Euclidean Geometry

EUCLID'S POSTULATES (5)

1. For every point A and for every point B not equal to A there exists a unique line that passes through A and B.
2. For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and such that segment CD is congruent to segment BE.
3. For every point O and every point A not equal to O, there exists a circle with center O and radius OA.
4. All right angles are congruent to each other.
5. (Euclid's Parallel Postulate) For every line l and for every point P that does not lie on l, there exists a unique line m passing through P that is parallel to l.

HILBERT'S AXIOMS (5 groups)

Undefined notions: point, line, incidence, betweenness, and congruence.

1. **Incidence Axioms (IA)**

IA 1. Given 2 distinct points there is a unique line incident with them.

IA 2. Given a line there exist at least 2 distinct points incident with it.

IA 3. There exist 3 distinct points not incident with the same line.

2. **Betweenness Axioms (BA)**

BA 1. If $A * B * C$ then also $C * B * A$ and A, B, C are distinct collinear points.

BA 2. Given 2 points P and Q there exist 3 points A, B, C such that $P * B * Q$ and $P * Q * C$ and $A * P * Q$.

BA 3. Given 3 collinear points, only one of them can be between the other two.

BA 4. (Plane Separation) For every line l and for every 3 points A, B, C not on l,

(a) If A, B are on the same side of l and B, C are on the same side of l, then A, C are on the same side of l.

(b) If A, B are on the opposite sides of l and B, C are on the opposite sides of l, then A, C are on the same side of l.

3. **Congruence Axioms (CA)**

CA 1. Given segment AB and any ray with vertex C, there is a unique point D on this ray such that $AB \approx CD$.

CA 2. If $AB \approx CD$ and $AB \approx EF$ then $CD \approx EF$.

CA 3. Given $A * B * C$ and $A' * B' * C'$, if $AB \approx A'B'$ and $BC \approx B'C'$ then $AC \approx A'C'$.

CA 4. Given $\angle D$ and any ray AB there is a unique ray AC on each half-plane of the line AB such that $\angle BAC \approx \angle D$.

CA 5. If $\angle A \approx \angle B$ and $\angle A \approx \angle C$ then $\angle B \approx \angle C$.

CA 6. (SAS Criterion) If 2 sides and the included angle of a triangle are congruent to those of another triangle, respectively, then the two triangles are congruent.

4. **Continuity Axioms (CtA)**

CtA 1. (Circular Continuity Principle) If a circle has one point inside and one point outside another circle, then the two circles intersect in two points.

CtA 2. (Archimedes' Axiom) Given segment CD and any ray AB there is a number n and a point E on this ray such that $n \times CD \approx AE \geq AB$.

5. **Parallelism Axiom (PA)**

PA 1. (Hilbert's Parallel Axiom) Given a line l and a point P not on l, there is at most one line through P which is parallel to l.

(SOME) THEOREMS of EUCLIDEAN GEOMETRY (E)

	Statement	Pf based on	reference	PA
	Basic facts:			
E1	Given a triangle $\triangle ABC$ and a line l s.t. $A, B, C \notin l$, if l intersects AB then l intersects either AC or BC.	BA4		-
E2	An angle and its supplement add to π. **Corollary:** Supplements of congruent angles are congruent.			-
E3	Vertical angles are equal.	E2		-
	Congruence of triangles, isosceles trianges:			
E4	(SAS) If $AB = A'B'$, $AC = A'C'$ and $\angle BAC = \angle B'A'C'$ then $\triangle ABC$ is congruent to $\triangle A'B'C'$.	=CA6	2.2	-
E5	(ASA) If $AB = A'B'$, $\angle BAC = \angle B'A'C'$ and $\angle ABC = \angle A'B'C'$ then $\triangle ABC$ is congruent to $\triangle A'B'C'$.	CA1, CA4	2.1	-
E6	(Thales's thm) If $AB = BC$ then $\angle BAC = \angle BCA$.	E4 (SAS)	3	-
E7	If $\angle BAC = \angle BCA$ then $AB = BC$.	E5 (ASA)		-
E8	If $AB = BC$ and M is a midpoint of AC then BM bisects $\angle ABC$ and is orthogonal to AC. ("in isosceles triangle a median is an altitude and an angle bisector").	E4 (SAS)	3.3	-
E9	Given a line l and a point A there exists a unique line l' perpendicular to l and containing A.	CA4, Cor. E2		-
E10	(SSS) If $AB = A'B'$, $AC = A'C'$ and $BC = B'C'$ then $\triangle ABC$ is congruent to $\triangle A'B'C'$.	E9, E4 (SAS) ,E8	2.3	-
	Parallel lines:			
E11	For lines a, b, c, if $a \| b$ and $b \| c$ then $a \| c$.	PA		+
E12	Distinct lines l and l' are parallel iff some transversal line creates included interior angles adding to exactly π. **Corollary:** $l \| l'$ iff alternate interior angles are equal.	BA4, E5 (ASA), E2		+
E13	Angle sum of any triangle equals to π.	E12, BA3, BA4		+
	Lines in triangles :			
E14	The locus of points on the same distance from A and B is a line, this line coincides with the perpendicular bisector to AB.	E8, E4 (SAS)		-
E15	The perpendicular bisectors of the sides of a triangle are concurrent. **Corollary:** Each triangle has a circumscribed circle.	E14	8	-
E16	The angle bisectors in a triangle are concurrent. **Corollary:** Each triangle has an inscribed circle.	E5 (ASA)	11	-
E17	(Orthocentre property) The altitudes in a triangle are concurrent.	E12, E15	13	+

	Similarity:			
E18	In a parallelogram opposite sides are equal.	E12, E5 (ASA)	4.1	+
E19	Let $\angle BOA$ be an angle, let $A_1, A_2, A_3 \in OA$, $B_1, B_2, B_3 \in OB$ be points satisfying $A_1B_1 \vert\vert A_2B_2 \vert\vert A_3B_3$. If $A_1A_2 = A_2A_3$ then $B_1B_2 = B_2B_3$.	E18, E5 (ASA)		+
E20	If ABC is a triangle and D, E are points on AB, AC s.t. $DE \vert\vert BC$ then $AD : AB = AE : AC$. **Converse thm** also holds: if $AD : AB = AE : AC$ then $DE \vert\vert BC$.	E19	5	+
E21	The medians of a triangle are concurrent (at a point G called centroid) and $AG = \frac{2}{3}AM$ for a median AM of $\triangle ABC$.	E19	12	+
E22	Triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar (i.e. $\angle A_i = \angle B_i$) if and only if then they are proportional (i.e. $\frac{A_1A_2}{B_1B_2} = \frac{A_2A_3}{B_2B_3} = \frac{A_1A_3}{B_1B_3}$).	E20	6	+
E23	Let CD be an altitude of a right triangle ABC ($\angle C = \frac{\pi}{2}$) then $AD \cdot DB = CD^2$.	E22		+
E24	(Pythagoras's thm) In a right triangle ABC ($\angle C = \frac{\pi}{2}$) holds $AB^2 = AC^2 + BC^2$.	E23	7	+
E25	(Triangle inequality) $AB + BC \geq AC$.	E24		+
	Angles in a circle:			
E26	(Angles in a semicircle) Let AB be a diameter of a circle $\mathcal{C}$, and let $P \in \mathcal{C}$ be point ($P \neq A, B$). Then $\angle APB = \frac{\pi}{2}$.	E8, E13		+
E27	(Angle at centre) Let AB be an arc of a circle $\mathcal{C}$, centre O, and let $P \in \mathcal{C}$ be a point contained in the same halfplane as O with respect to the line AB. Then $\angle AOB = 2\angle APB$.	E7, E13		+
E28	(Angle in the same segment are equal) Let AB be an arc of a circle $\mathcal{C}$, and let $P_1, P_2 \in \mathcal{C}$ be two points in the same halfplane as O with resp. to AB. Then $\angle AP_1B = \angle AP_2B$.	E27		+
E29	(Opposite angles of a cyclic quadrilateral add to π) If $ABCD$ is a quadrilateral with vertices on a circle then $\angle ABC + \angle CDA = \pi$.	E27, E13		+
	Sine and cosine rules:			
E30	(Sine rule) $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$	Def. of sin		+
E31	(Cosine rule) $c^2 = a^2 + b^2 - 2ab\cos\gamma$	E24		+

Remarks:

1. A. D. Gardiner, C.J. Bradley, *Plane Euclidean Geometry*, UKMT, Leeds 2012.
2. References are given to http://www.unitedthc.com/TUT/Geometry/geometry.htm
3. The third column contains hints to (one of the many possible!) proofs. (In many cases we choose proofs different from ones in the references.
4. Last column indicates use of the parallel axiom (PA) in the proof. Some statement marked "+" are still valid in the absence of PA!
5. For the detailed treatment of axiomatic fundations of Euclidean geometry see M. J. Greenberg, *Euclidean and Non-Euclidean Geometries*, San Francisco: W. H. Freeman, 2008.