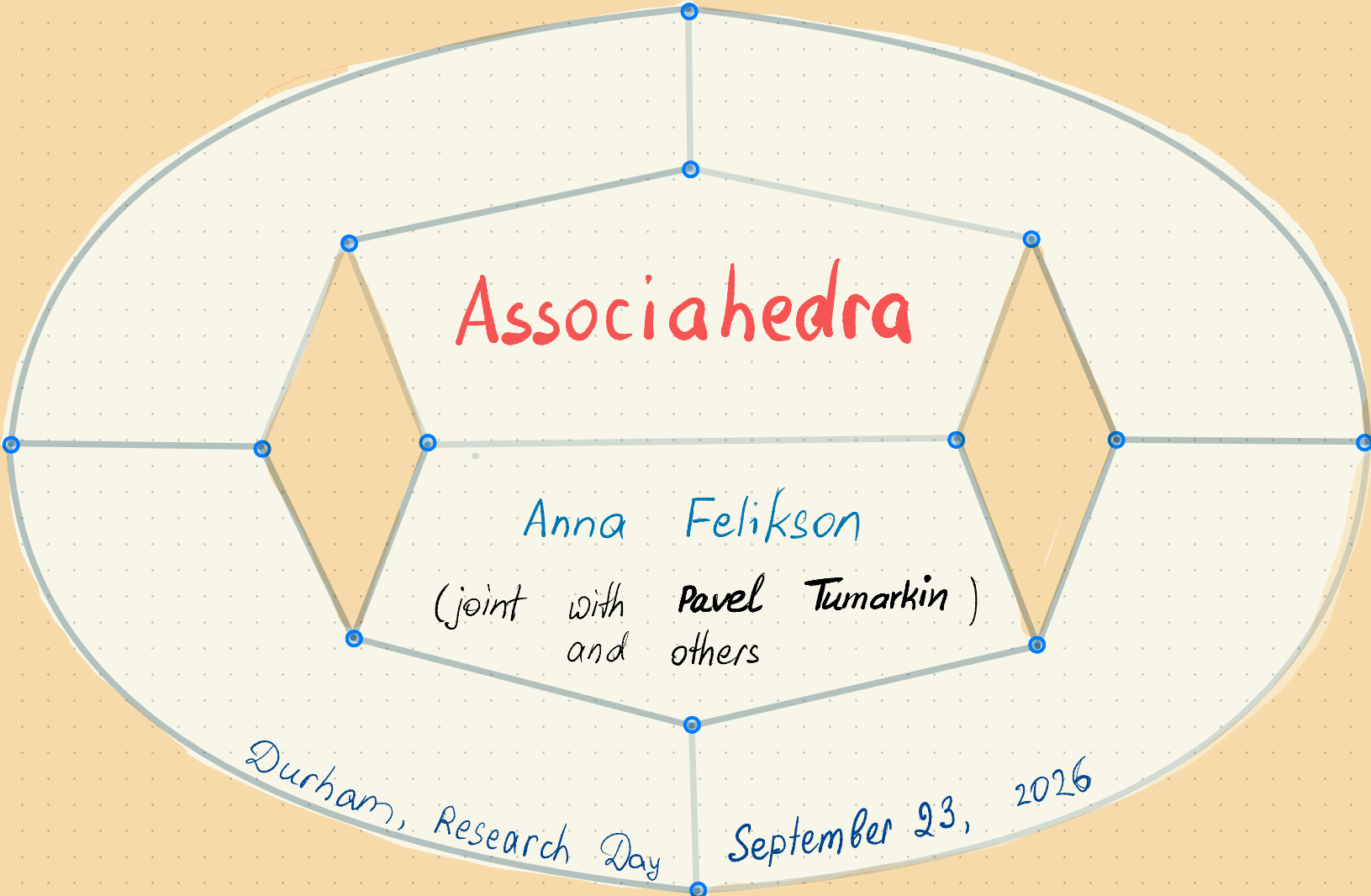




Associahedra

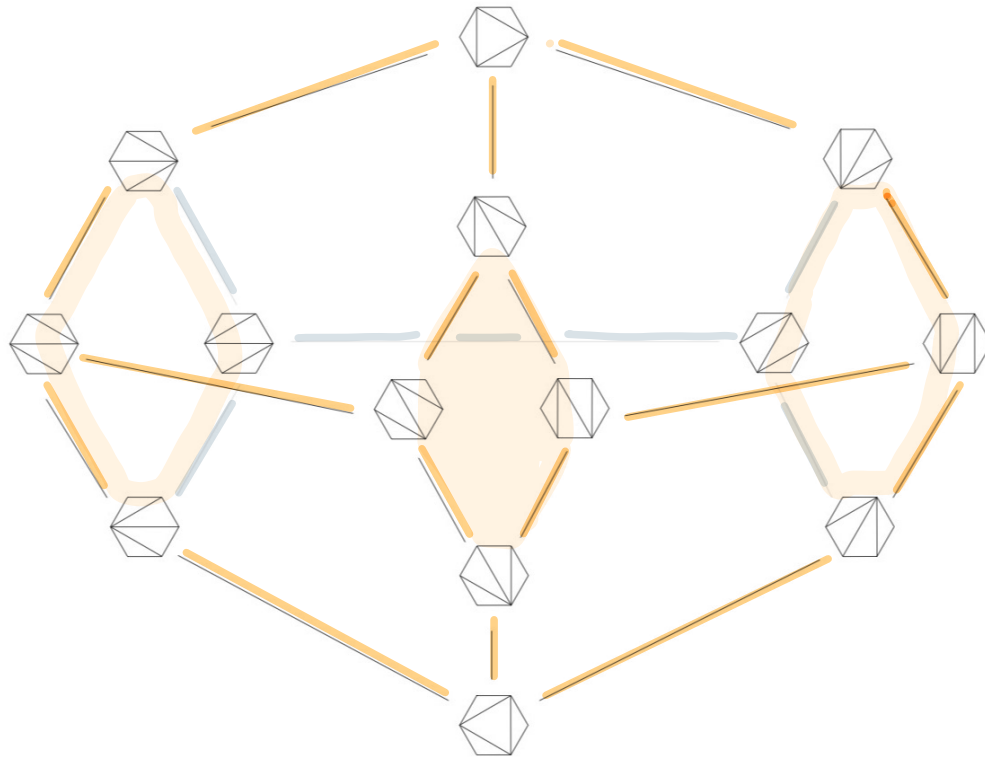
Anna Felikson

(joint with Pavel Tumarkin
and others)

Durham, Research Day

September 23, 2026

What is associahedron?



*n-dimensional
associahedron:*

vertices $\leftrightarrow$ triangulations
of $(n+3)$ -gon

edge $\leftrightarrow$  flip

faces $\leftrightarrow$ partial
triangulations

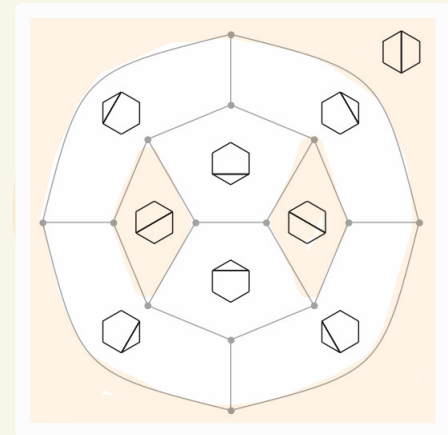


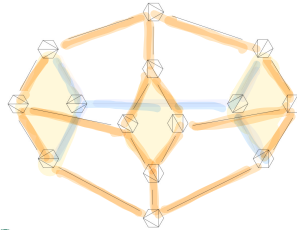
Figure from Fomin, Reading

Moduli spaces and topology
Stasheff, Keller

combinatorics of polytopes, trees, posets,
Tamari, Loday, Reading, Coxeter groups

Cluster algebras
Fomin, Zelevinsky,
Chapoton, Reading
Hohlweg, Lange, Thomas
Pilaud, Stella

Associahedron



Tamari '51
Stasheff '63

operads and rewriting theory
Stasheff, Loday, Vallette, Devadoss

Combinatorial Hopf algebras
Loday, Ronco, Chatel, Pilaud, Pons

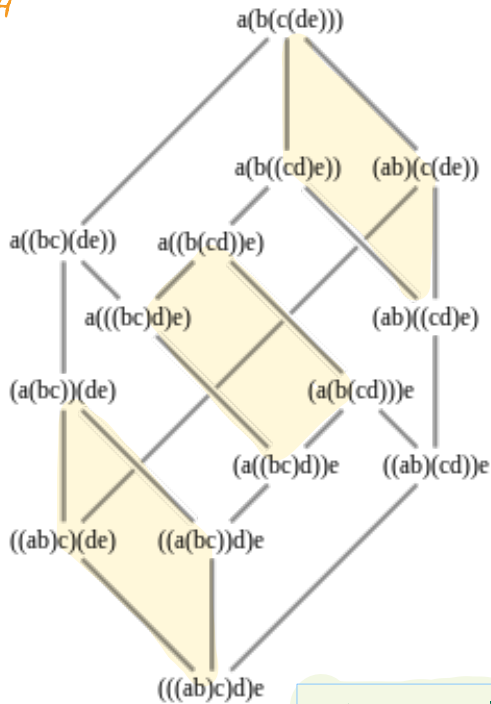
Diagonal harmonics
Bergeron, Préville-Ratelle, Viennot

Quiver representation theory
Baumann, Kamnitzer, Tingley
Bazier-Matte, Chapelier-Laget
Dauville, Mousavand, Thomas, Yıldırım
Padrol Palu, Pilaud, Plamondon
Aoki, Higashitani, Iyama, Kase, Mizuno

Physics of scattering amplitudes
Arkani-Hamed, Bai, He, Yan

1. Tamari lattice (Tamari's thesis, 1951)

$n=4$



$$x(yz) - (xy)z$$

- List all parenthesations of $n+1$ letters
- Connect:
 - $x(yz) - (xy)z$ (associativity)
- Partial order
 - $x(yz) > (xy)z$

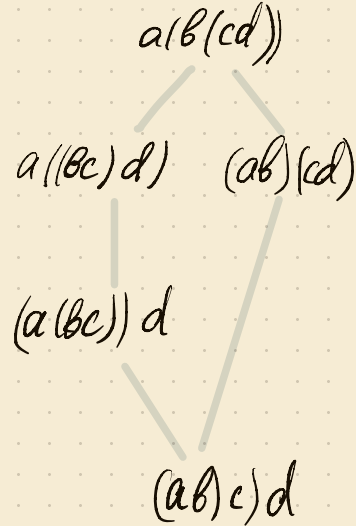
- Get a lattice with C_n elements (Catalan number)
 ($\forall x, y$ have greatest common predecessor and least common successor)

Same can be done with other Catalan objects (triangulations of n -gon, binary trees with n leaves, etc.)

$n=2$

$$\begin{array}{c} a(bc) \\ | \\ (ab)c \end{array}$$

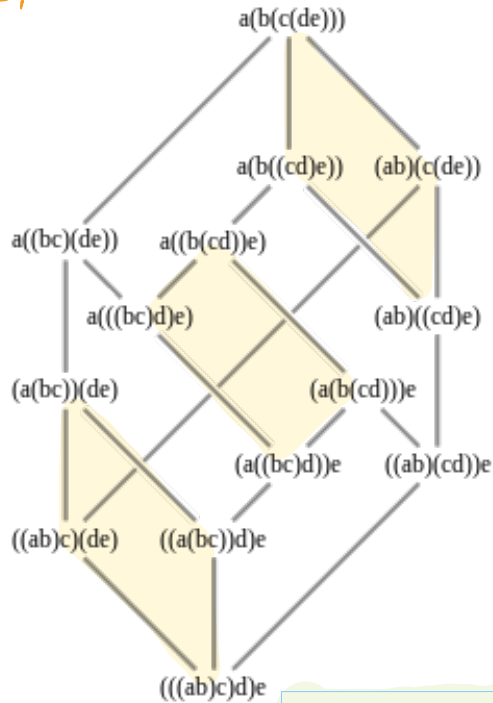
$n=3$



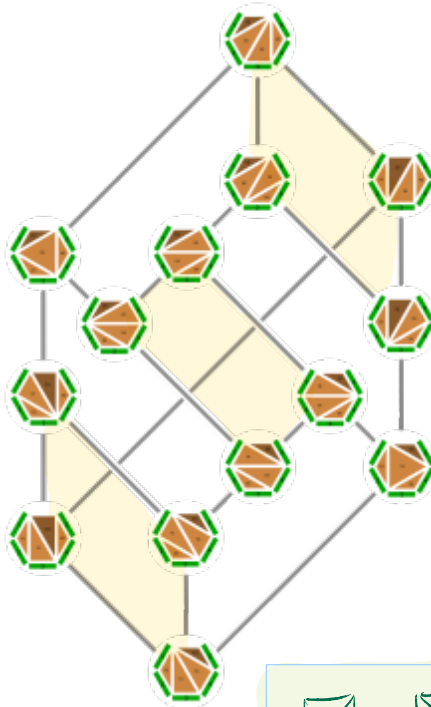
1. Tamari lattice

(Tamari's thesis, 1951)

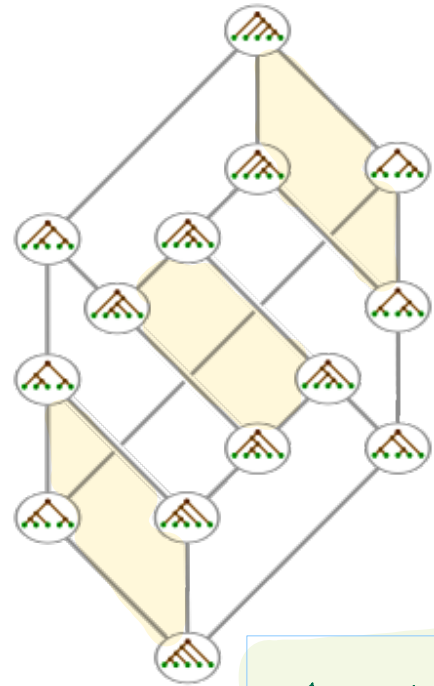
$n=4$



$$x(yz) - (xy)z$$



$$\square - \square$$



$$\triangle - \triangle$$

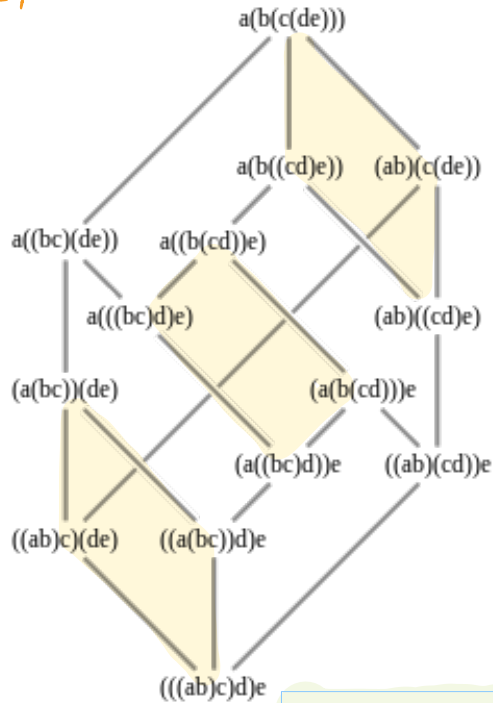
Same can be done with other Catalan objects
(triangulations of n -gon, binary trees with n leaves, etc.)

Figures from
Wikipedia

1. Tamari lattice

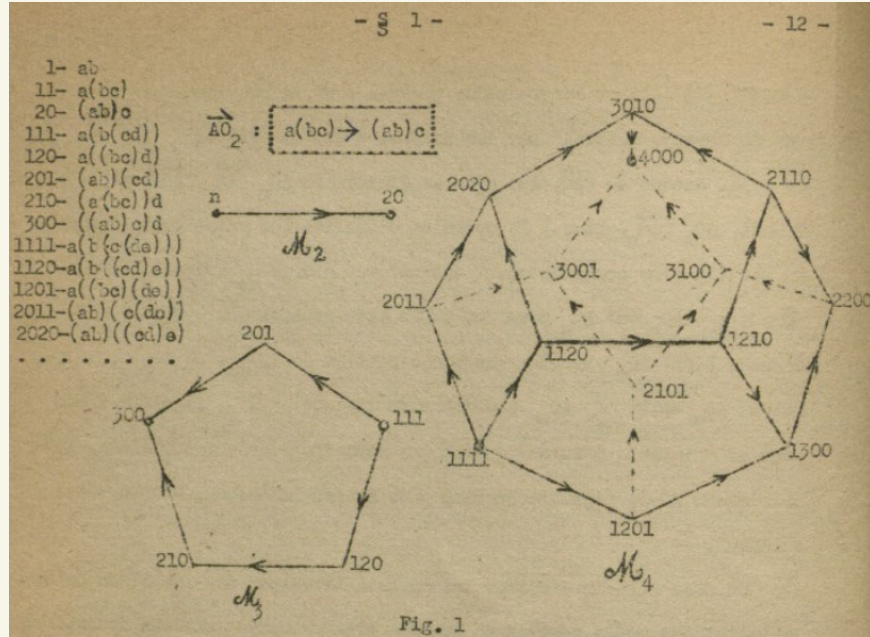
(Tamari's thesis, 1951)

$n=4$



$$x(yz) - (xy)z$$

Tamari noticed that the poset looks like a one-skeleton of a polyhedron:



Tamari's thesis

Q: Can it be realized as a polyhedron, i.e. a convex hull of C_n points in $\mathbb{R}^{n-1}$?

2. Is it possible to realize a graph as a polyhedron?

- $\dim = 2$: trivial question:

a graph G is a
1-skeleton of a
polygon

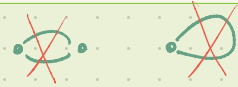
$\longleftrightarrow$ G is connected and
every vertex of G
has valency 2

- $\dim = 3$:

Steinitz's thm (1922)

G is the graph
of 3-dim polytope $\Leftrightarrow$

G is simple
planar
3-connected



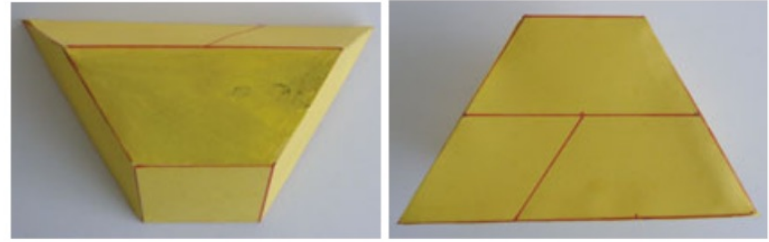
can be drawn on the plane
without self-intersections

G cannot be disconnected
by removing 2 vertices

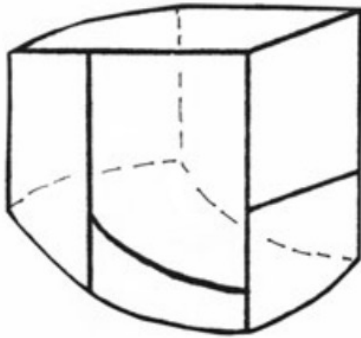
- $\dim \geq 4$: No such theorem!
No even conjecture!

2. Is it possible to realize a graph as a polyhedron?

Dov Tamari gave coordinates of vertices giving a polyhedron, but not all vertices were extremal points



Today's model of Tamari's description



Jim Stasheff in 1963 got a similar (curvilinear) polytope when giving a homotopy theoretic characterisation of based loop spaces.

$$x(yz) - (xy)z$$

→ "Stasheff polytope"



2. Is it possible to realize a graph as a polyhedron?

- Mark Haiman (1984), "Constructing the associahedron"
 - named the polytope;
 - gave the first construction for associahedra in any dimension;
 - never published the draft.

Mythical polytope

The associahedron is a mythical polytope whose face structure represents the lattice of partial parenthesizations of a sequence of variables, in a way to be made precise below. The purpose of these notes is to give an explicit construction of such a polytope.

2. Is it possible to realize a graph as a polyhedron?

- Mark Haiman (1984), "Constructing the associahedron"

- named the polytope;
- gave the first construction for associahedra in any dimension;
- never published the draft.

I never published the manuscript because it was soon made obsolete, first by a simpler and more symmetric construction of such a polytope by Carl Lee, and subsequently by the more general theory of secondary polytopes (the associahedron is secondary polytope of the

- Since then, many realizations constructed:

1989: Lee

1994: Gelfand, Kapranov, Zelevinsky

1995: Tonk

1995: Stasheff

2002 Chapoton, Fomin, Zelevinsky

2004 Loday

2009 Devadoss

2009 Postnikov

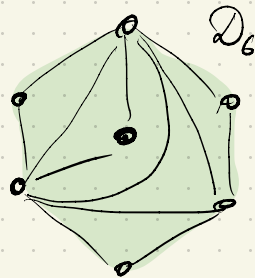


3. Generalizations

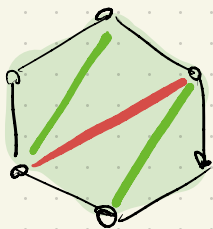
$\mathcal{D}_n$ Triangulate other surfaces (not polygons)

- Punctured polygon

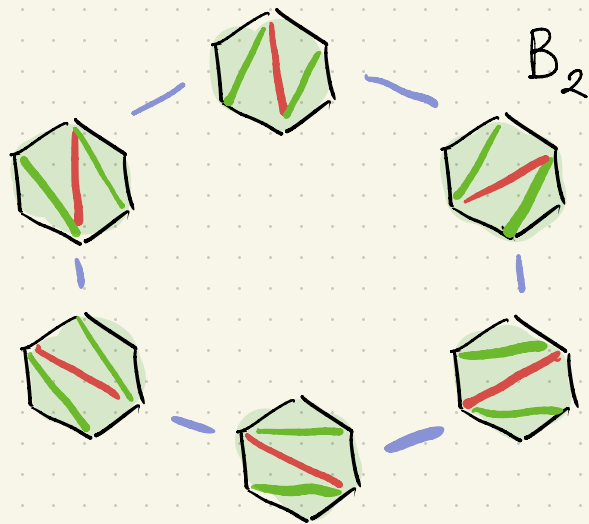
[For all other oriented surfaces
get infinitely many triangulations]



$\mathcal{B}_n$ Triangulate a $2n$ -gon
symmetrically



Slip symmetric diagonals
simultaneously



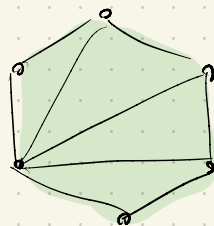
3. Generalizations

- 2001: Fomin, Zelevinsky

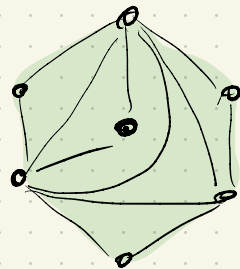
Used root systems to introduce generalized associahedra of types

$A_n, B_n = C_n, D_n, E_6, E_7, E_8, F_4, G_2$

For type A get Tamari - Stasheff associahedron



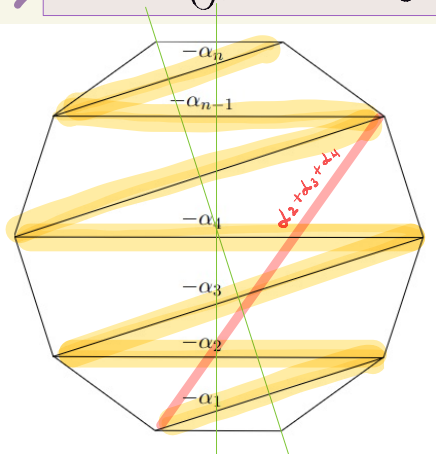
A_6



D_6

positive and negative simple

How?



- Diagonals of a polygon are labelled by roots
- Diag. in zig-zag triangulation $\leftrightarrow$ - (simple roots)
- Diagonal crossing $-\alpha_{i_1}, \dots, -\alpha_{i_m} \leftrightarrow \sum_{k=1, \dots, m} \alpha_{i_k}$
- faces of associahedron $\leftrightarrow$ sets of compatible roots $\leftrightarrow$ sets of non-crossing diagonals

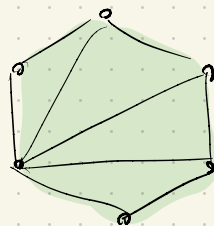
3. Generalizations

- 2001: Fomin, Zelevinsky,

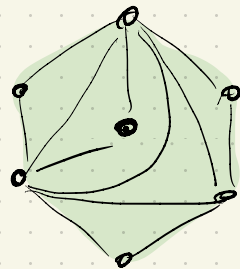
Used root systems to introduce
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For type A get Tamari - Stasheff associahedron

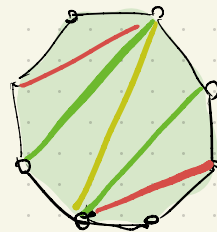


A_6



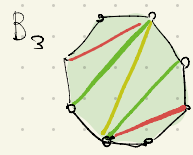
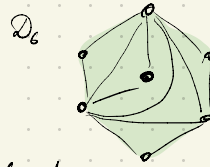
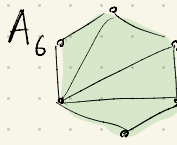
D_6

- 2002: Chapoton, Fomin, Zelevinsky:
they all are polytopes!



B_3

3. Generalizations

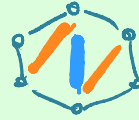


- 2001: Fomin, Zelevinsky, defined associahedra of types $A_n, B_n=C_n, D_n, E_6, E_7, E_8, F_4, G_2$
- 2002: Chapoton, Fomin, Zelevinsky: they all are polytopes!
- 2007: Fomin, Reading: defined associahedra of **non-crystallographic** types (H_3, H_4, I_2)
- 2024: AF, PT, Emine Yildirim (Leeds $\rightarrow$ Sofia $\rightarrow$ Istanbul)

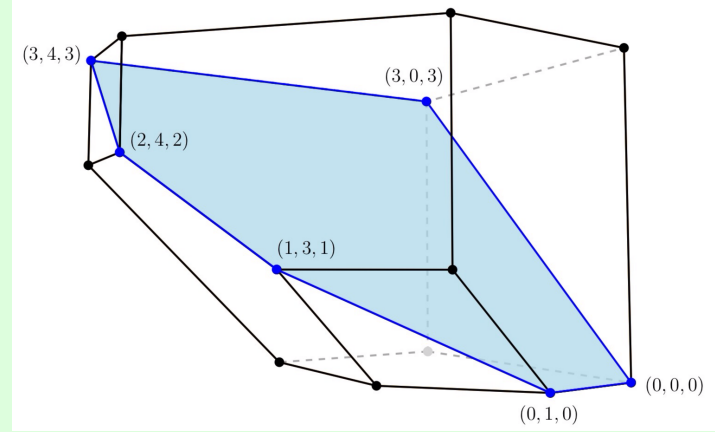
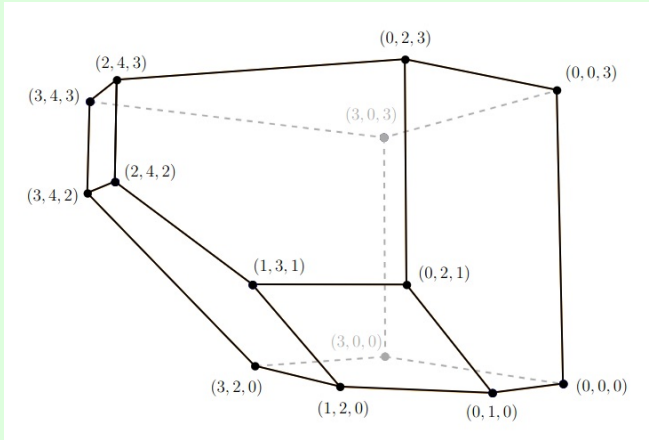
- Thm 1.
- Non-crystallographic associahedra are polytopes.
 - Associahedra of types $B_n=C_n, F_4, G_2, H_3, H_4, I_2$ can be obtained from "symmetric" associahedra of types ADE by crossing with some plane.

3. Generalizations

A_3



B_2



- 2024: AF, PT, Emine Yildirim (Leeds → Sofia → Istanbul)

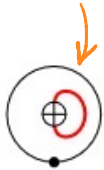
- Thm 1.
- Non-crystallographic associahedra are polytopes.
 - Associahedra of types $B_n = C_n, F_4, G_2, H_3, H_4, I_2$ can be obtained from "symmetric" associahedra of types ADE by crossing with some plane.

4. Triangulations of Möbius band

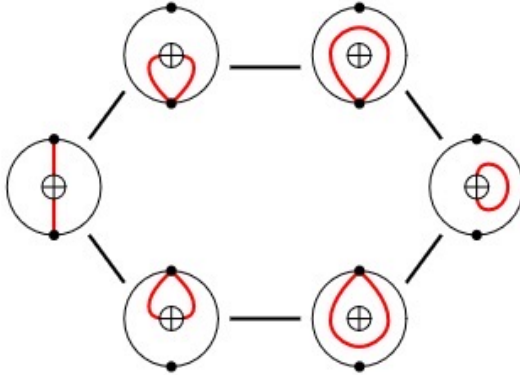
2014, Dupont - Palesi: quasi-triangulations for non-orientable surfaces

Diagrams from Wilson '15

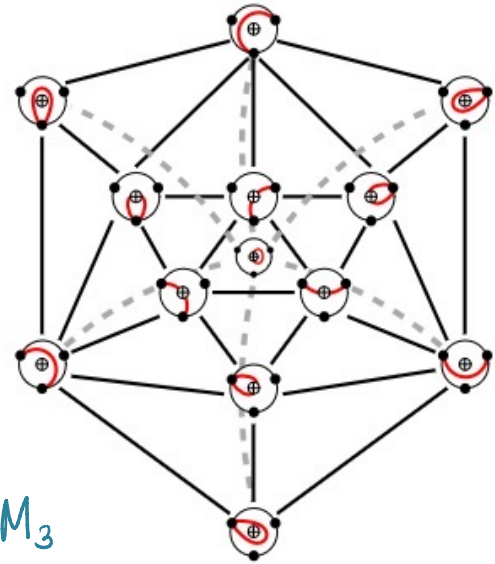
quasi-arc



M_1



M_2



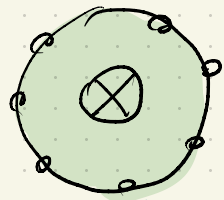
M_3

The quasi-arc complexes of M_1 , M_2 and M_3 .

→ quasi-arc complexes (only finite for Möbius bands)

4. Triangulations of Möbius band

2014, Dupont - Palesi: quasi-triangulations for non-orientable surfaces



n points on boundary

→ quasi-arc complexes (only finite for Möbius bands M_n)

2015, Jon Wilson: quasi-arc complexes of M_n are spherical (and shellable)

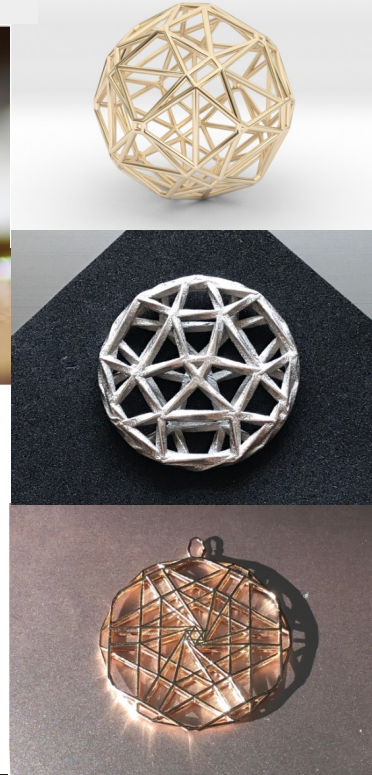
But are they polytopal?

2026(?), AF, PT, Salvatore Stella (Rome):

Thm 2 Yes!

← It is obtained by crossing some polyhedral but infinite object in higher dimension with some plane

Commercial break: Herbert Gangl – 3D prints associahedra (and other math. jewellery)



The Gems of Hypolytos

By Herbert Gangl

HG is a mathematician at Durham University, UK. He uses 'ideal tessellations' of hyperbolic 3-space as a rich source of beautiful new polytopes. Variations on these polytopes give rise to decorative objects and even jewellery.

https://www.instagram.com/3d_printed_jewellery/

online
coming
shop
soon →

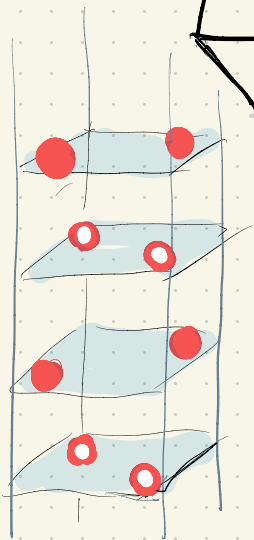
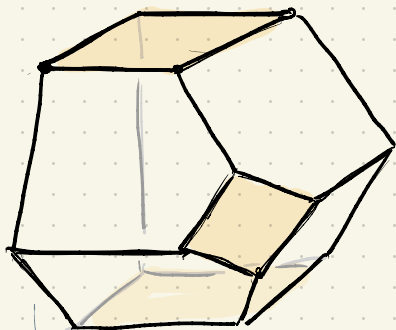
castimize.com

5. Tiling space by associahedra

with PT and
Zack Greenberg
[MPI - Leipzig]

tiles the square
cylinder $\rightarrow$

Moreover, this tiling
is by Voronoi cells
for an (easy to describe)
set of points



In progress: hope to generalize for higher dimensions
(use combinatorics of cluster algebras)

Some known tilings by associahedra follow from
Andy Tenks "Relating the associahedra and permutohedra", 1995

