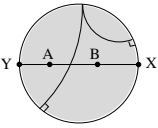
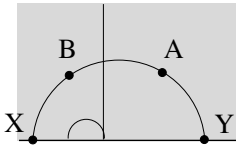
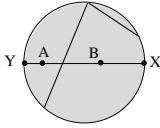


### Models of hyperbolic geometry

model	Poincaré disk	Upper half-plane	Klein disk	two-sheet hyperboloid
$\mathbb{H}^2$	$\{z \in \mathbb{C} \mid  z  < 1\}$	$\{z \in \mathbb{C} \mid \text{Im}z > 0\}$	$\{z \in \mathbb{C} \mid  z  < 1\}$	$\{v \in \mathbb{R}^{2,1} \mid \langle v, v \rangle = -1, v_3 > 0\}$ where $\langle v, u \rangle = v_1u_1 + v_2u_2 - v_3u_3$
$\partial\mathbb{H}^2$ (absolute)	$\{z \in \mathbb{C} \mid  z  = 1\}$	$\{z \in \mathbb{C} \mid \text{Im}z = 0\}$	$\{z \in \mathbb{C} \mid  z  = 1\}$	$\{v \in \mathbb{R}^{2,1} \mid \langle v, v \rangle = 0, v_3 > 0\}$ $v \sim \lambda v$
lines				$\{v \mid \langle v, a \rangle = 0\}$ where $\langle a, a \rangle > 0$
distance	$d(A, B) =  \ln[A, B, X, Y] $ $X, Y$ = the “endpoints” of the line $AB$		$d(A, B) = \frac{1}{2} \ln[A, B, X, Y] $	$d(A, B) = \frac{1}{2} \ln[A, B, X, Y] $ cross-ratio of four lines*
formula	$\cosh d(u, v) = 1 + \frac{ u-v ^2}{2\text{Im}(u)\text{Im}(v)}$			$Q = \left  \frac{\langle u, v \rangle^2}{\langle u, u \rangle \langle v, v \rangle} \right $ if $\langle u, u \rangle < 0, \langle v, v \rangle < 0$ $Q = \cosh^2 d(pt, pt)$ if $\langle u, u \rangle < 0, \langle v, v \rangle > 0$ $Q = \sinh^2 d(pt, line)$ if $\langle u, u \rangle > 0, \langle v, v \rangle > 0$ $Q < 1$ , intersecting lines $Q = \cos^2 \alpha$ $Q = 1$ , parallel lines $Q > 1$ , ultraparallel lines $Q = \cosh^2 d(line, line)$
isometries**	Möbius transformations		Projective tr	Linear transformations of $\mathbb{R}^{2,1}$
orientation-preserving isometries		$\frac{az+b}{cz+d}$ $a, b, c, d \in \mathbb{R}, ad - bc = 1$		
orientation-reversing isometries		$\frac{a\bar{z}+b}{c\bar{z}+d}$ $a, b, c, d \in \mathbb{R}, ad - bc = -1$		
reflections	Euclidean inversions or reflections			$r_a(v) = v - 2\frac{\langle v, a \rangle}{\langle a, a \rangle}a$
circles	Euclidean circles		ellipses	plane sections of the hyperboloid
angles	angles=Euclidean angles		distorted angles good for right angles***	

\* Cross-ratio of four lines lying in one plane and passing through one point is the cross-ratio of four points at which these lines are intersected by an arbitrary line  $l$  (it does not depend on  $l$ !).

\*\*We only list the type of the transformations not specifying that they preserve the model.

\*\*\* See the next page.

**\*\*\*Right angles in the Klein model.**

Let  $l$  be a hyperbolic line.

Let  $\bar{l}$  be a Euclidean line containing the segment which represents  $l$  in the Klein model.

Let  $X_1(l)$  and  $X_2(l)$  be the endpoints of  $l$  (intersections of  $\bar{l}$  with the unit circle).

Let  $t_1(l)$  and  $t_2(l)$  be tangent lines to the unit circle at the points  $X_1(l)$  and  $X_2(l)$ .

Let  $T(l) = t_1(l) \cap t_2(l)$  (if  $t_1 \parallel t_2$ , i.e.  $l$  is represented by a diameter, then  $T(l)$  is a point at infinity).

**Thm.**  $l'$  is orthogonal to  $l$  if and only if  $T(l) \in l'$ .

In particular, if  $l$  is represented by a **diameter**, then  $l' \perp l$  if and only if  $\bar{l}' \perp \bar{l}$  (in Euclidean sense).

