Durham University
Anna Felikson

Geometry 2024/2025

**Questions for Problems classes**

Here are some questions which will be probably discussed in the Problems Class (subject to change!)

# 1 Reflections on the plane, Geometric constructions

## Problems Class 1 (17 October, 2023)

1. Let $R_{A,\varphi}$ and $R_{B,\psi}$ be rotations with $0 < \varphi, \psi \leq \pi/2$. Find the type of the composition $f = R_{B,\psi} \circ R_{A,\varphi}$.

   *Hint:* This is an example of using reflections to study compositions of isometries (write everything as a composition of reflections, make you choice so that some of them cancel!).

2. Let $A$ and $B$ be two given points in one half-plane with respect to a line $l$. How to find a shortest path, which starts at $A$ then travels to $l$ and returns to $B$? (How to find the point where this path will reach the line $l$?)

3. Ruler and compass constructions: perpendicular bisector, perpendicular from a point to a line, midpoint of a segment, angle bisector, inscribed and circumscribed circles for a triangle.

# 2 Group actions on $\mathbb{E}^2$

## Problems Class 2 (31 October, 2023)

0. Let $g_1, \dots, g_n$ be isometries of $\mathbb{E}^2$. Let $G = \langle g_1, \dots, g_n \rangle$ be the group generated by $g_1, \dots, g_n$ (i.e. the minimal group containing all of $g_1, \dots, g_n$). Show that the group $G$ acts on $\mathbb{E}^2$.

1. Let $G$ be a group generated by two reflections on $\mathbb{E}^2$. When $G$ is discrete?

2. Let $T$ be a triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$. Let $r_1, r_2, r_3$ be the reflection with respect to the sides of $T$, and let $G$ be the group generated by $r_1, r_2, r_3$. In the lecture we have checked that $G : \mathbb{E}^2$ discretely. Find the fundamental domain of this action.

3. Find the orbit-space for the action introduced in Question 2.

4. Let $X$ be a regular triangle on $\mathbb{E}^2$. Let $r_1$ and $r_2$ be two distinct reflections taking $X$ to itself. Find the fundamental domain of the action $G : X$. Find also the orbit-space.

5. Let $G$ be a group generated by rotation through angle $\frac{2\pi}{3}$ on the plane. Find the orbit-space of the action $G : \mathbb{E}^2$. Are there closed geodesics in this orbit-space? Are there bi-infinite open geodesics?

# 3 Spherical Geometry

## Problems Class 3 (14 November, 2023)

1. Let $G : S^2$ be an action. $G$ acts discretely if and only if $|G| < \infty$.
2. Let $G : X$ be an action and suppose that $F$ is its fundamental domain. Then one can show that the action $G : X$ is discrete.
3. Let $g$ be a reflection, $h \in Isom(S^2)$. $h$ is a reflection if and only if there exists $f \in Isom(S^2)$ such that $fgf^{-1} = h$.
4. Let $S^2$ be a sphere of radius 1. Show that the length of a circle or(spherical) radius $r$ equals to $2\pi \sin r$.

   **Remark:** for the sphere of radius $R$, the length of the circle of radius $r$ will be $2\pi R \sin(\frac{r}{R})$. When $R \to \infty$ we see that $\frac{r}{R} \to 0$ and, hence, $2\pi R \sin(\frac{r}{R}) \to 2\pi r$.
5. Let $S^2$ be a sphere of radius $R$. Let $\alpha$ and $\beta$ be two parallel planes crossing $S^2$. Find the area of the part of $S^2$ lying between the planes $\alpha$ and $\beta$.
6. One can also discuss ruler and compass constructions, as in $\mathbb{E}^2$.

# 4 Projective geometry

## Problems Class 4 (28 November, 2023)

1. Find a projective transformation $f$ which takes

$$\begin{aligned} A = (1:0:0) &\text{ to } (0:0:1) \\ B = (0:1:0) &\text{ to } (0:1:1) \\ C = (0:0:1) &\text{ to } (1:0:1) \\ D = (1:1:1) &\text{ to } (1:1:1) \end{aligned}$$

   Find the image of $X = AD \cap BC$ under this transformation.
2. Find $[A, B, C, D]$ for the points above. (Does it exist?)
   For $E = (1:1:0)$, $F = (1:2:0)$ find $[A, B, E, F]$.
3. Check explicitly, that the transformation $f$ from Question 1 preserves the value of $[A, B, E, F]$.
4. Let $A_1, A_2, A_3, A_4$ be points on a line $a$, let $B_1, B_2, B_3, B_4$ be points on a line $b$. Denote by $p_i$ the line through $A_i$ and $B_i$. Show that if the lines $p_1, p_2, p_3, p_4$ are concurrent, then the points $A_{i+1}B_i \cap A_iB_{i+1}$ $(i = 1, 2, 3)$ are collinear.
5. Formulate and prove the statement dual to the one in Question 4.

# 5 Möbius geometry

## Problems Class 5 (30 January, 2025)

1. Find the type of Möbius transformation $f(z) = 1/z$.
2. Let $f$ and $g$ be inversions with respect to two intersecting circles. Show that $g \circ f = f \circ g$ if and only if $Fix_f \perp Fix_g$.
3. Let $\mathcal{C}_1, \dots, \mathcal{C}_5$ be circles all passing through the points $A$ and $B$ on the plane. Show that there exists a circle $\gamma$ orthogonal to all of $\mathcal{C}_i$.
4. Prove Ptolemy's theorem: given a quadrilateral $ABCD$ inscribed into a circle, one has

$$|AB| \cdot |CD| + |BC| \cdot |AD| = |AC| \cdot |BD|.$$