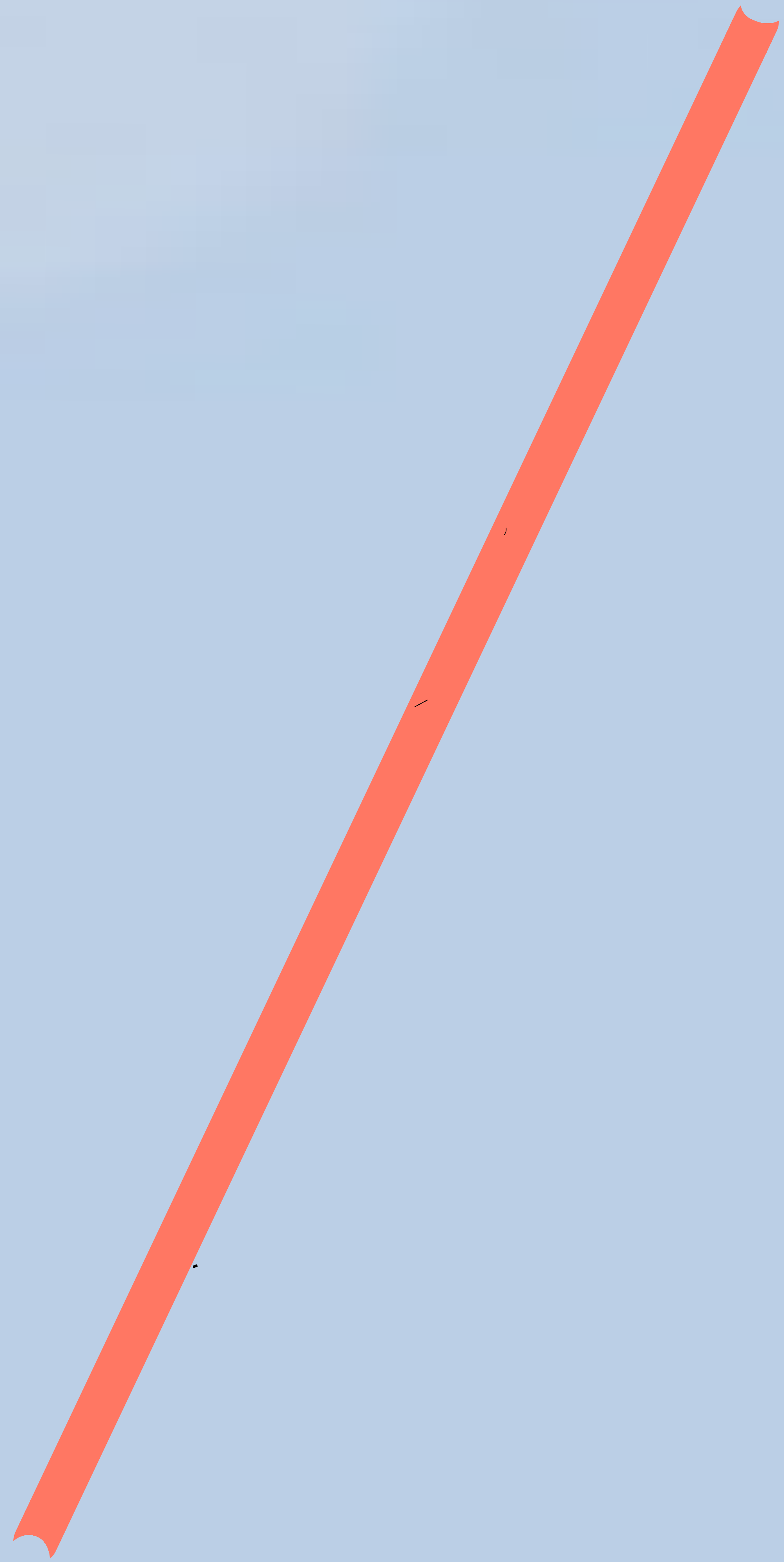
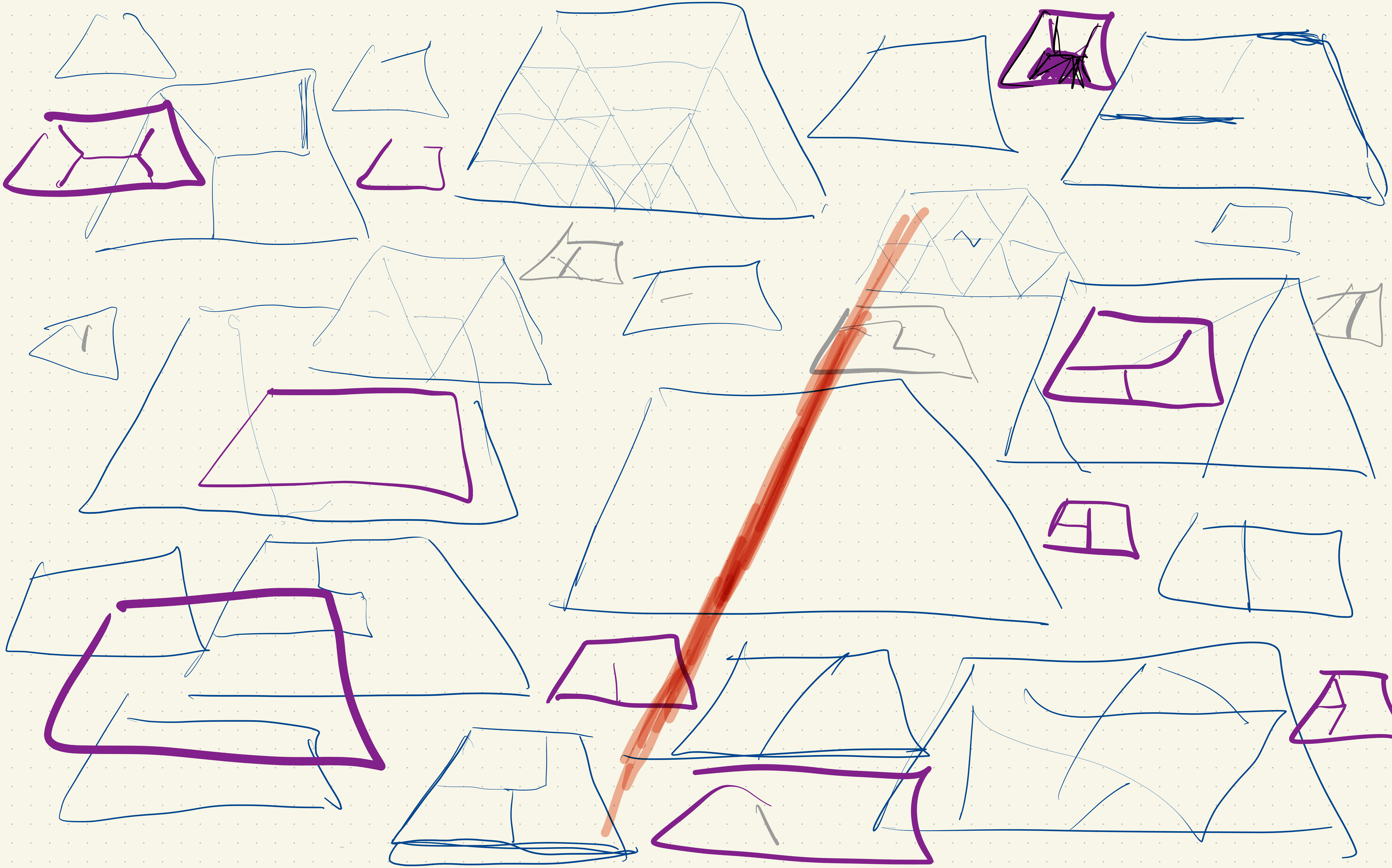


Find more than one solution.







$a < b$

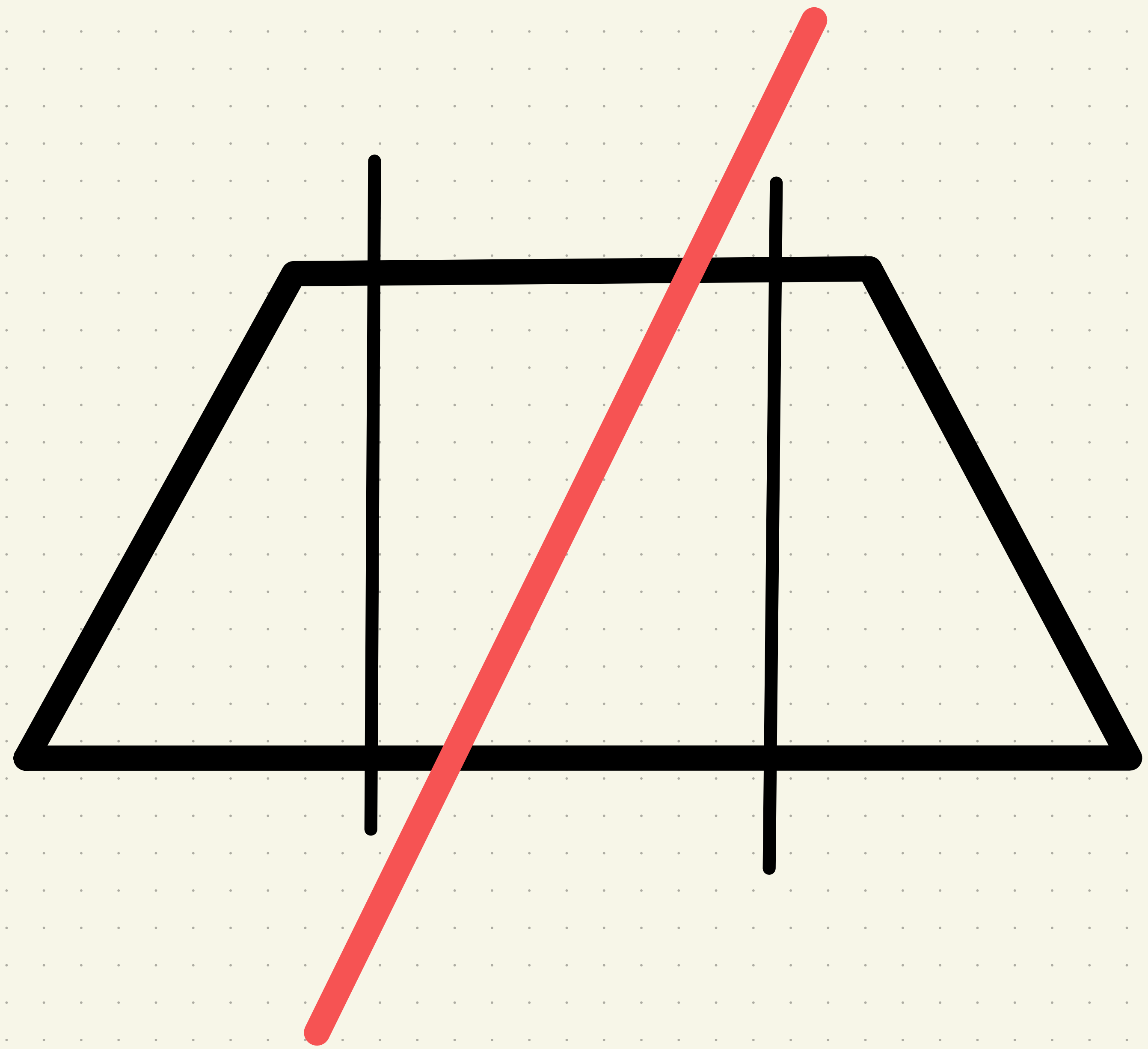
$$\begin{cases} 3a + b = 1 \\ a + 3b = 2 \end{cases}$$

$$\begin{cases} 3a + b = 1 \\ -8a = -1 \end{cases}$$

$$\begin{cases} a = \frac{1}{8} \\ b = \frac{5}{8} \end{cases}$$

Thanks to
my teachers!

- Yurii Burman
- Yaroslav Blanter
- Inessa Raskina

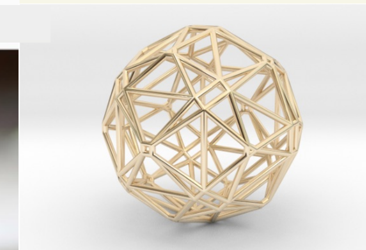


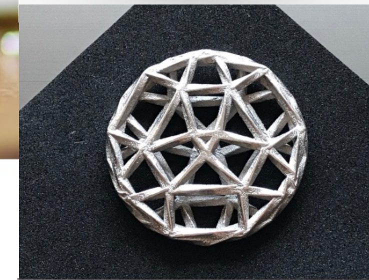
Commercial break: Herbert Gangle – 3D printed math. jewellery

The Gems of Hypolytos

By Herbert Gangl

HG is a mathematician at Durham University, UK. He uses `ideal tessellations' of hyperbolic 3-space as a rich source of beautiful new polytopes. Variations on these polytopes give rise to decorative objects and even jewellery.

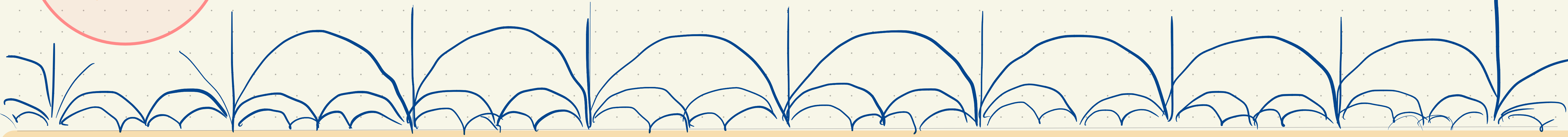




https://www.instagram.com/3d_printed_jewellery/

online shop coming soon → castimize.com

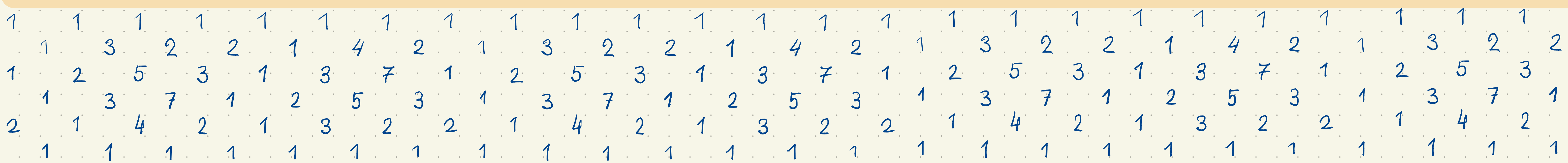
II Friezes on surfaces



Anna Felikson (Durham)

Pavel Tumarkin (Durham)

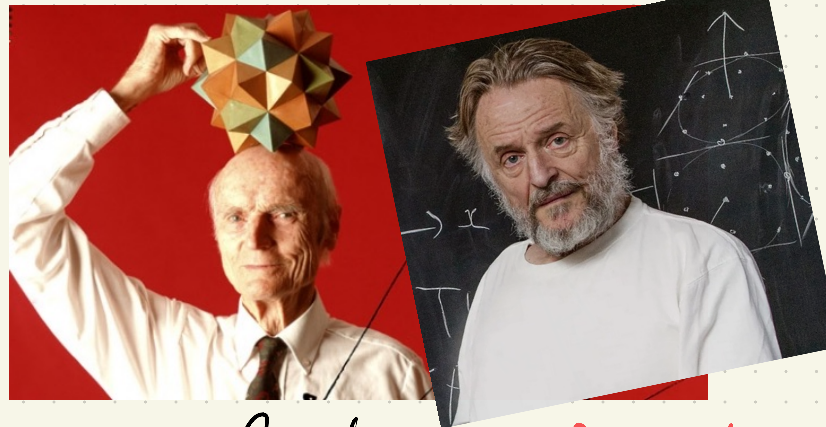
arXiv: 2410.13511v1



1. Frieze patterns

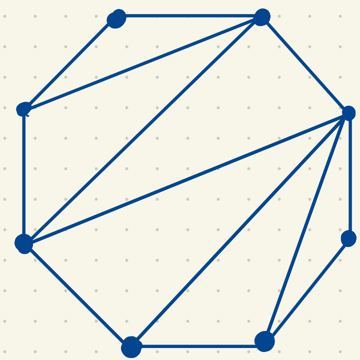
[early 70s]

Conway, Coxeter: Every frieze comes from a triangulated polygon



Coxeter Conway

$$
\begin{array}{ccccccccccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
2 & & 1 & & 3 & & 4 & & 1 & & 2 & & 2 & & 3 & & 2 & & 1 & & 3 \\
 & 1 & & 2 & & 11 & & 3 & & 1 & & 3 & & 5 & & 5 & & 1 & & 2 & \\
2 & & 1 & & 7 & & 8 & & 2 & & 1 & & 7 & & 8 & & 2 & & 1 & & 7 \\
 & 1 & & 3 & & 5 & & 5 & & 1 & & 2 & & 11 & & 3 & & 1 & & 3 & \\
1 & & 2 & & 2 & & 3 & & 2 & & 1 & & 3 & & 4 & & 1 & & 2 & & 2 \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$



- Periodicity, glide symmetry
- Integers
- Positivity

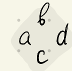
$$
\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}
\qquad ad-bc=1
$$

1. Frieze patterns

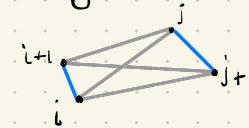
[early 70s]

Conway, Coxeter: Every frieze comes from a triangulated polygon

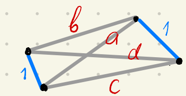
- $\begin{Bmatrix}\text{numbers} \\ \text{in frieze}\end{Bmatrix} \longleftrightarrow$ diagonals

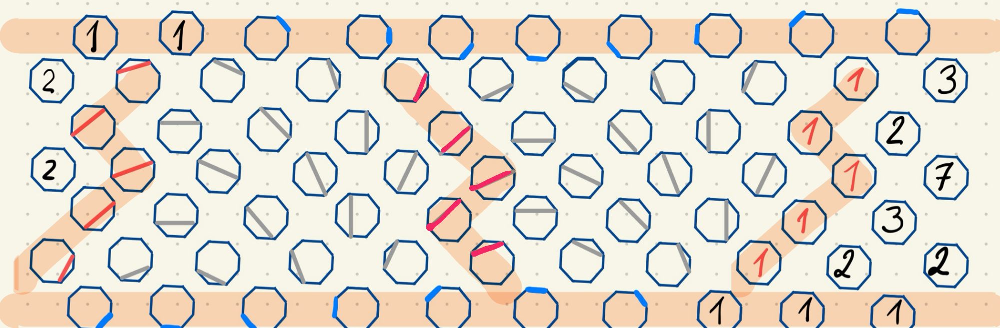


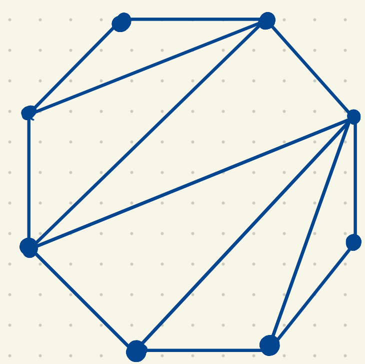
{diamonds} $\longleftrightarrow$



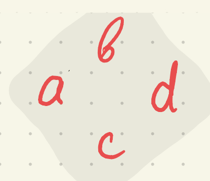
- $\{ad - bc = 1\} \longleftrightarrow$ Ptolemy thm







- Periodicity, glide symmetry
- Integers
- Positivity



$ad - bc = 1$

1. Frieze patterns

[early 70s]

Conway, Coxeter: Every frieze comes from a triangulated polygon

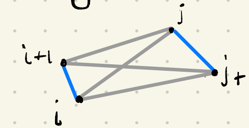
1	1	1	1	1	1	1	1	1	1	
2	1	3	4	1	2	2	3	2	1	3
1	2	11	3	1	3	5	5	1	2	
2	1	7	8	2	1	7	8	2	1	7
1	3	5	5	1	2	11	3	1	3	
1	2	2	3	2	1	3	4	1	2	2
1	1	1	1	1	1	1	1	1	1	

- Periodicity, glide symmetry
- Integers
- Positivity

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \qquad ad - bc = 1$$

- $\left\{\begin{matrix}\text{numbers}\\ \text{in frieze}\end{matrix}\right\} \longleftrightarrow$ diagonals

$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$ $\{\text{diamonds}\} \longleftrightarrow$



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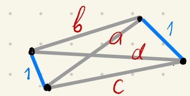
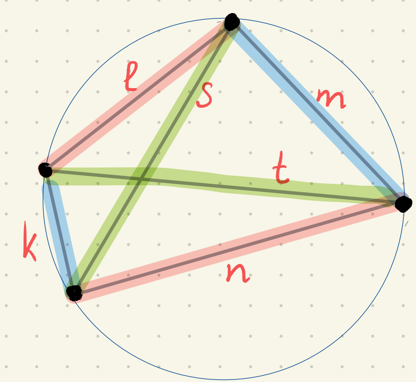


Ptolemy Thm:



$$st = km + ln$$

1. Frieze patterns

Thm (Conway, Coxeter '1973)

Every frieze pattern of finite width n comes from a triangulated $(n+3)$-gon (with 1^s assigned to sides and arcs in the triangulation)

Frieze patterns

Continued fractions

Enumerative combinatorics (dissections of polygons)

Ptolemy's thm

Representation theory

Cluster algebras

Moduli space of points in $\mathbb{RP}^1$

Hyperbolic geometry

Linear difference equations

Gauss' Pentagramma Mirificum

Sophie Morier-Genoud, „Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics" 2015

2. More general definition of a frieze:

Let (S,M) be a marked surface

E = set of all arcs on S

<u>Def</u> A frieze on (S,M) is a map $F: E \to \mathbb{R}$ for each (tagged) arc $\gamma \in E$ s.t. a Ptolemy relation holds for every quadrilateral

← Ring homomorphism of cluster algebra $A(S)$ to $\mathbb{R}$

- F is positive if $F: E \to \mathbb{R}_+$
  integer if $F: E \to \mathbb{Z}$

Requirement: $F(\gamma) = 1 \quad \forall \gamma \in \partial S$

<u>Ex</u> A surface S with a triangulation T defines a $\underbrace{\text{frieze}}_{\text{positive, integer}}$ by $F(\alpha) = 1 \quad \forall \alpha \in T$

2. More general definition of a frieze:

A frieze F on (S, M) is unitary
if $\exists$ triangulation T : $F(\alpha) = 1 \quad \forall \alpha \in T$

Frieze =
$F: \gamma^{eS} \mapsto \mathbb{R}$
+ Ptolemy

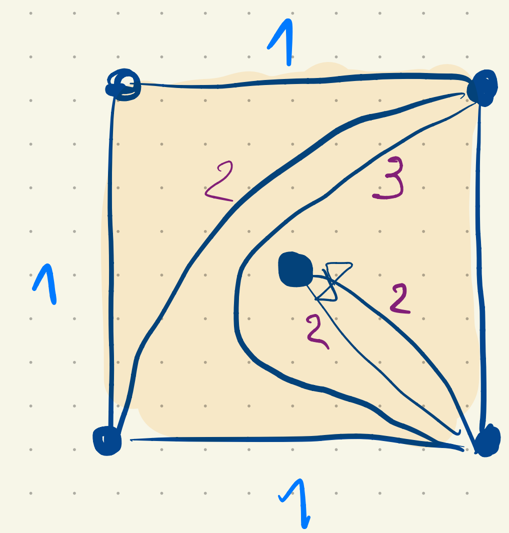
Frieze $\to$
positive,
integer

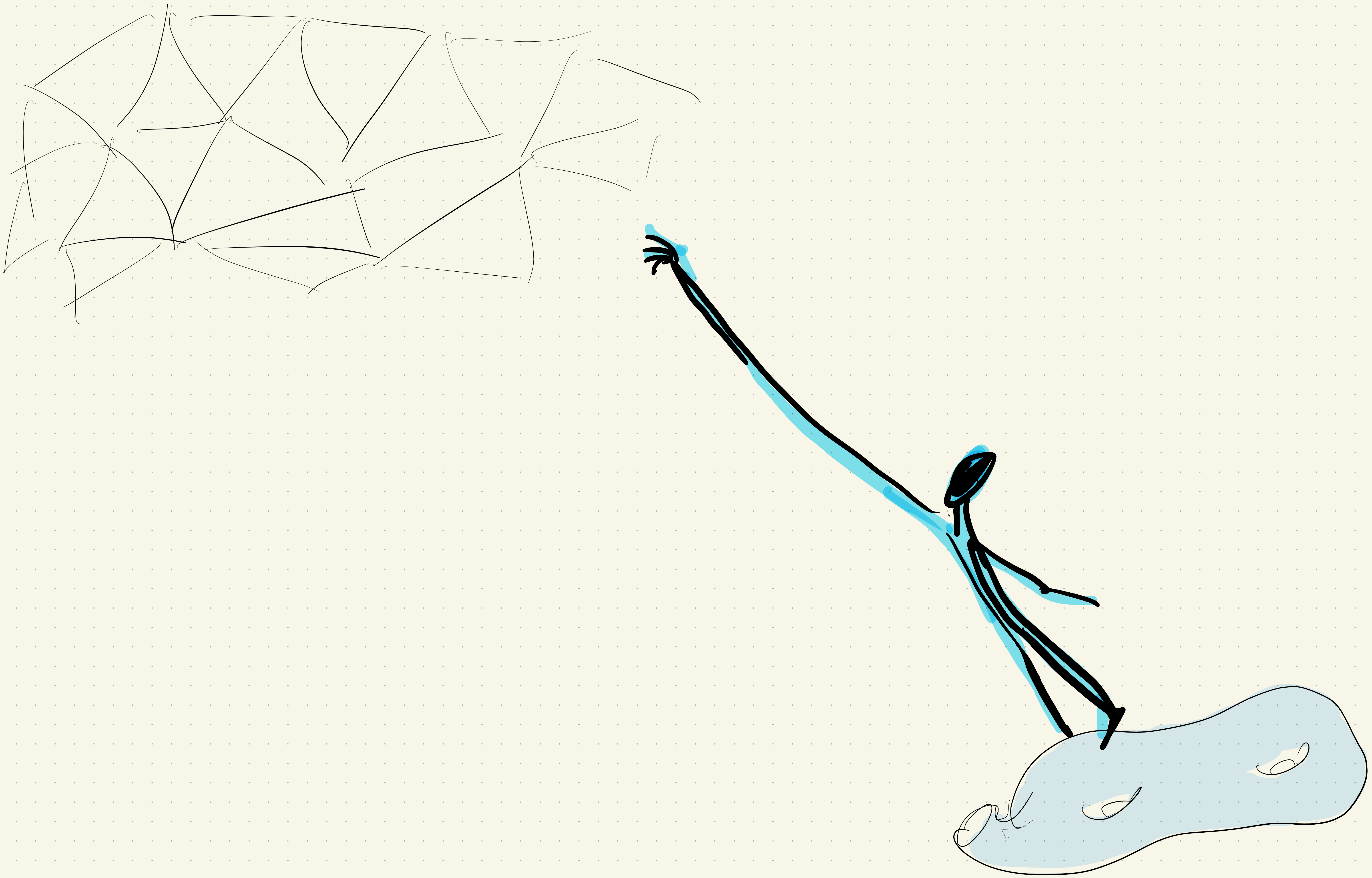
$F(\gamma) = 1$ if $\gamma \in \partial S$

Q: *Are all friezes on a given surface unitary?*

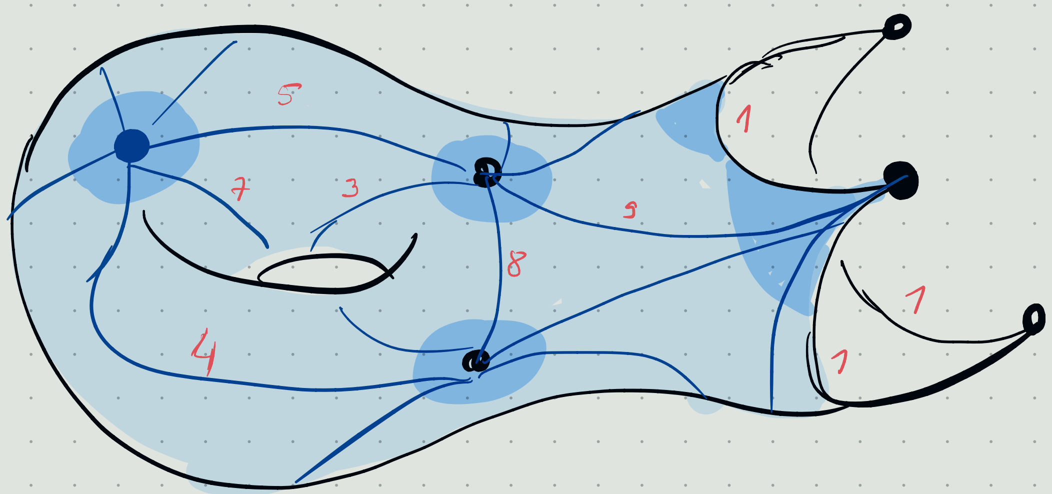
- Conway, Coxeter '73: **YES**, if S = polygon (type A)
- Thomas '2009 $\leftarrow D_4$; Fontaine, Plamondon '2016 **No**, if S = punctured polygon (type D)
- Gunawan, Schiffler '2018 **YES**, if S = annulus (type $\tilde{A}$)
- Çanakçi, AF, PT, García Elsener '2022 **YES**, if S = pair of pants

1 1 1 1
2 2 2 2
3 3 3 3
2 2 2 2
2 2 2 2

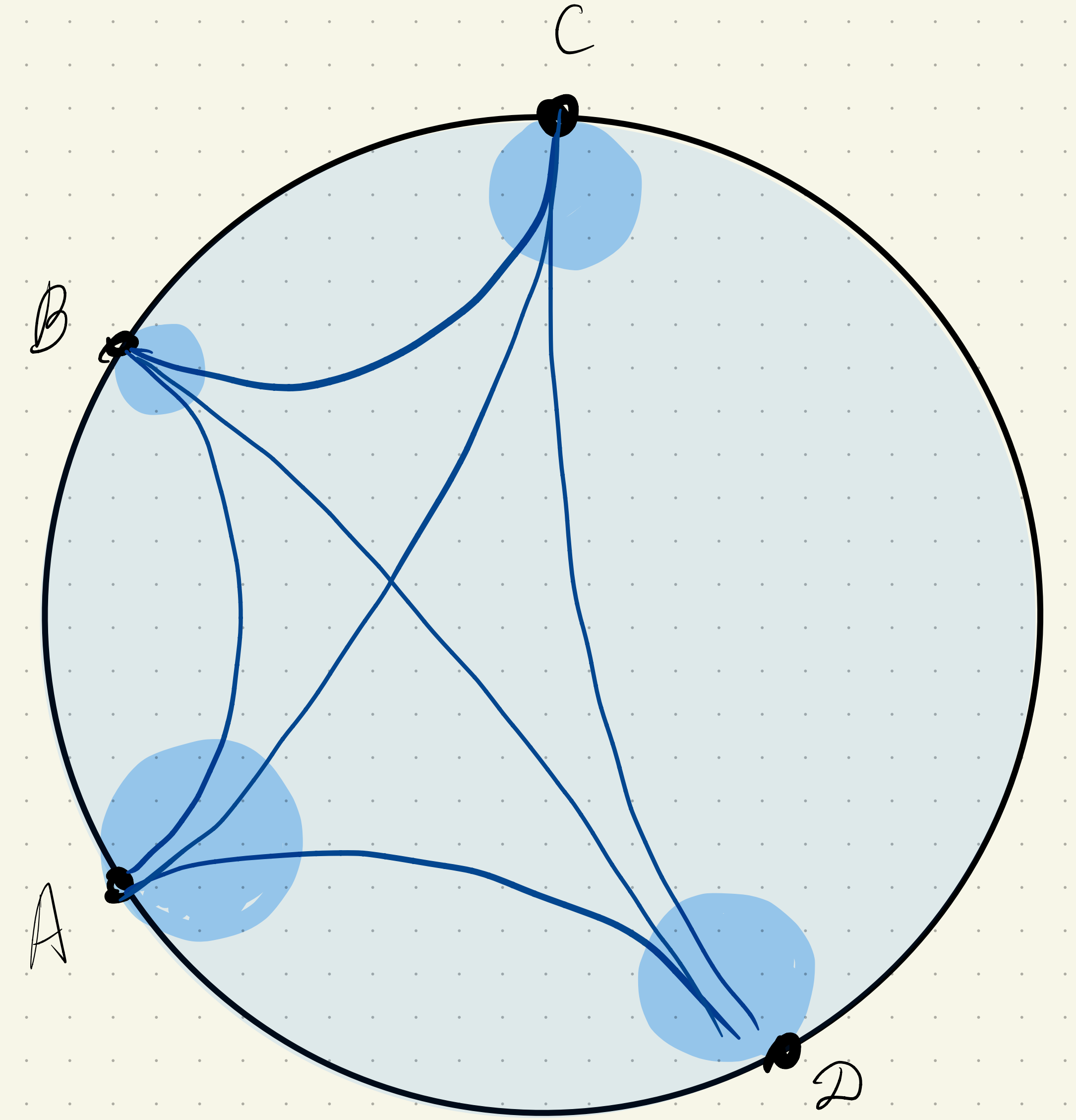




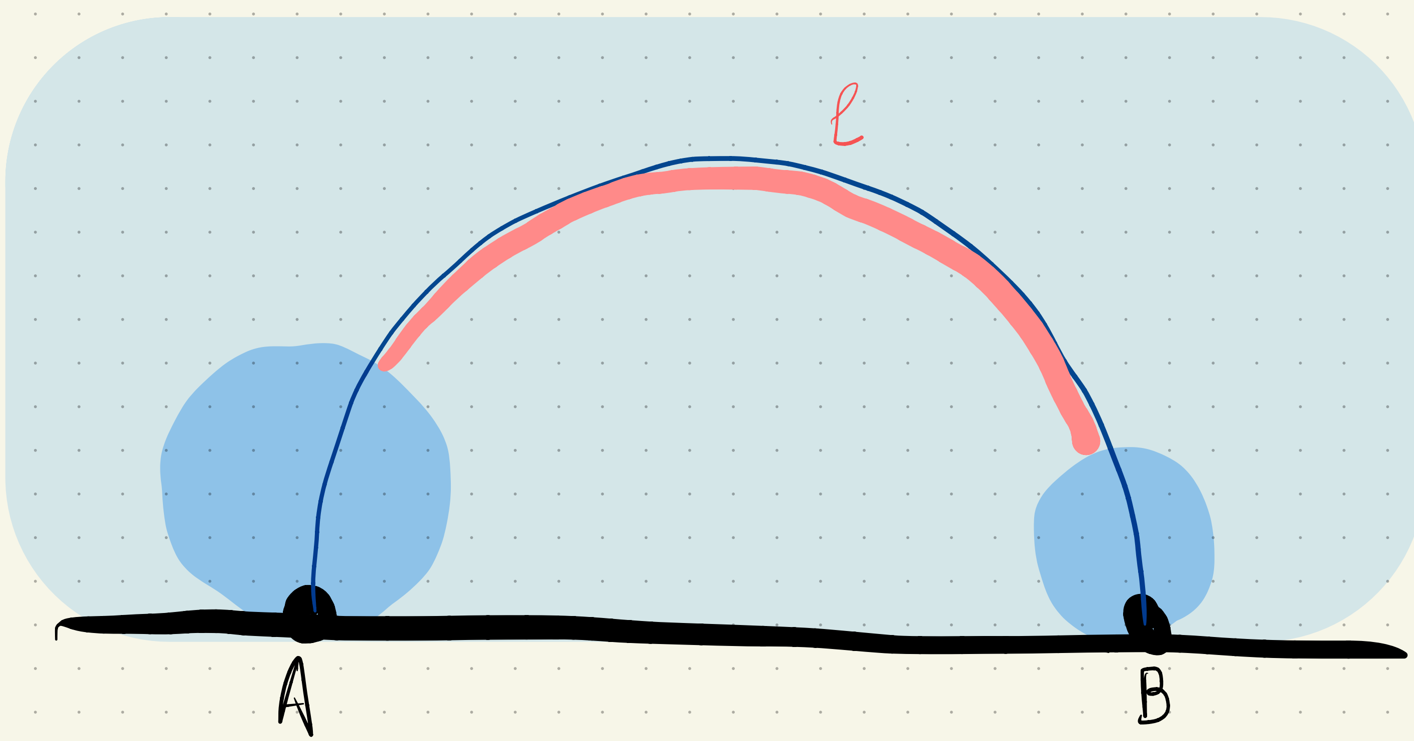
Decorated hyperbolic structure = hyp. str + horocycle at every marked point = Frieze on the surface

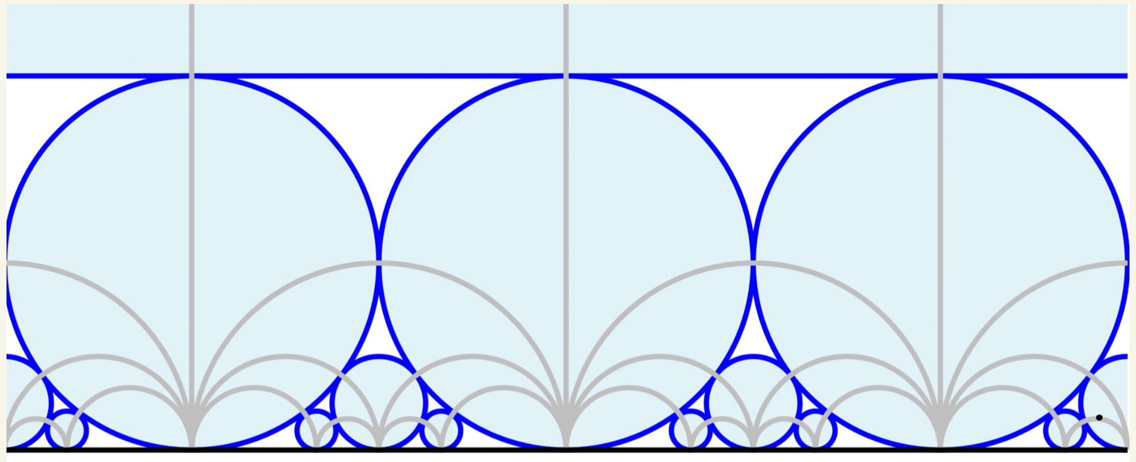


$$\lambda_{AC} \cdot \lambda_{BD} = \lambda_{AB}\lambda_{CD} + \lambda_{BC}\lambda_{AD}$$



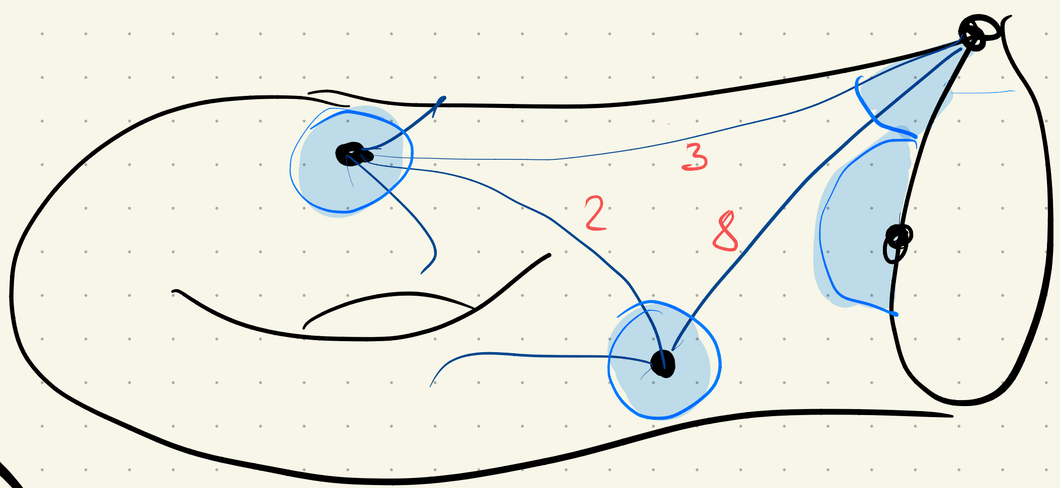
$\lambda_{AB} = e^{\ell/2}$ – Penner's lambda length





Hyperbolic plane with Farey triangulation

Uniformization



Frieze on surface = decorated hyperbolic structure

Need to prove

- Marked points lift to $\frac{p}{q} \in \mathbb{Q}$
- Horocycles lift to Ford circles

Thm (F, Tumarkin '24)

Let S be a surface with boundary without punctures. Then:

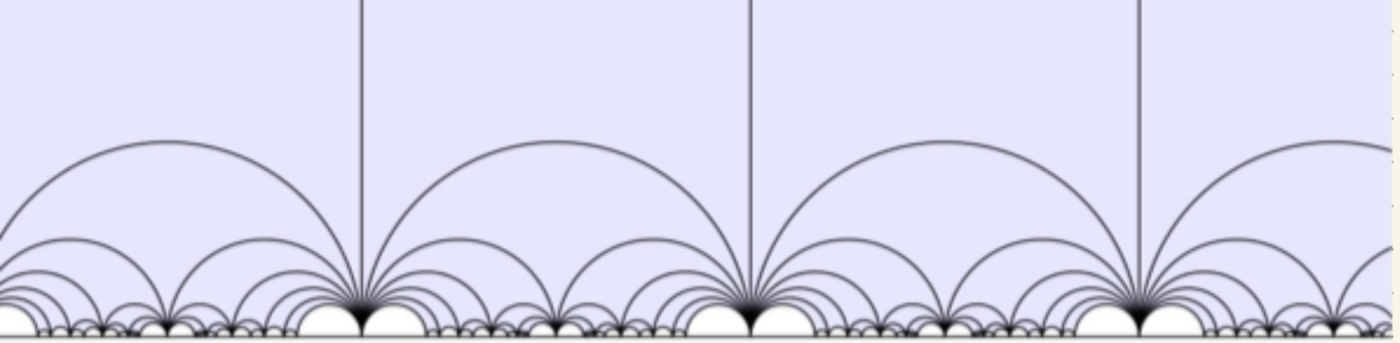
- All friezes on S are unitary [with unitary triangulation coming from Farey graph]
- Friezes on S (up to $MCG(S)$) $\xleftrightarrow{1-1}$ combinatorial types of ideal triangulations on S

On punctured surfaces with boundary:

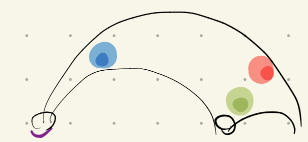
- Almost unitary
- $\#$ Friezes $< \infty$ (up to $MCG(S)$)

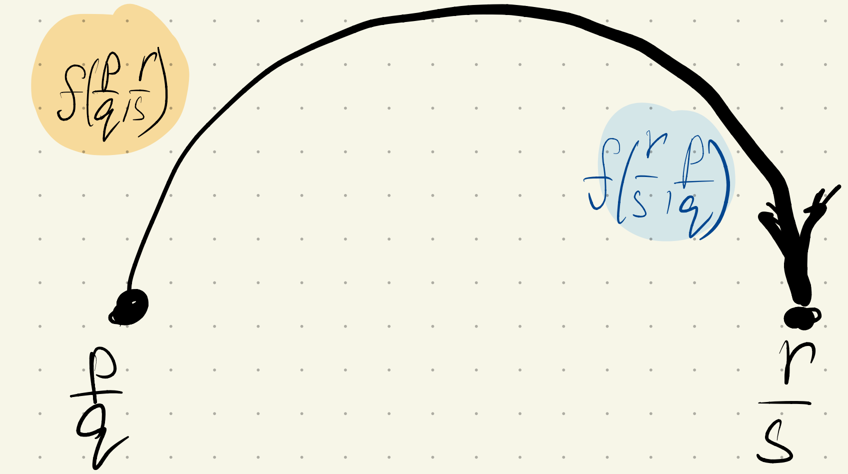
For surfaces without boundary: OPEN

THANKS!



Bonus:





Functions on Farey graph mod 12:

$\begin{pmatrix} p \\ q \end{pmatrix} \sim \begin{pmatrix} p' \\ q' \end{pmatrix}$ if $\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ mod 12

Want: $f(p,q,r,s)$ s.t. $f\left(\frac{p}{q},\frac{r}{s}\right) \overset{\text{mod } 12}{=} f\left(\frac{p'}{q'},\frac{r'}{s'}\right)$ if $\begin{pmatrix} p \\ q \end{pmatrix} \sim \begin{pmatrix} p' \\ q' \end{pmatrix}$, $\begin{pmatrix} r \\ s \end{pmatrix} \sim \begin{pmatrix} r' \\ s' \end{pmatrix}$

Non Example: $f(p,q,r,s) = p$: $f(p',q',r',s') = p' = ap + bq \equiv \pm p \not\equiv p$ if $a \equiv -1$

linear funct.

Prop $f(p,q,r,s) = g(p^2, q^2, r^2, s^2, pq, rs, ps, qr)$ does work.

Pf: $(p')^2 = (ap + \underset{\equiv 0}{b}q)^2 \equiv a^2p^2 \equiv p^2$ $\quad p'q' = (ap + \underset{\equiv 0}{b}q)(\underset{\equiv 0}{c}p + dq) \equiv ad\,pq \equiv pq$

Ex: $f = 2pq + 2rs$